Limit of a function of two variables: Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of** $f(x, y)$ **as** (x, y) **approaches** (a, b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$|f(x, y) - L| < \epsilon \quad \text{whenever} \quad (x, y) \in D \text{ and } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

Therefore, this definition says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance between (x, y) and a, b sufficiently small, but not zero.

Since the definition only refers to distance, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b). Hence, if the limit of $f(x, y)$ as (x, y) goes to (a, b) along a path C_1 is different from the limit of the function as (x, y) approaches (a, b) along another path C_2, then the limit $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ does not exist.

EXAMPLE 1: Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3 y}{x^6 + y^2}$ if it exists.

As with functions of one variable, the calculations of limits for functions of two variables can be simplified by the use of properties of limits. The limit laws that were discussed in section 2.3 can be extended to functions of two variables. The limit of a sum is the sum of the limits, the limit of a product is the product of the limits, etc. We also have the following

$$\lim_{(x,y)\to(a,b)} x = a, \qquad \lim_{(x,y)\to(a,b)} y = b, \qquad \text{and} \qquad \lim_{(x,y)\to(a,b)} c = c.$$

The Squeeze Theorem also holds.

Continuous: A function f of two variables is **continuous at** (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

We say that f is **continuous on** D if f is continuous at every point $(a, b) \in D$.

A polynomial function of two variables is a sum of terms of the form $cx^m y^n$, where c is a constant and m and n are nonnegative integers. A **rational function** is a ratio of polynomials.

NOTES:

- All polynomials are continuous on $\mathbb{R}^2$.
- A rational function is continuous on its domain.

EXAMPLE 2: Find the limit, if it exists, or show that the limit does not exist.

1. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2 + \sin^2 y}{2x^2 + y^2}$

2. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2}$

3. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2 + 2y^2}$

EXAMPLE 3: Determine the set of points at which the function is continuous.

1. $f(x, y) = e^{x^2 y} + \sqrt{x + y^2}$

2. $f(x, y) = \sin^{-1}(x^2 + y^2)$

Homework: pp 944–945; 5–19 odd, 27–35 odd