Definition: If $f$ is a continuous function defined for $a \leq x \leq b$, we divide $[a, b]$ into $n$ subintervals of equal width $\Delta x = \frac{b - a}{n}$. We let $x_0 = a$, $x_1$, $x_2$, $\ldots$, $x_n = b$ be the endpoints of these subintervals. Next, we choose sample points $x^*_1$, $x^*_2$, $\ldots$, $x^*_n$ in these subintervals, so $x^*_i$ lies in the $i$-th subinterval $[x_{i-1}, x_i]$. Then the **definite integral** of $f$ from $a$ to $b$ is

$$
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x^*_i) \Delta x
$$

Now, we want to consider the volume of 3-dimensional solids. Consider the following:

If we approximate the volume of the solid by $n$ such discs of width $\Delta x$ and radius $f(x^*_i)$ gives

$$
\text{Volume of solid} \approx \sum_{i=1}^{n} \pi [f(x^*_i)]^2 \Delta x
$$

$$
= \pi \sum_{i=1}^{n} [f(x^*_i)]^2 \Delta x
$$

This approximation is better as $n \to \infty$. Thus,

$$
\text{Volume of solid} = \lim_{n \to \infty} \pi \sum_{i=1}^{n} [f(x^*_i)]^2 \Delta x
$$

$$
= \pi \int_a^b [f(x)]^2 \, dx
$$
The Disc Method

To find the volume of a solid of revolution with the disc method, use one of the following:

<table>
<thead>
<tr>
<th>Horizontal Axis of Revolution</th>
<th>Vertical Axis of Revolution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V = \int_a^b \pi [f(x)]^2 , dx$</td>
<td>$V = \int_c^d \pi [f(y)]^2 , dy$</td>
</tr>
</tbody>
</table>

NOTE: You can determine the variable of integration by placing a representative rectangle in the plane region perpendicular to the axis of revolution. If the width of the rectangle is $\Delta x$, integrate with respect to $x$, and if the width of the rectangle is $\Delta y$, integrate with respect to $y$.

The disc method can be extended to cover solids of revolution with holes by replacing the representative disc by a representative washer. Then the volume of each washer is given by

$$ \text{Volume of each washer} = \pi \left( [\text{outer radius}]^2 - [\text{inner radius}]^2 \right) \cdot \text{(width)}.$$

Therefore, volume of the resulting solid is given by

$$ V = \int_a^b \pi \left( [\text{outer radius}]^2 - [\text{inner radius}]^2 \right) \, dx $$