SECTION 4.7—OPTIMIZATION PROBLEMS

GUIDELINES FOR SOLVING OPTIMIZATION PROBLEMS

(a) If possible, draw a picture to illustrate the problem and label the the pertinent parts.

(b) Write down any relationships between the variables and constants in the problem.

(c) Express the quantity to be maximized or minimized as a function of just one of the variables, making sure that the domain of the function is physically reasonable.

(d) Find the critical numbers of the function.

(e) If the domain of the function is a closed interval, find the values of the function at the critical numbers and at the endpoints of the domain. The absolute maximum is the largest of these values, and the absolute minimum is the smallest. (See Section 4.1)

(f) If the domain of the function is not a closed interval, the problem may not have a solution. If a solution does exist, it must occur at one of the critical numbers or at an endpoint that is in the domain. Use the first and second derivatives of the function to help analyze the behavior of its graph and determine any absolute extrema.

Solve the following optimization problems:

1. Find two positive real numbers whose product is 16 and whose sum is a minimum.
2. Farmer MacDonald has 300 feet of chicken wire to use for constructing a rectangular pen for the chickens. The edge of a 300 foot long barn is to be used as one side of the pen, so the wire is needed only to fence in the remaining three sides. How can this be accomplished in a way that will maximize the amount of space the birds have to roam?
3. A rectangle is to be inscribed in a semicircle with radius 2 inches. Find the dimensions of the rectangle that encloses the maximum area.
4. A rectangular plot of ground containing 432 square feet is to be fenced off in a large lot and a fence is to be constructed down the middle of the lot to separate it into equal parts. Find the dimensions of the plot that requires the minimal amount of fencing.
5. Suppose that the fence being used to enclose the plot of ground described in Problem 4 costs $\$ 10$ per foot and the fence used to divide the plot into parts costs $\$ 5$ per foot. Find the dimensions of the plot that requires the least expense for fencing.
6. A rectangular printed page is to have margins 2 inches wide at the top and the bottom and margins 1 inch wide on each of the two sides. If the page is to have 35 square inches of printing, determine the minimum possible area of the page itself.
7. An open rectangular box is made from a piece of cardboard 8 inches wide and 8 inches long by cutting a square from each corner and bending up the sides. Find the dimensions of the box with the largest volume.

8. An outdoor track is to be built with two straight sides and semicircles at the ends, and it is to have a perimeter of 440 yards. Find the dimensions for the track that maximize the area of the rectangular portion of the field enclosed by the track.

9. A Norman window is a window in the shape of a rectangle with a semicircle attached at the top. Assuming that the perimeter of the window is 12 feet, find the dimensions that allow the maximum amount of light to enter.

10. Suppose a window has the shape of a rectangle with an equilateral triangle attached at the top. Assuming that the perimeter of the window is 12 feet, find the dimensions that allow the maximum amount of light to enter.

11. A ring of radius a carries a uniform electric charge Q. The electric field intensity at the point $x \geq 0$ along the axis of the ring is given by

$$E(x) = \frac{Qx}{(x^2 + a^2)^{3/2}}.$$

At what point on the axis is the electric field the greatest?

12. Find the points on the line $y = 3x - 2$ that is closest to the origin.

13. Find the points on the line $y = 2x - 4$ that are closest to the point $(1, 3)$.

14. A cylindrical can with top and bottom has volume 900 cm^3. Find the radius of the can with the smallest possible surface area.

ANSWERS

1. 4 and 4
2. 75 by 150 feet
3. $\sqrt{2}$ by $2\sqrt{2}$ inches
4. $18\sqrt{2}$ by $12\sqrt{2}$ feet
5. $6\sqrt{15}$ by $72/\sqrt{15}$ feet
6. $43 + 4\sqrt{70}$ square inches
7. 16/3 by 16/3 by 4/3 inches
8. radius is $110/\pi$ yards, length is 110 yards
9. radius and height are $12/(4 + \pi)$ feet
10. base is $12/(6 - \sqrt{3})$ feet, height is $(18 - 6\sqrt{3})/(6 - \sqrt{3})$ feet
11. $x = a/\sqrt{2}$
12. $(3/5, \ -1/5)$
13. $(3, 2)$
14. radius is $\sqrt[3]{450/\pi}$