Section 2.5: Higher-Ordered Derviatives

NOTATION:

- **Second Derivative:**

$$f''(x) \qquad \text{or} \qquad \frac{d^2}{dx^2}f(x) \qquad \text{or} \qquad y'' \qquad \text{or} \qquad \frac{d^2y}{dx^2}$$

- **Third Derivative:**

$$f'''(x) \qquad \text{or} \qquad \frac{d^3}{dx^3}f(x) \qquad \text{or} \qquad y''' \qquad \text{or} \qquad \frac{d^3y}{dx^3}$$

- **n-th Derivative:**

$$f^{(n)}(x) \qquad \text{or} \qquad \frac{d^n}{dx^n}f(x) \qquad \text{or} \qquad y^{(n)} \qquad \text{or} \qquad \frac{d^ny}{dx^n}$$

Example 1. Find (and simplify) the second derivative of $f(x) = \dfrac{2x+1}{3x-2}$.

Solution. We first find (and simplify) f':

$$f(x) = \frac{2x+1}{3x-2}$$

$$f'(x) = \frac{(3x-2)[2] - (2x+1)[3]}{(3x-2)^2}; \qquad \text{simplify the numerator}$$

$$f'(x) = \frac{6x-4-6x-3}{(3x-2)^2}$$

$$f'(x) = \boxed{\frac{-7}{(3x-2)^2}}$$

We now find (and simplify) f'':

$$f'(x) = \frac{-7}{(3x-2)^2}$$

$$f''(x) = \frac{(3x-2)^2[0] - (-7)\left[2(3x-2)(3)\right]}{\left((3x-2)^2\right)^2}; \qquad \text{multiply out the numerator}$$

$$f''(x) = \frac{42\cancel{(3x-2)}}{\cancel{(3x-2)^4}}; \qquad \text{cancel the common factor } (3x-2)$$

$$f''(x) = \boxed{\frac{42}{(3x-2)^3}}$$

Example 2. Find (and simplify) the second derivative of $f(x) = \dfrac{2x}{(x+5)^2}$.

Solution. We first find (and simplify) f':

$$f(x) = \frac{2x}{(x+5)^2}$$

$$f'(x) = \frac{(x+5)^2[2] - 2x[2(x+5)(1)]}{\left((x+5)^2\right)^2}$$

$$f'(x) = \frac{2(x+5)^2 - 4x(x+5)}{(x+5)^4}; \qquad \text{factor the numerator}$$

$$f'(x) = \frac{2(x+5)\big((x+5) - 2x\big)}{(x+5)^4}$$

$$f'(x) = \frac{2\cancel{(x+5)}(-x+5)}{\cancel{(x+5)^4}}; \qquad \text{cancel the common factor}$$

$$f'(x) = \boxed{\frac{-2(x-5)}{(x+5)^3}}$$

The second derivative is on the next page.

We now find (and simplify) f''. To make the derivative calculation easier, we will first multiply out the numerator:

$$f'(x) = \frac{-2(x-5)}{(x+5)^3} = \frac{-2x+10}{(x+5)^3}$$

$$f''(x) = \frac{(x+5)^3[-2] - (-2x+10)\left[3(x+5)^2(1)\right]}{\left((x+5)^3\right)^2}$$

$$f''(x) = \frac{-2(x+5)^3 - 3(-2x+10)(x+5)^2}{(x+5)^6}; \qquad \text{factor the numerator}$$

$$f''(x) = \frac{\cancel{(x+5)^2}\big(-2(x+5) - 3(-2x+10)\big)}{\cancel{(x+5)^6}}; \qquad \text{cancel the common factor}$$

$$f''(x) = \frac{-2(x+5) - 3(-2x+10)}{(x+5)^4}; \qquad \text{simplify the numerator}$$

$$f''(x) = \frac{-2x - 10 + 6x - 30}{(x+5)^4}$$

$$f''(x) = \frac{4x-40}{(x+5)^4}$$

$$f''(x) = \boxed{\frac{4(x-10)}{(x+5)^4}}$$

Example 3. Evaluate $\left.\dfrac{d^2}{dx^2}\left(\dfrac{4}{3}\pi x^3\right)\right|_{x=2}$

Notation. Consider the points $\big(a, f(a)\big)$ and $\big(x, f(x)\big)$ on the graph of $y = f(x)$. Let

$$\Delta x = x - a = \text{change in } x, \quad \text{and}$$

$$\Delta y = f(x) - f(a) = \text{change in } y.$$

Definition. The following are three equivalent ways of defining the derivative at $x = a$:

1. $f'(a) = \displaystyle\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

2. $f'(a) = \displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

3. $f'(a) = \displaystyle\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

Definition. The **average rate of change of** y **with respect to** x over the interval $[a, x]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a},$$

and the **instantaneous rate of change of** y **with respect to** x at $x = a$ is

$$(\text{instantaneous rate of change at } x = a) \;=\; \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \left.\frac{dy}{dx}\right|_{x=a}.$$

Result. The derivative $f'(a)$ is the *instantaneous rate of change* of $y = f(x)$ with respect to x when $x = a$. That is, the derivative measures how "fast" the y-values are changing when $x = a$. If $f'(a)$ is a large number (in absolute value), then the function is changing very rapidly when $x = a$. If $f'(a)$ is a small number (in absolute value), then the function is changing very slowly when $x = a$.

Some applications of the derivative in the physical sciences

- **Physics**. If $s(t)$ gives the distance a particle moving in a straight line travels at time t, then the **instantaneous velocity** of the particle at time t is

$$\text{velocity } = v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{\text{change in distance}}{\text{change in time}} = \frac{ds}{dt}.$$

- **Physics**. If $W(t)$ measures the amount of work done by a motor at time t, then the **power** done by the motor at time t is

$$\text{power } = P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \lim_{\Delta t \to 0} \frac{\text{change in work}}{\text{change in time}} = \frac{dW}{dt}.$$

  That is, power is the rate at which work is done.

- **Electricity**. If $Q(t)$ gives the amount of charge in a wire at time t, then the **current** i at time t is

$$\text{current } = i = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \lim_{\Delta t \to 0} \frac{\text{change in charge}}{\text{change in time}} = \frac{dQ}{dt}.$$

Example 4. During each cycle, the vertical displacement s (in cm) of the end of a robot arm is given by the equation

$$s = -3t^2 + 16t + 8, \qquad 0 \leq t \leq 5,$$

where t is the time (in seconds). Find the instantaneous velocity of the end of the robot arm when $t = 2.0\,\text{s}$.

Example 5. The electric power P (in Watts W) as a function of the current i (in amps A) in a certain circuit is given by the equation

$$P = 16i^2 + 60i, \qquad 0 \leq i \leq 3.$$

Find the instantaneous rate of change of P with respect to i when $i = 1.0\,\text{A}$.

Example 6. After t hours a passenger train is $s(t) = 24t^2 - 2t^3$ miles due west of its starting point for $0 \leq t \leq 12$.

(a) Find its velocity at time $t = 4$ hours.

(b) Find its acceleration at time $t = 2$ hours.

EXERCISES

Find (and simplify) the second derivative f''.

1. $f(x) = x^3 - 7x^2 + 3x - 5$
2. $f(x) = 2x^2 - 7x + 4\sqrt{x} + 3$
3. $f(x) = \sqrt{8x + 5}$
4. $f(x) = \dfrac{4x + 1}{x - 2}$
5. $f(x) = \dfrac{3x - 5}{x + 4}$
6. $f(x) = \dfrac{x^2 + 2}{x^2 - 4}$
7. $f(x) = \dfrac{x}{(x - 6)^2}$

8. Refining crude oil into various products, such as gasoline, heating oil, and plastics, requires heating and cooling the oil at different rates. Suppose that the temperature of the oil in degrees Fahrenheit at time t hours is given by the equation

$$T = t^2 - 7t + 150, \qquad 0 \le t \le 8.$$

   (a) Find the instantaneous rate of change of the temperature with respect to time.

   (b) Find the instantaneous rate of change at $t = 6$ hours and interpret this result.

   (c) Find the instantaneous rate of change at $t = 2$ hours and interpret this result.

9. The population of a town is after t weeks is given by the equation

$$P = 3t^2 - 12t + 200, \qquad 0 \le t \le 20.$$

   (a) Find the instantaneous rate of change of the population with respect to time.

   (b) Find the instantaneous rate of change of the population after 1 week and interpret this result.

   (c) Find the instantaneous rate of change of the population after 5 weeks and interpret this result.

10. A ball is thrown vertically upward from the top of a building 96 feet tall with an initial velocity of 80 feet per second. The height of the ball is given by the equation

$$h = -16t^2 + 80t + 96, \qquad 0 \le t \le 6,$$

where h is measured in feet and t is time measured in seconds.

   (a) Find the the instantaneous velocity of the ball.

   (b) Find the instantaneous velocity of the ball after 1.0 second.

11. The number of bacteria in a culture t hours after treatment with an antibiotic is given by

$$N = -t^2 + 12t + 1000, \qquad 1 \le t \le 30.$$

   (a) Find the instantaneous rate of change of the bacteria population with respect to time.

   (b) Find the instantaneous rate of change of the bacteria population after 2 hours and interpret this result.

   (c) Find the instantaneous rate of change of the bacteria population after 20 hours and interpret this result.

12. Nitroglycerin is often prescribed to enlarge blood vessels that have become too constricted. If the cross-sectional area (in cm^2) of a blood vessel t hours after nitroglycerin is administered is

$$A = 0.01t^2, \qquad 1 \le t \le 5,$$

find the instantaneous rate of change of the cross-sectional area 4 hours after the administration of nitroglycerin and interpret this result.

13. The value (in thousands of dollars) of a certain car is given by the equation

$$V = \frac{48}{t+3}, \qquad 0 \le t \le 20,$$

where t is measured in years.

(a) Find the the instantaneous rate of change of V with respect to time t.

(b) Find the instantaneous rate of change of V with respect to time t after 3 years.

14. An automobile dealership finds that the number of cars that it sells on day t of an advertising campaign is

$$N = -t^2 + 10t, \qquad 0 \le t \le 7.$$

(a) Find the instantaneous rate of change of the number of cars with respect to time.

(b) Find the instantaneous rate of change of the number of cars on day 3 and interpret this result.

(c) Find the instantaneous rate of change of the number of cars on day 6 and interpret this result.

15. In a psychology experiment, a person can memorize x words in $t = 2x^2 - x$ seconds.

(a) Find the instantaneous rate of change of the number of seconds with respect to the number of words.

(b) Find $\left.\dfrac{dt}{dx}\right|_{x=5}$ and interpret this as an instantaneous rate of change.

16. The number of people newly infected on day t of a flu epidemic is

$$P = 13t^2 - t^3, \qquad 0 \le t \le 13.$$

(a) Find the instantaneous rate of change of the number of people infected with respect to time.

(b) Find the instantaneous rate of change on day 5 and interpret this result.

(c) Find the instantaneous rate of change on day 10 and interpret this result.

ANSWERS

1. $f''(x) = 6x - 4$

2. $f''(x) = 4 - \dfrac{1}{x^{3/2}}$

3. $f''(x) = \dfrac{-16}{(8x+5)^{3/2}}$

4. $f''(x) = \dfrac{18}{(x-2)^3}$

5. $f''(x) = \dfrac{-34}{(x+4)^3}$

6. $f''(x) = \dfrac{12(3x^2+4)}{(x^2-4)^3}$

7. $f''(x) = \dfrac{2(x+12)}{(x-6)^4}$

8. (a) $\dfrac{dT}{dt} = 2t - 7$

   (b) $\left.\dfrac{dT}{dt}\right|_{t=6} = 5^\circ$ F/hr; after 6 hrs, the temperature is increasing at the rate of 5° F per hour.

   (c) $\left.\dfrac{dT}{dt}\right|_{t=2} = -3^\circ$ F/hr; after 2 hrs, the temperature is decreasing at the rate of 3° F per hour.

9. (a) $\dfrac{dP}{dt} = 6t - 12$

   (b) $\left.\dfrac{dP}{dt}\right|_{t=1} = -6$ people/week; after 1 week, the population is decreasing at a rate of 6 people per week.

   (c) $\left.\dfrac{dP}{dt}\right|_{t=5} = 18$ people/week; after 5 weeks, the population is increasing at a rate of 18 people per week.

10. (a) $v = \dfrac{dh}{dt} = -32t + 80$

(b) $\left.\dfrac{dh}{dt}\right|_{t=1.0} = 48$ ft/sec

11. (a) $\dfrac{dN}{dt} = -2t + 12$

(b) $\left.\dfrac{dN}{dt}\right|_{t=2} = 8$ bacteria/hour; after 2 hours, the number of bacteria in the culture are increasing at a rate of 8 bacteria per hour.

(c) $\left.\dfrac{dN}{dt}\right|_{t=20} = -28$ bacteria/hour; after 20 hours, the number of bacteria in the culture are decreasing at a rate of 28 bacteria per hour.

12. $\left.\dfrac{dA}{dt}\right|_{t=4} = 0.08\,\text{cm}^2/\text{hr}$; four hours after administrating nitroglycerin, the cross-sectional area of the blood vessel is increasing at a rate of $0.08\,\text{cm}^2$ per hour.

13. (a) $\dfrac{dV}{dt} = \dfrac{-48}{(t+3)^2}$

(b) $\left.\dfrac{dV}{dt}\right|_{t=3} = -1.33333$ thousands of dollars/year; when the car is 3 years old, its value is decreasing at a rate of \$1333.33 per year.

14. (a) $\dfrac{dN}{dt} = -2t + 10$

(b) $\left.\dfrac{dN}{dt}\right|_{t=3} = 4$ cars/day; on day 3 of the advertising campaign, the number of cars that the dealership sells is increasing at a rate of 4 cars per day.

(c) $\left.\dfrac{dN}{dt}\right|_{t=6} = -2$ cars/day; on day 6 of the advertising campaign, the number of cars that the dealership sells is decreasing at a rate of 2 cars per day.

15. (a) $\dfrac{dt}{dx} = 4x - 1$

(b) $\left.\dfrac{dt}{dx}\right|_{x=5} = 19$ seconds/word; when 5 words have been memorized, the memorization time is increasing at the rate of 19 seconds per word.

16. (a) $\dfrac{dP}{dt} = 26t - 3t^2$

(b) $\left.\dfrac{dP}{dt}\right|_{t=5} = 55$ people/day; on day 5 of a flu epidemic, the number of people infected is increasing at a rate of 55 people per day.

(c) $\left.\dfrac{dP}{dt}\right|_{t=10} = -40$ people/day; on day 10 of a flu epidemic, the number of people infected is decreasing at a rate of 40 people per day.