Section 3.3: Optimization

Definition. Let f be a function with domain D.

- Then f has an **absolute maximum** at $x = c$ if $f(c) \geq f(x)$ for all x in D. The number $f(c)$ is the **absolute maximum value** of f on D (occurring at $x = c$).
- Then f has an **absolute minimum** at $x = c$ if $f(c) \leq f(x)$ for all x in D. The number $f(c)$ is the **absolute minimum value** of f on D (occurring at $x = c$).
- Maximum and minimum values are called **extrema**.

Optimizing Continuous Functions on a Closed Interval

A continuous function f on a closed interval $[a, b]$ has both an absolute maximum and and absolute minimum. To find them:

1. Find all critical values of f in $[a, b]$.
2. Evaluate f at the critical values and at the endpoints a and b.

The largest and smallest values found in step 2 will be the absolute maximum and absolute minimum values of f on $[a, b]$.

Example 1. Find the absolute extreme values of $f(x) = 2x^4 - 8x^3 + 30$ on $[-1, 4]$.

Definitions: Common Economic Functions

- **Total cost function** C: describes the total cost $C(x)$ of producing x units of an item.

- **Average cost function** AC: describes the cost per unit when x units are produced.

$$\boxed{\text{Average cost} = AC(x) = \frac{C(x)}{x}}$$

- **Price function** (or **demand function**) p: describes the price that must be charged in order to sell x units.

- **Total revenue function** R: describes the total revenue $R(x)$ produced when x units of an item are sold.

$$\boxed{R(x) = x \cdot p(x)}$$

- **Total profit function** P: describes the total profit $P(x)$ realized from the sale of x units of an item.

$$\boxed{P(x) = R(x) - C(x) = x \cdot p(x) - C(x)}$$

- **Marginal cost function** MC: describes the rate of change of the cost relative to the change in the number of units produced. That is, marginal cost is the derivative of the total cost function.

$$\boxed{MC(x) = C'(x)}$$

  For large values of x, marginal cost is approximately the cost of producing one more unit:

$$MC(x) \approx C(x+1) - C(x) = \text{cost of producing one more unit}$$

  For example, if $MC(1000) = 0.09$, then the cost to produce the 1001st unit is approximately \$0.09.

- **Marginal Revenue** MR **and Marginal Profit** MP: analogous to marginal cost.

$$\boxed{MR(x) = R'(x)} \qquad \text{and} \qquad \boxed{MP(x) = P'(x)}$$

Example 2. A manufacturer of tennis rackets finds that the total cost $C(x)$ in dollars of manufacturing x rackets per day is given by

$$C(x) = 400 + 4x + 0.0001x^2.$$

Each racket can be sold at a price of p dollars, where p is related to x by the demand equation $p = 10 - 0.0004x$.

(a) If all rackets that are manufactured can be sold, find the daily level of production that will yield a maximum profit for the manufacturer.

(b) Find the maximum profit.

Example 3. A retired potter can produce china pitchers at a cost of \$5 each. She estimates her price function to be $p(x) = 17 - 0.5x$, where p is the price at which exactly x pitchers will be sold per week.

(a) Find the number of pitchers she should produce to maximize profit.

(b) Find the price that she should charge in order to maximize profit.

(c) Find the maximum profit.

Example 4. A farmer wants to make two identical rectangular enclosures along a straight river. If he has 600 yards of fence, and if the sides along the river need no fence, what should be the dimensions of each enclosure if the total area is to be maximized?

Example 5. A copier company finds that copiers that are x years old require, on average, $f(x) = 1.2x^2 - 4.7x + 10.8$ repairs annually for $0 \leq x \leq 5$. Find the year that requires the least repairs, rounding your answer to the nearest year.

EXERCISES

1. Suppose that total cost function for manufacturing a certain product is

$$C(x) = 0.002x^2 + 24$$

dollars, where x represents the number of units produced. Find the level of production that will minimize the average cost. Round answer to the nearest whole unit.

2. Suppose the quantity demanded per week of a certain dress is related to the unit price p by the demand function $p(x) = \sqrt{800 - x}$, where p is in dollars and x is the number of dresses made. To maximize the revenue, how many dresses should be made and sold each week? Round to the nearest whole unit.

3. The quantity demanded each month of a certain wristwatch is related to the unit price by the equation

$$p(x) = \frac{50}{0.01x^2 + 1}, \quad 0 \leq x \leq 20,$$

where p is measured in dollars and x is measured in thousands of units. To yield the maximum revenue, how many watches must be sold?

ANSWERS

1. 110 units

2. 533 dresses

3. $x = 10$ (i.e., 10,000 watches)