The Cross Product (definition 1): If $\mathbf{a} = \langle a_1,\ a_2,\ a_3\rangle$ and $\mathbf{b} = \langle b_1,\ b_2,\ b_3\rangle$, then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector $\mathbf{a} \times \mathbf{b}$ given by

$$\boxed{\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2,\, a_3b_1 - a_1b_3,\, a_1b_2 - a_2b_1\rangle}$$

A **determinant of order 2:** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

A **determinant of order 3:** can be defined in terms of second-order determinants as

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

The Cross Product (definition 2): The cross product of vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

NOTE: The cross product is also called the **vector product**.

EXAMPLE 1: Calculate $\mathbf{a} \times \mathbf{b}$.

1. $\mathbf{a} = \langle 5,\, 1,\, 4\rangle$ and $\mathbf{b} = \langle -1,\, 3,\, -2\rangle$

2. $\mathbf{a} = \mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k}$

Theorem 1. The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

NOTE: This theorem states that if $\mathbf{a}$ and $\mathbf{b}$ are represented by directed line segments with the same initial point, then the cross product $\mathbf{a} \times \mathbf{b}$ points in the direction perpendicular to the plane through $\mathbf{a}$ and $\mathbf{b}$.

Theorem 2. If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $0 \leq \theta \leq \pi$, then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\ |\mathbf{b}|\ \sin\theta.$$

NOTE: The length of the cross product $\mathbf{a}\times\mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

Corollary 1. Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}.$$

EXAMPLE 2: Find two unit vectors orthogonal to both $\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $2\mathbf{i}+\mathbf{k}$.

EXAMPLE 3: Find a vector orthogonal to the plane through the points $P = (2,\ 1,\ 5)$, $Q = (-1,\ 3,\ 4)$, and $R = (3,\ 0,\ 6)$.

Properties of the Cross Product

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Scalar triple product of vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is given by $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. It is useful to note that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

NOTE: The volume of the parallelepiped determined by the vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ is the magnitude (absolute value) of their scalar triple product. That is,

$$\boxed{V = |\ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}\ )|}$$

If the volume is found to be zero, then the vectors must lie in the same plane; that is, they are **coplanar**.

EXAMPLE 4: Find the volume of the parallelepiped determined by the vectors $\mathbf{a} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, and $\mathbf{c} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$.

EXAMPLE 5: Determine whether the points $P = (1,\ 0,\ 1)$, $Q = (2,\ 4,\ 6)$, $R = (3,\ -1,\ 2)$, and $S = (6,\ 2,\ 8)$ lie in the same plane.