Sequence: A **sequence** is a list of numbers written in a definite order:

$$a_1,\ a_2,\ a_3, \ldots,\ a_n, \ldots$$

The sequence $\{a_1,\ a_2,\ a_3, \ldots,\ a_n, \ldots\}$ can also be denoted as

$$\{a_n\} \qquad \text{or} \qquad (a_n)_n \qquad \text{or} \qquad \{a_n\}_{n=1}^{\infty}$$

EXAMPLE 1: List the first five terms of the sequence $a_n = \dfrac{n+1}{3n-1}$.

EXAMPLE 2: Find a formula for the general term a_n of the sequence

$$\left\{-\frac{1}{4},\ \frac{2}{9},\ -\frac{3}{16},\ \frac{4}{25}, \ldots\right\}$$

assuming that the pattern of the first few terms continues.

Limit of a sequence: A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \longrightarrow \infty} a_n = L \qquad \text{or} \qquad a_n \longrightarrow L \text{ as } n \longrightarrow \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If the limit, we say that the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

- $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$
- $\lim\limits_{n\to\infty} (a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n$
- $\lim\limits_{n\to\infty} ca_n = c \lim\limits_{n\to\infty} a_n$
- $\lim\limits_{n\to\infty} (a_n \cdot b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$
- $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ if $\lim\limits_{n\to\infty} b_n \neq 0.$
- $\lim\limits_{n\to\infty} a_n^p = \left(\lim\limits_{n\to\infty} a_n\right)^p$ if $p > 0$ and $a_n > 0$
- **Squeeze Theorem for Sequences**: If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.
- If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.
- If $\lim\limits_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim\limits_{n\to\infty} a_n = L$.

EXAMPLE 3: Determine whether the sequence converges or diverges. If it converges, find the limit.

1. $a_n = \dfrac{n+1}{3n-1}$
2. $a_n = \cos\left(\dfrac{2}{n}\right)$
3. $a_n = (-1)^n$
4. $\left\{\dfrac{\ln n}{\ln 2n}\right\}$
5. $a_n = \ln(n-1) - \ln n$

Important Property: The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if} \quad -1 < r < 1 \\ 1 & \text{if} \quad r = 1. \end{cases}$$

Increasing and Decreasing Sequences: A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$. A sequence is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. It is called **monotonic** if it is either increasing or decreasing.

Bounded Sequences: A sequence $\{a_n\}$ is **bounded above** if there is a number M such that $a_n \leq M$ for all $n \geq 1$. It is **bounded below** if there is a number m such that $m \leq a_n$ for all $n \geq 1$. If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

Monotonic Sequence Theorem: Every bounded, monotonic sequence is convergent.

EXAMPLE 4: Determine if the following sequences are increasing, decreasing, or neither. Determine if the following sequences are bounded above, bounded below, bounded, or not bounded.

1. $a_n = \dfrac{1}{5^n}$

2. $a_n = \dfrac{2n-3}{3n+4}$

3. $a_n = ne^{-n}$

4. $a_n = n + \dfrac{1}{n}$