MATH 12003 CALCULUS AND PARAMETRIC EQUATIONS Section 11.2

- **TANGENTS**

Parametric Form of the Derivative: If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \qquad \text{provided } \frac{dx}{dt} \neq 0.$$

Occasionally we need to find the second derivative $\frac{d^2y}{dx^2}$. This is given by

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

QUESTION 1: When does the curve have a horizontal tangent?

QUESTION 2: When does the curve have a vertical tangent?

EXAMPLE 1: For the curve given by $x = \sqrt{t}$ and $y = \frac{1}{4}\left(t^2 - 4\right)$ where $t \geq 0$, find the slope of the tangent line at $(2, 3)$.

EXAMPLE 2: Find an equation of the tangent to the curve defined by $x = \cos\theta + \sin 2\theta$ and $y = \sin\theta + \cos 2\theta$ at the point corresponding to $\theta = 0$.

EXAMPLE 3: Given $x = t^3 - 12t$ and $y = t^2 - 1$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. For which values of t is the curve concave upward?

EXAMPLE 4: Find the points on the curve $x = \cos 3\theta, \quad y = 2\sin\theta$ where the tangent line is horizontal or vertical.

Homework: pp 702–703; #1–19 odd, 25, 29

- **ARC LENGTH**

  Arc length of C: If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt.$$

EXAMPLE 1: Find the length of the curve $x = 3\left(\cos t + t \sin t\right),\ y = 3\left(\sin t - t\cos t\right),$ $0 \le t \le \pi$.

EXAMPLE 2: Find the length of the curve $x = e^t + e^{-t}, \quad y = 5 - 2t, \quad 0 \le t \le 3$.

Homework: pg 703; #37–47 (skip 43)

- **AREA OF A SURFACE**

Area of a Surface of Revolution: If a smooth curve C given by $x = f(t)$ and $y = g(t)$ does not cross itself on an interval $a \leq t \leq b$, then the area S of a surface of revolution formed by revolving C about the coordinate axes is given by the following:

$$S = 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \text{Revolution about the } x\text{−axis: } g(t) \geq 0$$

$$S = 2\pi \int_a^b f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \text{Revolution about the } y\text{−axis: } f(t) \geq 0$$

EXAMPLE 1: Find the area of the surface obtained by rotating the curve given by $x = 3t - t^3,\ y = 3t^2,\ \ 0 \leq t \leq 1$ about the x−axis.

EXAMPLE 2: Find the surface area generated by rotating the curve given by $x = e^t - t$, $y = 4e^{t/2},\ \ 0 \leq t \leq 1$ about the y−axis.

Homework: pp 703–704; #57, 59, 61, 65