CYLINDRICAL COORDINATES: A point $P$ in 3-dimensional space is represented by the ordered triple $(r, \theta, z)$, where $r$ and $\theta$ are the polar coordinates of the projection of $P$ onto the $xy$–plane and $z$ is the directed distance from the $xy$–plane to $P$.

To find the cylindrical coordinates for a point $(x, y, z)$, we use the equations

\[
r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}, \quad z = z
\]

To find the rectangular coordinates for a point $(r, \theta, z)$ we recall our conversions for polar coordinates and use

\[
x = r \cos \theta, \quad y = r \sin \theta, \quad z = z
\]

EXAMPLE 1:

(a) Plot the point whose cylindrical coordinates are $(1, \frac{\pi}{4}, e)$. Find the rectangular coordinates of the point.

(b) Find cylindrical coordinates of the point with rectangular coordinates $(1, 1, 2)$. 
Our primary interest in this coordinate system is that it allows us to express various surfaces in a more simplified manner. It is especially useful for problems involving symmetry about an axis, and the $z-$axis is chosen to coincide with this axis of symmetry. Eventually, this will allow us to simplify certain integrals.

**EXAMPLE 2:** Describe the surface of constant $r$, $\theta$, or $z$.

(a) The equation $\theta = \frac{\pi}{2}$ defines a plane; in fact, in this case the $xy-$plane.

(b) The equation $z = 5$ defines a horizontal plane at a “height” of 5 units.

(c) The equation $r = 4$ defines a cylinder of radius 4 with the axis of symmetry the $z-$axis.

**EXAMPLE 3:** Write the equation $r = 8 \sin \theta$ into rectangular coordinates.

**SPHERICAL COORDINATES:** The spherical coordinates $(\rho, \theta, \phi)$ of a point $P$ in space are where $\rho$ is the distance from the origin $O$ to $P$, $\theta$ is the same angle as in the cylindrical coordinates, and $\phi$ is the angle between the positive $z-$axis and the line segment $OP$. Therefore,

$$\rho \geq 0 \quad \text{and} \quad 0 \leq \phi \leq \pi.$$ 

The spherical coordinate systems is especially useful in problems where there is symmetry at a specific point and the origin is placed at this point.
To convert from spherical to rectangular is a little more complicated. First, since \( \rho \) is the distance between the origin and \( P = (x, y, z) \), we obtain \( \rho^2 = x^2 + y^2 + z^2 \). Suppose that \( P \) has \( z > 0 \) and let \( P' = (x, y, 0) \) and let \( Q = (0, 0, z) \) on the \( z \)-axis.

Consider the right triangle formed by \( P, Q \) and the origin. The angle at the origin in this triangle is by definition \( \phi \). One leg of the triangle is the portion of the \( z \)-axis between the origin and the point \( Q \) which has length \( z \). The other leg is the line segment from \( Q \) to \( P \) which using the distance formula has length \( r = \sqrt{x^2 + y^2} \). Using right triangle trigonometry we have \( \cos \phi = \frac{r}{\rho} \) and \( \sin \phi = \frac{z}{\rho} \). Thus, \( r = \rho \sin \phi \) and \( z = \rho \cos \phi \).

For \( P' = (x, y, 0) \) we see that \( r = \sqrt{x^2 + y^2} \) is the distance from \( P' \) to the origin. As in polar coordinates, we have \( x = r \cos \theta \) and \( y = r \sin \theta \). Using \( r = \rho \sin \phi \), we have \( x = \rho \sin \phi \cos \theta \) and \( y = \rho \sin \phi \sin \theta \).

Therefore, to convert from spherical coordinates to rectangular coordinates, we use the following equations:

\[
\begin{align*}
x &= \rho \sin \phi \cos \theta, \\
y &= \rho \sin \phi \sin \theta, \\
z &= \rho \cos \phi
\end{align*}
\]

To convert from rectangular to spherical coordinates, we use

\[
\begin{align*}
\rho^2 &= x^2 + y^2 + z^2, \\
\cos \phi &= \frac{z}{\rho}, \\
\cos \theta &= \frac{x}{\rho \sin \phi}
\end{align*}
\]

**EXAMPLE 4:**

(a) Find the spherical coordinates of the point with rectangular coordinates \((-1, 1, \sqrt{6})\)

(b) Find the rectangular coordinates of the point with spherical coordinates \((5, \pi, \pi/2)\).

(c) Find the cylindrical coordinates of a point with spherical coordinates \((4, \pi/4, \pi/3)\).
EXAMPLE 5: Describe in words, the surface whose equation is given.

(a) The equation $\rho = 5$ is a sphere centered at the origin with radius 5.

(b) The equation $\theta = \frac{\pi}{3}$ is a half-plane including the $z-$axis and intersecting the $xy-$plane in the half-line $y = \sqrt{3}x, \ x > 0$.

(c) The equation $\phi = \frac{\pi}{2}$ gives the $xy-$plane since $z = 0$ and there is no restriction on $x$ or $y$.

(d) The equation $\phi = \frac{\pi}{3}$ gives a half-cone with vertex at the origin and axis the positive $z-$axis.

EXAMPLE 6: Sketch the solid described by the given inequalities:

(a) $2 \leq \rho \leq 3, \ \frac{\pi}{2} \leq \phi \leq \pi$

(b) $0 \leq \phi \leq \frac{\pi}{3}, \ \rho \leq 2$

Homework: pp 878–879; 3–45 odd, 55, 58, 59, 61