Arc Length: Suppose that a curve has the vector equation $\mathbf{r}(t) = \langle f(t),\, g(t),\, h(t)\rangle,\ \ a \le t \le b$, where f', g', and h' are continuous. If the curve is traversed exactly once as t increases from a to b, then it can be shown that its **arc length**, or length, is given by

$$L = \int_a^b \|\mathbf{r}'(t)\|\, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt.$$

EXAMPLE 1: Find the length of the curve $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j} + \ln t\mathbf{k}$ for $1 \le t \le e$.

Arc Length Function: Suppose that a curve C is a piecewise-smooth curve given by a vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k},\ \ a \le t \le b$, and C is traversed exactly once as t increases from a to b. The **arc length function** s is defined as

$$s(t) = \int_a^t \|\mathbf{r}'(u)\|\, du = \int_a^t \sqrt{[f'(u)]^2 + [g'(u)]^2 + [h'(u)]^2}\, du.$$

NOTES:

- The arc length function is nonnegative. It measures the distance from $r(a)$ to $r(t)$.
- Using the above definition along with part I of The Fundamental Theorem of Calculus, we obtain
$$\frac{ds}{dt} = \|\mathbf{r}'(t)\|.$$
- Arc length is independent of the parametrization that is used.

Curvature: The curvature of C at a given point is a measure of how quickly the curve changes direction at that point. In particular, it is defined to be the magnitude of the rate of change of the unit tangent vector with respect to arc length. Thus, the curvature of a curve is given to be

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|$$

where $\mathbf{T}$ is the unit tangent vector given by $\dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$.

NOTES:

- Using the chain rule, we have that $\dfrac{d\mathbf{T}}{dt} = \dfrac{d\mathbf{T}}{ds}\dfrac{ds}{dt}$. Therefore, we can calculate arc length using $\kappa = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$.

- Although the above formula can be used in all cases to calculate curvature, it is often more convenient to find the curvature of the curve given by the vector function $\mathbf{r}$ by

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

- For the special case of a plane curve with equation $y = f(x)$, we have

$$\kappa(t) = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}}$$

EXAMPLE 2: Find the curvature of $\mathbf{r}(t) = \langle e^t \cos t,\, e^t \sin t,\, t \rangle$ at the point $(1, 0, 0)$.

EXAMPLE 3: Find the curvature of $x^2 + y^2 = a^2$.

Principal unit normal vector: The principal unit normal vector (or unit normal), denoted $\mathbf{N}(t)$, is defined as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

Note that $\mathbf{T}'$ is orthogonal to $\mathbf{T}$ since $\|\mathbf{T}\| = 1$. (Recall the example that we did in section 14.2).

Binormal vector: The binormal vector, denoted $\mathbf{B}(t)$, is defined as $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$. Note that it is orthogonal to both $\mathbf{T}$ and $\mathbf{N}$. Also, it is a unit vector.

Normal Plane: The plane determined by the normal vector $\mathbf{N}$ and binormal vector $\mathbf{B}$ at a point P on the curve C is called the normal plane. It consists of all lines that are orthogonal to the tangent vector $\mathbf{T}$.

EXAMPLE 4: Find the equation of the normal plane of the curve $x = t$, $y = t^2$, $z = t^3$ at the point (1, 1, 1).

Homework: pp 904–905; 1–5 odd, 11, 13, 17–23 odd, 39, 41