# Section 2.1: Limits Algebraically

**Recall.** A function $f$ is continuous at $x = a$ provided the graph of $y = f(x)$ does not have any holes, jumps, or breaks at $x = a$. (That is, the function is connected at $x = a$.)

If $f$ is continuous at $x = a$, then

$$\lim_{x \to a} f(x) = f(a).$$

That is, the value of the limit equals the value of the function.

**Result.** Almost all of the functions you are familiar with are continuous at every number in their domain. In particular, the following functions (and any combinations of these functions) are continuous at every number in their domain:

- polynomials $\big(\text{e.g., } f(x) = x^3 - 2x^2 + 7x - 2\big)$.
- rational functions $\left(\text{e.g., } f(x) = \dfrac{x^2 - 3x - 9}{x^2 - 2x - 3}\right)$
- radical functions $\big(\text{e.g., } f(x) = \sqrt{2x - 5}\big)$
- exponential functions $\big(\text{e.g., } f(x) = e^{3x}\big)$
- logarithmic functions $\big(\text{e.g., } f(x) = \ln(3x - 8)\big)$

Hence, to find the limit of any of the above function as $x$ approaches $a$, we simply evaluate that function at $x = a$.

**Example 1.** Find $\lim\limits_{x \to 3} x^2 + 4x + 1$.

Solution. The function $f(x) = x^2 + 4x + 1$ is continuous at all values of $x$. (Just think of the graph of $y = x^2 + 4x + 1$: it is a parabola and there are no holes, breaks, or jumps in the graph. Hence, it is continuous at all values of $x$.) Therefore, to evaluate the limit, we simply evaluate the function:

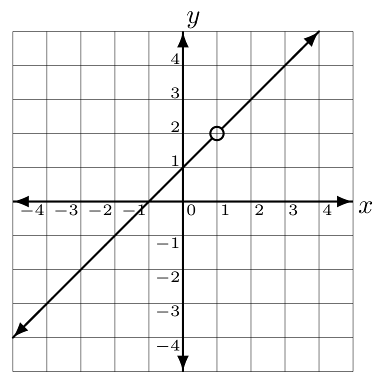
$$\lim_{x \to 3} x^2 + 4x + 1 = (3)^2 + 4(3) + 1 = \boxed{22}.$$

**Example 2.** Find $\displaystyle\lim_{x\to 1}\frac{x^2-1}{x-1}$.

SOLUTION. The function $f(x)=\frac{x^2-1}{x-1}$ is not continuous at $x=1$ since $f(1)=\left(\frac{0}{0}\right)$. Therefore, to find the limit, we must perform some algebra and eliminate the $\left(\frac{0}{0}\right)$ condition. In this case, we simplify the fraction:

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}}=\lim_{x\to 1}x+1=(1)+1=\boxed{2}$$

The graph of $y=\frac{x^2-1}{x-1}$ is given to the right. Notice that the graph has a hole at $x=1$ and hence is not continuous there.




**Example 3.** Find $\displaystyle\lim_{x\to 2}\frac{\frac{x+2}{x}-2}{x-2}$.

SOLUTION. The function $f(x)=\frac{\frac{x+2}{x}-2}{x-2}$ is not continuous at $x=2$ since $f(2)=\left(\frac{0}{0}\right)$. Therefore, to find the limit, we must perform some algebra and eliminate the $\left(\frac{0}{0}\right)$ condition. In this case, we simplify the complex fraction:

$$\begin{aligned}\lim_{x\to 2}\frac{\frac{x+2}{x}-2}{x-2}=\lim_{x\to 2}\frac{\frac{x+2}{x}-\frac{2x}{x}}{x-2}=\lim_{x\to 2}\frac{\frac{x+2-2x}{x}}{x-2}&=\lim_{x\to 2}\frac{\frac{-x+2}{x}}{x-2}\\&=\lim_{x\to 2}\frac{-x+2}{x}\cdot\frac{1}{x-2}\\&=\lim_{x\to 2}\frac{-1\cancel{(x-2)}}{x}\cdot\frac{1}{\cancel{x-2}}\\&=\lim_{x\to 2}\frac{-1}{x}\\&=\boxed{-\frac{1}{2}}\end{aligned}$$

Graphically, $y=\dfrac{\frac{x+2}{x}-2}{x-2}$ will have a hole at $x=2$ and hence will not be continuous there.

**Example 4.** Find $\lim_{x\to 4} \dfrac{x^2 - 2x - 8}{x - 4} =$

**Example 5.** Find $\lim_{x\to -4} \dfrac{x^3 + 4x^2 - x - 4}{x^2 + 5x + 4} =$

**Example 6.** Find $\lim_{x\to 0} \dfrac{\dfrac{1}{x+2} - \dfrac{1}{2}}{x} =$

**Example 7.** Find $\lim_{x\to 9} \dfrac{x-9}{\sqrt{x}-3} =$

# Supplemental Exercises

Find the following limits:

1. $\lim_{x\to 3} x^2 + 2x - 7 =$

2. $\lim_{x\to 5} \dfrac{x^2 - 2x - 15}{x - 5} =$

3. $\lim_{x\to 1} \dfrac{4x^4 - 5x^2 + 1}{x^2 + 2x - 3} =$

4. $\lim_{x\to 2} \dfrac{(2x+1)^2 - 25}{x - 2} =$

5. $\lim_{x\to 1} \dfrac{\frac{2x}{x+1} - 1}{x - 1} =$

6. $\lim_{x\to -2} \dfrac{x^4 - 2x^2 - 8}{x^2 - x - 6} =$

7. $\lim_{x\to 0} \dfrac{x^2 + 7x + 6}{x + 3} =$

8. $\lim_{x\to 2} \dfrac{x^3 + x^2 - 4x - 4}{x^2 + x - 6} =$

9. $\lim_{x\to -1} \dfrac{\frac{1}{x} + 1}{x + 1} =$

10. $\lim_{x\to 0} \dfrac{(x+1)^2 - 1}{x} =$

11. $\lim_{x\to -3} \dfrac{2x^2 + 2x - 12}{x^2 + 4x + 3} =$

12. $\lim_{x\to 2} \dfrac{(3x-2)^2 - (x+2)^2}{x - 2} =$

13. $\lim_{x\to 2} \dfrac{\frac{2}{x^2} - \frac{1}{2}}{x - 2} =$

14. $\lim_{x\to 1} \dfrac{x - 1}{\sqrt{x} - 1} =$

15. $\lim_{x\to -2} \dfrac{\frac{x}{x+4} + 1}{x + 2} =$

16. $\lim_{x\to 3} \dfrac{x^2 - 2x - 3}{x + 5} =$

# ANSWERS

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|---|---|---|---|---|---|---|---|
| 1. | 8 | 5. | 1/2 | 9. | −1 | 13. | −1/2 |
| 2. | 8 | 6. | 24/5 | 10. | 2 | 14. | 2 |
| 3. | 3/2 | 7. | 2 | 11. | 5 | 15. | 1 |
| 4. | 20 | 8. | 12/5 | 12. | 16 | 16. | 0 |