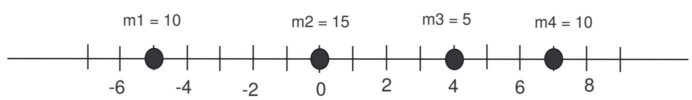
Moments and Center of Mass: One-Dimensional System: Let the point masses $m_1, m_2, \dots, m_n$ be located at $x_1, x_2, \dots, x_n$.

- The **moment about the origin** is $M_0 = m_1x_1 + m_2x_2 + \cdots + m_nx_n$.
- The **center of mass** is $\overline{x} = \dfrac{M_0}{m}$, where $m = m_1 + m_2 + \cdots + m_n$ is the total mass of the system.

EXAMPLE: Find the center of mass of the linear system shown below:



Moments and Center of Mass: Two-Dimensional System: Let the point masses $m_1, m_2, \dots, m_n$ be located at $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$.

- The **moment about the $y-$axis** is $M_y = m_1x_1 + m_2x_2 + \cdots + m_nx_n$.
- The **moment about the $x-$axis** is $M_x = m_1y_1 + m_2y_2 + \cdots + m_ny_n$.
- The **center of mass** $(\overline{x}, \overline{y})$ (or **center of gravity**) is

$$\overline{x} = \frac{M_y}{m} \qquad \text{and} \qquad \overline{y} = \frac{M_x}{m}$$

where $m = m_1 + m_2 + \cdots + m_n$ is the total mass of the system.

EXAMPLE: Find the center of mass of the system of point masses $m_1 = 6, m_2 = 3, m_3 = 2$, and $m_4 = 9$, located at $(3, -2)$, $(0, 0)$, $(-5, 3)$, and $(4, 2)$.

A **planar lamina** is a thin, flat plate of material of uniform density. (For a planar lamina density is a measure of mass per unit of area.)

Moments and Center of Mass of a Planar Lamina: Let f and g be continuous functions such that $f(x) \geq g(x)$ on $[a, b]$ and consider the planar lamina of uniform density ρ bounded by the graphs of $y = f(x)$ and $y = g(x)$, and $a \leq x \leq b$.

- The **moments about the $x-$axis and $y-$axis** are

$$M_x = \int_a^b \frac{\rho}{2} \left(f^2(x) - g^2(x) \right) \, dx$$
$$M_y = \int_a^b \rho x \left(f(x) - g(x) \right) \, dx$$

- The **center of mass** $(\overline{x}, \overline{y})$ is given by

$$\overline{x} = \frac{M_y}{m} \qquad \text{and} \qquad \overline{y} = \frac{M_x}{m},$$

  where $m = \int_a^b \rho \left(f(x) - g(x) \right) \, dx$ is the mass of the region.

EXAMPLE: Find the center of mass of the lamina of uniform density ρ bounded by the graph of $f(x) = 4 - x^2$ and the $x-$axis.

Since the density ρ is a common factor of both the moments and the mass, it cancels out of the quotients representing the coordinates of the center of mass. Thus, the center of mass of a lamina of uniform density depends only on the shape of the lamina and not on its density. For this reason, the point $(\overline{x}, \overline{y})$ is sometimes called the center of mass of a region in the plane, or the **centroid** of the region. In other words, to find the centroid of a region in the plane, you simply assume that the region has a constant density of $\rho = 1$ and compute the corresponding center of mass.

EXAMPLE: Find the centroid of the region bounded by the graphs of $f(x) = 4 - x^2$ and $g(x) = x + 2$.

The Theorem of Pappus: Let R be a region in a plane and let L be the line in the same plane such that L does not intersect the interior of R. If r is the distance between the centroid of R and the line, then the volume V of the solid of revolution formed by revolving R about the line is

$$V = 2\pi r A$$

where A is the area of R. (Note that $2\pi r$ is the distance traveled by the centroid as the region is revolved about the line.)

EXAMPLE: Find the volume of the torus formed by revolving the circular region bounded by $(x-2)^2 + y^2 = 1$ about the $y-$axis.