The Dot Product: If $\mathbf{a} = \langle a_1,\ a_2,\ a_3\rangle$ and $\mathbf{b} = \langle b_1,\ b_2,\ b_3\rangle$, then the **dot product** of **a** and **b** is the number $\mathbf{a}\cdot\mathbf{b}$ given by

$$\boxed{\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3}$$

NOTES:

- To find the dot product we multiply corresponding components together and then add.
- $\mathbf{a}\cdot\mathbf{b}$ is not a vector; it is a real number.
- The dot product is sometimes called the inner product or scalar product.
- The definition given above can be extended to any two vectors **a** and **b** in V_n.

EXAMPLE 1: Given $\mathbf{a} = \langle 3,\ -2,\ 5\rangle$ and $\mathbf{b} = \langle 4,\ 2,\ -6\rangle$, find $\mathbf{a}\cdot\mathbf{b}$.

Properties of the Dot Product

1. $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$
3. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$
4. $(c\mathbf{a})\cdot\mathbf{b} = c\,(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(c\mathbf{b})$
5. $\mathbf{0}\cdot\mathbf{a} = 0$

Theorem 1. If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}| \cos \theta$$

NOTES:

- This theorem gives a geometric interpretation of the dot product in terms of the angle θ between the representations of $\mathbf{a}$ and $\mathbf{b}$ that begin at the origin with $0 \le \theta \le \pi$.
- If $\mathbf{a}$ and $\mathbf{b}$ are parallel, then $\theta = 0$ or $\theta = \pi$.
- This theorem also allows us to find the angle between any two vectors.
- Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are **perpendicular** or **orthogonal** if the angle between them is $\theta = \dfrac{\pi}{2}$. Using the above theorem we have

$$\boxed{\mathbf{a} \text{ and } \mathbf{b} \text{ are orthogonal if and only if } \mathbf{a} \cdot \mathbf{b} = 0}$$

- We can think of the dot product as measuring the extent to which $\mathbf{a}$ and $\mathbf{b}$ point in the same direction. The dot product is positive if $\mathbf{a}$ and $\mathbf{b}$ point in the same general direction, zero if they are orthogonal, and negative if they point in generally opposite directions. When $\mathbf{a}$ and $\mathbf{b}$ point in exactly the same direction, $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|$. If they point in exactly opposite directions, then $\mathbf{a} \cdot \mathbf{b} = -\ |\mathbf{a}|\ |\mathbf{b}|$.

EXAMPLE 2: Find the angle between the vectors $\mathbf{a} = \langle 6, -3, 2 \rangle$ and $\mathbf{b} = \langle 2, 1, -2 \rangle$.

EXAMPLE 3: Determine if $\mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ are orthogonal, parallel, or neither.

EXAMPLE 4: For what values of b are the vectors $\langle -6,\, b,\, 2\rangle$ and $\langle b,\, b^2,\, b\rangle$ orthogonal?

EXAMPLE 5: Find two unit vectors that make an angle of 60° with $\mathbf{v} = \langle 3,\, 4\rangle$.