Partial Derivatives of a function of two variables: If $z = f(x, y)$, then the **first partial derivatives** of f with respect to x and y are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

This definition indicates that if $z = f(x, y)$, then to find f_x you consider y constant and differentiate with respect to x. Similarly, to find f_y you consider x constant and differentiate with respect to y.

Notation of first partial derivatives: For $z = f(x, y)$, the partial derivatives f_x and f_y are denoted by

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = z_x = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

and

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = z_y = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Partial Derivatives of functions of more than two variables: Partial derivatives can also be defined for functions of three or more variables. In general, if u is a function of n variables, $u = f(x_1, x_2, \ldots, x_n)$, its partial derivative with respect to the ith variable is defined by

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, x_2, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, x_2, \ldots, x_i, \ldots, x_n)}{h}.$$

We can also use the notation

$$\frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f$$

EXAMPLE 1: If $f(x, y) = x^5 + 3x^2y^4 + 3xy^4$ find the first partial derivative of f.

EXAMPLE 2: If $f(x, y, z, t) = \dfrac{xy^2}{t + 2z}$ find the first partial derivatives of f.

Higher Partial Derivatives: If f is a function of two variables, then its partial derivatives are also functions of two variables and therefore, we can consider their partial derivatives. If $z = f(x, y)$, then the **second partial derivatives** of f are given by

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

NOTE: The notation f_{xy} means that you first differentiate with respect to x and then with respect to y. The notation f_{yx} means that you differentiate with respect to y and then with respect to x. Partial derivatives of order 3 and higher can be defined similarly.

EXAMPLE 3: Find all second partial derivatives of $f(x, y) = \ln(3x + 5y)$

EXAMPLE 4: Find all second partial derivatives of $f(x, y) = x^4y^2 - 2xy^5$.

EXAMPLE 5: For $f(x, y, z) = x\ln(xy^2z^3)$ find f_{xyy} and f_{xyz}.

Laplace's Equation: Laplace's equation is the partial differentiation equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Note that this equation can also be stated as

$$u_{xx} + u_{yy} = 0.$$

Solutions of this equation are called **harmonic functions**. Harmonic functions play a role in problems of heat conduction, fluid flow, and electric potential.

EXAMPLE 6: Determine if $f(x, y) = e^y \sin x$ is a solution of Laplace's equation.

The wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

This describes the motion of a waveform which could be an ocean wave, a sound wave, a light wave, or a wave travelling along a vibrating string.

EXAMPLE 7: Show that $u = \sin(kx)\sin(akt)$ is a solution of the wave equation.

Homework: pp 956–957; 13–37 every other odd (eoo), 47–63 eoo, 68a,b,f