Math 12003 **The Shell Method** **Section 6.3**

Sometimes finding the volume of a 3-dimensional solid is very difficult using the disc method. Therefore, we may want to consider the shell method. Before we discuss the method, let us first review one of key parts of the formula. Consider the following cylindrical shell with inner radius r_1 and outer radius r_2, and height h.

Recalling that the volume of a cylinder is $2\pi(\text{radius})(\text{height})$, we find the volume of the above cylindrical shell by subtracting the volume of the inner cylinder from the volume of the outer cylinder; namely,

$$\begin{aligned} V &= \pi r_2^2 h - \pi r_1^2 h \\ &= \pi h \left(r_2^2 - r_1^2\right) \\ &= \pi h \left(r_2 + r_1\right)\left(r_2 - r_1\right) \\ &= 2\pi h \left(\frac{r_2 + r_1}{2}\right)\left(r_2 - r_1\right) \\ &= 2\pi h r \Delta r \end{aligned}$$

where $\Delta r = r_2 - r_1$ and $r = \dfrac{r_2 + r_1}{2}$. Notice that r is the average radius of the shell. Thus the volume of this cylindrical shell becomes $V = 2\pi r h \Delta r = (\text{circumference})(\text{height})(\text{thickness})$.

Now we are ready to discuss the shell method. Consider the solid obtained by rotating about the $y-$axis the region bounded by $y = f(x)$ (where $f(x) \geq 0$), $y = 0, x = a$, and $x = b$ as shown below.

Dividing into n subintervals $[x_{i-1}, x_i]$ of equal width Δx, let $\overline{x_i}$ be the midpoint of the ith subinterval. If this ith subinterval (represented as a rectangle) is rotated about the $y-$axis, its height is $f(\overline{x_i})$, its average radius is $\overline{x_i}$, and its thickness is Δx. Therefore, as discussed above the volume of this ith shell is $V_i = 2\pi\overline{x_i}f(\overline{x_i})\Delta x$. Therefore, the volume of the solid is found by summing all n such shells to obtain

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi\overline{x_i}f(\overline{x_i})\Delta x.$$

We know that this approximation becomes better as $n \to \infty$. Therefore, we get,

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi\overline{x_i}f(\overline{x_i})\Delta x = \int_a^b 2\pi x f(x)\, dx,$$

and we obtain the formula used for the shell method.

The Shell Method

To find the volume of a solid of revolution with the shell method, use one of the following:

Rotating about the x-axis	*Rotating about the $y-$axis*
$V = \int_c^d 2\pi y f(y)\, dy$	$V = \int_a^b 2\pi x f(x)\, dx$

NOTES:

- You can determine the variable of integration by placing a representative rectangle in the plane region *parallel* to the axis of revolution. If the width of the rectangle is Δx, integrate with respect to x, and if the width of the rectangle is Δy, integrate with respect to y.

- In the above formulas, the x or y represents the radius. If you rotate around a different vertical axis or horizontal axis, the formulas will need to be altered to illustrate this.