Absolute Convergence: A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

EXAMPLE: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ is absolutely convergent since $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^3}\right| = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent since it is a p-series with $p = 3$.

EXAMPLE: $\displaystyle\sum \frac{(-1)^n}{n}$ is not absolutely convergent since $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series which diverges.

Conditional Convergence: A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

EXAMPLE: $\displaystyle\sum \frac{(-1)^n}{n}$ is conditionally convergent since it is convergent by the alternating series test, but is not absolutely convergent as noted above. Therefore, it is possible for a series to be convergent but not absolutely convergent.

Theorem: If a series $\sum a_n$ is absolutely convergent, then it is convergent.

This theorem gives us another tool to determine whether a series converges.

EXAMPLE: Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converges or diverges.

THE RATIO TEST

- If $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

- If $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or if $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

- If $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the ratio test is inconclusive and therefore no conclusion can be made about the convergence or divergence of the series. In this case, use another test to determine convergence or divergence.

EXAMPLES: Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

1. $\displaystyle\sum_{n=1}^{\infty} e^{-n} n!$

2. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2\, 2^n}{n!}$

3. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

THE ROOT TEST

- If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent.

- If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

- If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the root test is inconclusive and therefore no conclusion can be made about the convergence or divergence of the series. In this case, use another test to determine convergence or divergence.

EXAMPLES: Determine if the series is absolutely convergent, conditionally convergent, or divergent.

1. $$\sum_{n=1}^{\infty} \left(\frac{4n+1}{5n-3}\right)^n$$

2. $$\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$$

3. $$\sum_{n=1}^{\infty} \frac{n^n}{5^{2+4n}}$$

4. $$\sum_{n=1}^{\infty} \frac{n^3(n+1)^n}{(2n)^n}$$

5. $$\sum_{n=2}^{\infty} \frac{(-5)^n}{2^n}$$