Consider the following combinations:

$$\binom{k}{n} = \frac{k!}{(k-n)!\,n!} \qquad \text{for } n = 0, 1, 2, \ldots, k.$$

Writing out the terms, we see that

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

If k is a positive integer and we multiply out $(a+b)^k$ term by term, the coefficient of $a^{k-n}b^n$ is given by $\binom{k}{n}$. Accordingly, we refer to $\binom{k}{n}$ as a **binomial coefficient**.

$$\boxed{(a+b)^k = \sum_{n=0}^{k} \binom{k}{n} a^{k-n}\, b^k}$$

EXAMPLE: $(x+y)^5 =$

If we take $a = 1$ and $b = x$ we have

$$(1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n$$

Sir Isaac Newton extended this to the case in which k is no longer a positive integer. For this case, $(1+x)^k$ is no longer a finite sum, but an infinite sum.

Theorem 1 (The Binomial Theorem). If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

where $\binom{k}{n} = \dfrac{k!}{(k-n)!\,n!}$ for $n \geq 1$ and $\binom{k}{0} = 1$.

EXAMPLES: Use the binomial series to expand the function as a power series. State the radius of convergence.

1. $f(x) = \dfrac{1}{(1+x)^4}$

2. $g(x) = \dfrac{1}{\sqrt[5]{32-x}}$