Definition: A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

where x is a variable and the c_n's are the coefficients of the series. For each fixed x, the above series is a series of constants for which we can determine convergence or divergence. Note that a power series may converge for some values of x and diverge for other values of x.

Definition: A **power series about** a is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where we adopt the convention that $(x-a)^0 = 1$ even when $x = a$. Notice that when $x = a$ all of the terms except the first one are zero. Hence, the power series about a always converges when $x = a$.

A power series in x can be viewed as a function of x, $f(x) = \sum_{n=0}^{\infty} c_n x^n$, where the *domain* of f is the set of all x for which the power series converges.

Theorem 1. For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only three possibilities:

- The series converges only when $x = a$.
- The series converges for all x.
- There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The number R in the above theorem is called the **radius of convergence** of the power series. When the series converges for only $x = a$ then we say that $R = 0$, and when it converges for all x we say that $R = \infty$.

The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges. When the series converges at a singles point, then the interval is a single point a. When the series converges for all x, the interval is $(-\infty, \infty)$. When the series converges for certain values of x the interval is $a - R < x < a + R$. Note that series might converge at one or both endpoints or it might diverge at both endpoints.

EXAMPLES: Find the radius of convergence and the interval of convergence of the series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x+1)^n}{2^n}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n3^n}$

4. $\displaystyle\sum_{n=1}^{\infty} n^n x^n$

5. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$