MATH 12003 Representations of functions as power series SECTION 12.9

Recall the following form of the geometric series with $a = 1$ and $r = x$:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}$$

One way to represent certain types of functions as a power series is to manipulate the above geometric series.

EXAMPLES: Express the following functions as the sum of a power series and find the interval of convergence.

1. $f(x) = \dfrac{3}{1-x^4}$

2. $g(x) = \dfrac{x}{4x+1}$

3. $h(x) = \dfrac{x^3}{x+2}$

The sum of a power series is a function $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ whose domain is the interval of convergence of the series. We now would like to be able to differentiate and integrate such functions. The following theorems tells us that we can do so by differentiating or integrating each individual term in the series. This is called **term by term differentiation and integration.**

Theorem 1. If the power series $\sum c_n (x-a)^n$ has a radius of convergence $R > 0$, then the function f defined by

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

is differentiable and continuous on the interval $(a - R, a + R)$ and

- $f'(x) = \sum_{n=1}^{\infty} n\, c_n\, (x-a)^{n-1}$
- $\int f(x)\, dx = C + \sum_{n=0}^{\infty} c_n \, \frac{(x-a)^{n+1}}{n+1}$

The radii of convergence of both the derivative and antiderivative are both R.

NOTE: The above theorem says that the radius of convergence remains the same. However, the interval of convergence may not. Check the endpoints to determine if the new function converges or diverges.

EXAMPLES: Find the power series representation of the following functions and find its radius of convergence.

1. $\dfrac{1}{(1-x)^2}$

2. $\dfrac{x^2}{(1-2x)^2}$

3. $\tan^{-1}(2x)$

4. $\ln(1-3x)$