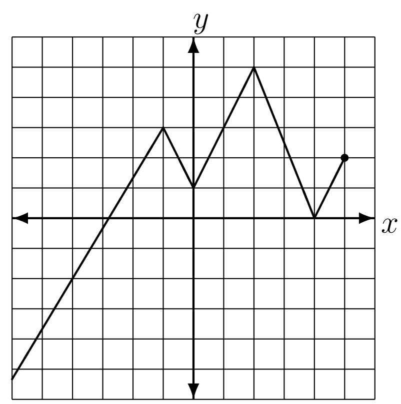
- A function f has an **absolute maximum** at $x = c$ if $f(c) \geq f(x)$ for all x in the domain of f. The number $f(c)$ is the absolute maximum.
- A function f has an **absolute minimum** at $x = c$ if $f(c) \leq f(x)$ for all x in the domain of f. The number $f(c)$ is the absolute minimum.
- A function f has a **local maximum** at $x = c$ if there exists an open interval I containing c such that $f(c) \geq f(x)$ for all $x \in I$.
- A function f has a **local minimum** at $x = c$ if there exists an open interval I containing c such that $f(c) \leq f(x)$ for all $x \in I$.

Example 1. Identify the absolute maximum, absolute minimum, local maximum(s), and local minimum(s) for the following function f, if they exist.



- **Fermat's Theorem:** If f has a local maximum or a local minimum at $x = c$, and if $f'(c)$ exists, then $f'(c) = 0$.
- NOTE: the converse of Fermat's Theorem is NOT true.
- A **critical number**, or **critical point**, of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.
- If f has a local maximum or local minimum at c, then c is a critical point of f.

Example 2. Find the critical points of the following functions.

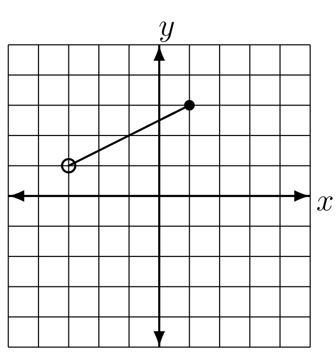
(a) $f(x) = x^3 + x^2 - x$

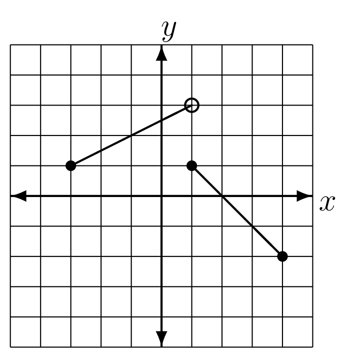
(b) $g(x) = x^{3/5}(4 - x)$

(c) $h(\theta) = 4\theta - \tan\theta$

Extreme Value Theorem: If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum $f(c)$ and absolute minimum $f(d)$ at some numbers c and d in $[a, b]$.

NOTE: The Extreme Value Theorem does not hold if the function is discontinuous on $[a, b]$ or if the interval is not closed.





How to find ABSOLUTE extrema of a function on a closed interval $[a, b]$.

STEP I: Find the critical number(s) of the function (say $x = c$) that are in the interval (a, b).

STEP II: Evaluate the **function** at the critical number(s). That is, calculate $f(c)$.

STEP III: Evaluate the **function** at each endpoint of $[a, b]$. That is, calculate $f(a)$ and $f(b)$.

STEP IV: The **absolute maximum** is the largest of the function values $f(c)$, $f(a)$, and $f(b)$.

STEP V: The **absolute minimum** is the smallest of the function values $f(c)$, $f(a)$, and $f(b)$.

Example 3. Find the absolute extrema of $f(x) = 3x^4 - 4x^3$ on the interval $[-1, 2]$.

STEP I: $f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$. So, the critical numbers are

$$12x^2(x-1) = 0 \quad \Longrightarrow \quad x = 0 \ \text{ or } \ x = 1,$$

each of which lies in the interval $(-1, 2)$.

STEP II: Evaluate the function at these critical numbers:
$f(0) = 3(0)^4 - 4(0)^3 = 0$,
$f(1) = 3(1)^4 - 4(1)^5 = -1$.

STEP III: Evaluate the function at the endpoints of the interval $[-1, 2]$:
$f(-1) = 3(-1)^4 - 4(-1)^3 = 7$,
$f(2) = 3(2)^4 - 4(2)^3 = 16$.

STEP IV: The largest function value is $f(2) = 16$. Hence $x = 2$, $f(2) = 16$ is the absolute maximum of f on the interval $[-1, 2]$.

STEP V: The smallest function value is $f(1) = -1$. Hence $x = 1$, $f(1) = -1$ is the absolute minimum of f on the interval $[-1, 2]$.