Derivative of vector function: The **derivative of r**, denoted $\mathbf{r}'$, is defined by

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if the limit exists.

NOTES:

- The vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve defined by $\mathbf{r}$ at the point P, provided that $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq \mathbf{0}$.

- The **tangent line** to the curve C at the point P is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t)$.

Theorem: If $\mathbf{r}(t) = \langle f(t),\, g(t),\, h(t)\rangle = f(t)\mathbf{i}+g(t)\mathbf{j}+h(t)\mathbf{k}$, where f, g, and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t),\, g'(t),\, h'(t)\rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$

EXAMPLE 1: Find the derivative of the vector function $\mathbf{r}(t) = at\cos 3t\mathbf{i} + b\sin^3 t\mathbf{j} + c\cos^3 t\mathbf{k}$.

EXAMPLE 2: Find parametric equations for the tangent line to the curve with the parametric equations

$$x = t^2 - 1 \qquad y = t^2 + 1 \qquad z = t + 1$$

at the point (3, 5, 3).

smooth: A curve given by $\mathbf{r}(t)$ on an interval I is smooth if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq 0$ except possibly at the endpoints of I.

piecewise smooth: A curve is piecewise smooth if it is made up of a finite number of smooth pieces.

Differentiation Rules

1. $\dfrac{d}{dt}\left[\mathbf{u}(t) + \mathbf{v}(t)\right] = \mathbf{u}'(t) + \mathbf{v}'(t)$

2. $\dfrac{d}{dt}\left[c\mathbf{u}(t)\right] = c\mathbf{u}'(t)$

3. $\dfrac{d}{dt}\left[f(t)\mathbf{u}(t)\right] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

4. $\dfrac{d}{dt}\left[\mathbf{u}(t) \cdot \mathbf{v}(t)\right] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

5. $\dfrac{d}{dt}\left[\mathbf{u}(t) \times \mathbf{v}(t)\right] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

6. $\dfrac{d}{dt}\left[\mathbf{u}(f(t))\right] = \mathbf{u}'(f(t))\, f'(t)$

EXAMPLE 3: Show that if $\|\mathbf{r}(t)\|$ is constant, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

EXAMPLE 4: Let $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k}$ and $\mathbf{u}(t) = 4t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$. Find $\dfrac{d}{dt}\left[\mathbf{r}(t) \cdot \mathbf{u}(t)\right]$ and $\dfrac{d}{dt}\left[\mathbf{r}(t) \times \mathbf{u}(t)\right]$.

Integration

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}$$

We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\,\Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where $\mathbf{R}$ is an antiderivative of $\mathbf{r}$, that is, $\mathbf{R}'(t) = \mathbf{r}(t)$. For indefinite integrals $\int \mathbf{r}(t)\,dt$ remember to add the vector constant of integration $\mathbf{C}$.

EXAMPLE 3: Evaluate $\displaystyle\int_0^1 \left(\frac{4}{1+t^2}\,\mathbf{j} + \frac{2t}{1+t^2}\,\mathbf{k}\right)\,dt$

Homework: pp 897-898; 9–25 odd, 33–39 odd, 45, 47