Function of Two Variables: A function of two variables is a rule that assigns to each ordered pair of real numbers \((x, y)\) in a set \(D\) a unique real number denoted \(f(x, y)\). The set \(D\) is the domain of \(f\) and its range is the set \(\{f(x, y) \mid (x, y) \in D\}\). So, a function of two variables is a function whose domain is a subset of \(\mathbb{R}^2\) and whose range is a subset of \(\mathbb{R}\). We sometimes write \(z = f(x, y)\) to make explicit the value that \(f\) takes at the general point \((x, y)\).

**Example 1:** Evaluate the following functions.

1. Let \(f(x, y) = \frac{3x + y}{2y - x}\). Find \(f(2, -1)\).

2. Let \(f(x, y) = e^{7xy^2}\). Find \(f(3, 2)\).

**Example 2:** Find and sketch the domain of the following functions.

1. \(f(x, y) = \frac{x - 3y}{x + 3y}\)

2. \(f(x, y) = f(x, y) = \sqrt{x^2 + y^2 - 1} + \ln(4 - x^2 - y^2)\)
The graph of a function of two variables is the set \( \{(x, y, z) \mid z = f(x, y) \text{ where } (x, y) \in D\} \). This means that the graph is a surface \( z = f(x, y) \) in \( \mathbb{R}^3 \).

**Example 3:** Sketch the graph of the following functions.

1. \( f(x, y) = \cos x \)

2. \( f(x, y) = 2 + y^2 \)

Another method for visualizing functions is a contour map on which points of constant elevation are joined to form level curves. The level curves of a function \( f \) of two variables are the curves with equations \( f(x, y) = k \), where \( k \) is a constant (in the range of \( f \)). A level curve \( f(x, y) = k \) is the set of all points in the domain of \( f \) at which \( f \) takes on a given value \( k \). In other words, it shows where the graph of \( f \) has a height of \( k \).

**Example 4:** Draw a contour map of \( f(x, y) = \sqrt{x^2 + y^2} \) showing several level curves.
Functions of three variables: A function of three variables is a rule that assigns to each ordered triple $(x, y, z)$ in the domain $D$ a subset of $\mathbb{R}^3$ a unique real number $w$ denoted by $f(x, y, z)$. (Note that similar definitions exist of functions of $n$ variables). The graph of such a function is defined to be $\{(x, y, z, w) \in \mathbb{R}^4 \mid w = f(x, y, z)\}$. However, drawing this is impossible since it exists in four-dimensional space. However, we can draw the level surfaces for such a function. These are the surfaces with equations $f(x, y, z) = k$ where $k$ is a constant. If the point $(x, y, z)$ moves along a level surface, the value of $f(x, y, z)$ remains fixed.

Example 5: Let $f(x, y, z) = 3 - xy + yz^2$. Find $f(-2, 1, 5)$.

Example 6: Find the level surfaces of the function $f(x, y, z) = x^2 + y^2 + z^2$.