Tangent plane: at a point P is the plane that most closely approximates the surface S near the point P.

Equation of the tangent plane: Suppose that f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P = (x_0, y_0, z_0)$ is

$$\boxed{z - z_0 = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0)}$$

or

$$\boxed{z = z_0 + f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0)}$$

EXAMPLE 1: Find an equation of the tangent plane to the surface $z = 9x^2 + y^2 + 6x - 3y + 5$ at the point $(1, 2, 18)$.

Linear Approximations: Suppose the surface $z = f(x, y)$ has the tangent plane $z = a(x - x_0) + b(y - y_0) + z_0$ at the point (x_0, y_0, z_0) as above. Then we say that the function

$$L(x, y) = a(x - x_0) + b(y - y_0) + z_0$$

or

$$L(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$$

is a **linear approximation** to $f(x, y)$ at the point (x_0, y_0).

EXAMPLE 2: The linear approximation to $f(x, y) = 9x^2 + y^2 + 6x - 3y + 5$ at the point $(1, 2)$ is $L(x, y) = 24x + y - 8$. Compare $L(1.01, 2.03)$ and $f(1.01, 2.03)$.

Recall that for a function in one variable $y = f(x)$ if x changes from a to $a + \Delta x$, we define the increment of y as $\Delta y = f(a + \Delta x) - f(a)$. We then showed that if f is differentiable at a, then

$$\Delta y = f'(a)\Delta x + \varepsilon \Delta x \qquad \text{where} \qquad \varepsilon \to 0 \;\text{ as }\; \Delta x \to 0.$$

For a function of two variable $z = f(x, y)$ if we let x change from a to $a + \Delta x$ and y changes from b to $b + \Delta y$, then the corresponding increment of z is defined as $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$. Therefore, the increment Δz represents the change in the value of f when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

Differentiable: If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

NOTE: A differentiable function is one for which the linear approximation is a good approximation when (x, y) is near (a, b). This means that the tangent plane approximates the graph of f well near the point of tangency.

Theorem: If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Differentials: Recall that for a differentiable function of one variable $y = f(x)$ we define the differential dx to be an independent variable. The differential of y is then defined as $dy = f'(x)dx$. For a function of two variables $z = f(x, y)$ we define the differentials dx and dy to be independent variables. Then the **differential** dz, also called **total differential**, is defined by

$$\boxed{dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy}$$

EXAMPLE 3: Find dz for $z = xe^{xy}$.

EXAMPLE 4: Find dw for $w = f(x, y, z) = xy^3 + yz^3$.

EXAMPLE 5: Let $z = x^2 + 3xy - y^2$. If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of dz and Δz.

Homework: pp 966–967; 1–5 odd, 11–17 odd, 23–29 odd