<u>Series</u>: Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots$, let s_n denote its nth partial sum:

$$s_n = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write

$$\sum_{n=1}^{\infty} a_n = s.$$

The number s is called the **sum of the series**. Otherwise, the series is called **divergent**.

Geometric Series: Each term of a geometric series is obtained from the preceding one by multiplying it by a common ratio r. Namely,

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0.$$

The geometric series converges when $|r| < 1$ and its sum is given by

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1.$$

When $|r| \geq 1$, the geometric series is divergent.

Harmonic Series: The harmonic series is given by

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

The harmonic series is a divergent series.

Theorem 1. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Theorem 2. Test for Divergence: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Theorem 3. If $\sum_n a_n$ and $\sum_n b_n$ are convergent series, then so are the series $\sum ca_n$, $\sum(a_n+b_n)$, and $\sum(a_n - b_n)$ and

- $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$
- $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$
- $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

EXAMPLES: Determine whether the series is convergent or divergent. If it is convergent, find its sum.

1. $\sum_{n=1}^{\infty} \frac{(-6)^{n-1}}{5^{n-1}}$
2. $\sum_{n=1}^{\infty} \frac{4^{n+1}}{5^n}$
3. $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}$
4. $\sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} + \frac{2}{3^{n-1}}\right)$