CYLINDRICAL COORDINATES: A point P in 3-dimensional space is represented by the ordered triple $(r,\ \theta,\ z)$, where r and θ are the polar coordinates of the projection of P onto the $xy-$plane and z is the directed distance from the $xy-$plane to P.

To find the cylindrical coordinates for a point $(x,\ y,\ z)$, we use the equations

$$\boxed{r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}, \qquad z = z}$$

To find the rectangular coordinates for a point $(r,\ \theta,\ z)$ we recall our conversions for polar coordinates and use

$$\boxed{x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z}$$

EXAMPLE 1:

(a) Plot the point whose cylindrical coordinates are $\left(1,\ \frac{\pi}{4},\ e\right)$. Find the rectangular coordinates of the point.

(b) Find cylindrical coordinates of the point with rectangular coordinates $(1,\ 1,\ 2)$.

Our primary interest in this coordinate system is that it allows us to express various surfaces in a more simplified manner. It is especially useful for problems involving symmetry about an axis, and the $z-$axis is chosen to coincide with this axis of symmetry. Eventually, this will allow us to simplify certain integrals.

EXAMPLE 2: Describe the surface of constant r, θ, or z.

(a) The equation $\theta = \frac{\pi}{2}$ defines a plane; in fact, in this case the $xy-$plane.

(b) The equation $z = 5$ defines a horizontal plane at a "height" of 5 units.

(c) The equation $r = 4$ defines a cylinder of radius 4 with the axis of symmetry the $z-$axis.

EXAMPLE 3: Write the equation $r = 8\sin\theta$ into rectangular coordinates.

SPHERICAL COORDINATES: The spherical coordinates (ρ, θ, ϕ) of a point P in space are where ρ is the distance from the origin O to P, θ is the same angle as in the cylindrical coordinates, and ϕ is the angle between the positive $z-$axis and the line segment OP. Therefore,

$$\rho \geq 0 \qquad \text{and} \qquad 0 \leq \phi \leq \pi.$$

The spherical coordinate systems is especially useful in problems where there is symmetry at a specific point and the origin is placed at this point.

To convert from spherical to rectangular is a little more complicated. First, since ρ is the distance between the origin and $P = (x,\, y,\, z)$, we obtain $\rho^2 = x^2 + y^2 + z^2$. Suppose that P has $z > 0$ and let $P' = (x,\, y,\, 0)$ and let $Q = (0,\, 0,\, z)$ on the $z-$axis.

Consider the right triangle formed by P, Q and the origin. The angle at the origin in this triangle is by definition ϕ. One leg of the triangle is the portion of the $z-$axis between the origin and the point Q which has length z. The other leg is the line segment from Q to P which using the distance formula has length $r = \sqrt{x^2 + y^2}$. Using right triangle trigonometry we have $\cos\phi = \dfrac{r}{\rho}$ and $\sin\phi = \dfrac{z}{\rho}$. Thus, $r = \rho\sin\phi$ and $z = \rho\cos\phi$.

For $P' = (x,\, y,\, 0)$ we see that $r = \sqrt{x^2 + y^2}$ is the distance from P' to the origin. As in polar coordinates, we have $x = r\cos\theta$ and $y = r\sin\theta$. Using $r = \rho\sin\phi$, we have $x = \rho\sin\phi\cos\theta$ and $y = \rho\sin\phi\sin\theta$.

Therefore, **to convert from spherical coordinates to rectangular coordinates**, we use the following equations:

$$\boxed{x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi}$$

To convert from rectangular to spherical coordinates, we use

$$\boxed{\rho^2 = x^2 + y^2 + z^2, \quad \cos\phi = \frac{z}{\rho}, \quad \cos\theta = \frac{x}{\rho\sin\phi}}$$

EXAMPLE 4:

(a) Find the spherical coordinates of the point with rectangular coordinates $\left(-1,\, 1,\, \sqrt{6}\right)$

(b) Find the rectangular coordinates of the point with spherical coordinates $(5,\, \pi,\, \pi/2)$.

(c) Find the cylindrical coordinates of a point with spherical coordinates $(4, \pi/4, \pi/3)$.

EXAMPLE 5: Describe in words, the surface whose equation is given.

(a) The equation $\rho = 5$ is a sphere centered at the origin with radius 5.

(b) The equation $\theta = \dfrac{\pi}{3}$ is a half-plane including the $z-$axis and intersecting the $xy-$plane in the half-line $y = \sqrt{3}x, \ x > 0$.

(c) The equation $\phi = \dfrac{\pi}{2}$ gives the $xy-$plane since $z = 0$ and there is no restriction on x or y.

(d) The equation $\phi = \dfrac{\pi}{3}$ gives a half-cone with vertex at the origin and axis the positive $z-$axis.

EXAMPLE 6: Sketch the solid described by the given inequalities:

(a) $2 \leq \rho \leq 3, \qquad \pi/2 \leq \phi \leq \pi$

(b) $0 \leq \phi \leq \dfrac{\pi}{3}, \qquad \rho \leq 2$

Homework: pp 878–879; 3–45 odd, 55, 58, 59, 61