The Chain Rule (Case 1): Suppose that $z = f(x, y)$ is a differentiable function of $x$ and $y$, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of $t$. Then $z$ is a differentiable function of $t$ and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}.$$

Example 1: Use the chain rule to find $\frac{dz}{dt}$ where $z = \sqrt{x^2 + y^2}$, $x = e^{2t}$, $y = e^{-2t}$.

Example 2: Suppose that $z = 4x^2y^3$ where $x = \sin t$ and $y = \cos t$. Find $\frac{dz}{dt}$ when $t = \pi$.

The Chain Rule (Case 2): Suppose that $z = f(x, y)$ is a differentiable function of $x$ and $y$, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of $s$ and $t$. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}.$$

Example 3: Use the chain rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ where $z = e^{xy} \tan y$, $x = s + 2t$, $y = s/t$. 
The Chain Rule (Generalized Version): Suppose that $u$ is a differentiable function of $n$ variables $x_1, x_2, \ldots, x_n$ and each $x_i$ is a differentiable function of the $m$ variables $t_1, t_2, \ldots, t_m$. Then $u$ is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i}$$

for each $t = 1, 2, \ldots, m$.

Example 4: If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s \sin t$, find the value of $\frac{\partial u}{\partial s}$ when $r = 2$, $s = 1$, $t = 0$.

The Chain Rule (Implicit Differentiation): If the equation $F(x, y) = 0$ defines $y$ implicitly as a differentiable function of $x$, then

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}, \quad F_y(x, y) \neq 0$$

If the equation $F(x, y, z) = 0$ defines $z$ implicitly as a differentiable function of $x$ and $y$, then

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}, \quad F_z(x, y, z) \neq 0$$
EXAMPLE 5: Find \( \frac{dy}{dx} \) given that \( y^5 + x^2y^3 = 1 - ye^{x^2} \).

EXAMPLE 6: Find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) given that \( 3x^2z - x^2y^2 + 2z^3 + 3yz - 5 = 0 \).

**Homework:** pp 974–975; 1–13 odd, 19–33 odd, 43, 47