<u>Definition</u>: If f is continuous function defined for $a \leq x \leq b$, we divide $[a, b]$ into n subintervals of equal width $\Delta x = \dfrac{b-a}{n}$. We let $x_0 = a,\ x_1,\ x_2,\ \ldots,\ x_n = b$ be the endpoints of these subintervals. Next, we choose sample points $x_1^*,\ x_2^*,\ \ldots,\ x_n^*$ in these subintervals, so x_i^* lies in the i-th subinterval $[x_{i-1}, x_i]$. Then the **definite integral of** f from a to b is

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Now, we want to consider the volume of 3-dimensional solids. Consider the following:

If we approximate the volume of the solid by n such discs of width Δx and radius $f(x_i^*)$ gives

$$\begin{aligned}\text{Volume of solid} &\approx \sum_{i=1}^{n} \pi \left[f(x_i^*)\right]^2 \Delta x \\ &= \pi \sum_{i=1}^{n} \left[f(x_i^*)\right]^2 \Delta x\end{aligned}$$

This approximation is better as $n \to \infty$. Thus,

$$\begin{aligned}\text{Volume of solid} &= \lim_{n\to\infty} \pi \sum_{i=1}^{n} \left[f(x_i^*)\right]^2 \Delta x \\ &= \pi \int_a^b \left[f(x)\right]^2\ dx\end{aligned}$$

The Disc Method

To find the volume of a solid of revolution with the disc method, use one of the following:

Horizontal Axis of Revolution

$V = \int_a^b \pi \left[f(x)\right]^2 \, dx$

Vertical Axis of Revolution

$V = \int_c^d \pi \left[f(y)\right]^2 \, dy$

NOTE: You can determine the variable of integration by placing a representative rectangle in the plane region *perpendicular* to the axis of revolution. If the width of the rectangle is Δx, integrate with respect to x, and if the width of the rectangle is Δy, integrate with respect to y.

The disc method can be extended to cover solids of revolution with holes by replacing the representative disc by a representative washer. Then the volume of each washer is given by

$$\text{Volume of each washer} = \pi \left[(\text{outer radius})^2 - (\text{inner radius})^2)\right] \cdot (\text{width}).$$

Therefore, volume of the resulting solid is given by

$$V = \int_a^b \pi \left([\text{outer radius}]^2 - [\text{inner radius}]^2\right) \, dx$$