Recall that if $z = f(x, y)$, then the partial derivatives f_x and f_y are defined as

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$
$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

and represent the rates of changes of z in the x and y directions; that is, in the direction of the unit vectors $\mathbf{i}$ and $\mathbf{j}$.

We know are interested in finding the derivative of this functions in any direction we wish. To do this, we let $\mathbf{u} = \langle a, b \rangle$ be a unit vector. Suppose we wish to walk exactly h units in the direction of $\mathbf{u}$; namely, $h\mathbf{u}$. If we start at the point (x_0, y_0) and walk h units in the direction of $\mathbf{u}$, we arrive at $(x_0 + ha, y_0 + hb)$. The difference of these two elevations is the elevation change. Therefore, the rate of change we seek is

$$\lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}.$$

Directional Derivative: The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

We think of this as the slope of $z = f(x, y)$ at (x_0, y_0) if we face in the direction of $\mathbf{u}$. Note that $D_{\mathbf{i}} f = f_x$ and $D_{\mathbf{j}} f = f_y$.

When computing the directional derivative of a function defined by a formula, we generally use the following theorem.

Theorem If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$\boxed{D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b}$$

If the unit vector $\mathbf{u}$ make an angle θ with the positive $x-$axis, then we can write $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ and the formula in the above theorem becomes

$$\boxed{D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta}$$

EXAMPLE 1: Let $f(x, y) = x^3 + 3xy^2$. Find $D_{\mathbf{u}}f(x, y)$ at $(2, 1)$ if $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$.

EXAMPLE 2: Let $f(x, y) = \sqrt{x^2 + y^2}$. Find the directional derivative at the point $(4, 4)$ in the direction of $\theta = 45^\circ$ and $\theta = 135^\circ$.

Gradient: Note that the directional derivative $D_{\mathbf{u}}f(x,y)$ is actually the dot product of the vector $\langle f_x(x,y), f_y(x,y)\rangle$ and the vector $\mathbf{u} = \langle a, b\rangle$. The first vector is important and occurs in various contexts. Therefore, it is given a name, the **gradient** of f, and a special symbol ∇f. Therefore, if f is a function of two variable x and y, then the **gradient** of f is the vector function ∇f defined by

$$\textbf{grad } f = \nabla f = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

Using this notation, the directional derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$.

EXAMPLE 3: Let $f(x,y) = x^2y^3 - 4y$. Compute $\nabla f(2,-1)$ and $D_{\mathbf{u}}f(2,-1)$ where $\mathbf{u} = 2\mathbf{i} + 5\mathbf{j}$.

NOTE: All above definitions and formulas can be extended to functions of three variables.

Maximizing the directional derivative: Suppose we consider all possible directional derivatives of f at a given point. In which of these directions does f change fastest and what is the maximum rate of change?

Theorem: Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(x,y)$ is $\|\nabla f(x,y)\|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(x,y)$.

EXAMPLE 4: Suppose that the temperature at a point is given by $T(x, y, z) = \frac{80}{1 + x^2 + 2y^2 + 3z^2}$, where T is measured in Celsius and x, y, z in meters.

- In which direction does the temperature increase fastest at the point $(1, 1, -2)$?

- What is the maximum rate of increase?

Tangent Planes to Level Surfaces: It can be shown that the gradient vector at (x_0, y_0, z_0), is perpendicular to the tangent vector to any curve C on the surface that passes through (x_0, y_0, z_0). Thus, if $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, we define the **tangent plane to the level surface** $F(x, y, z) = k$ **at** (x_0, y_0, z_0) as the plane that passes through (x_0, y_0, z_0) and has normal vector $\nabla F(x_0, y_0, z_0)$. Therefore, the equation of this tangent plane is

$$\boxed{F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0}$$

Normal Line: The **normal line** to the surface S at the point (x_0, y_0, z_0) is the line passing through (x_0, y_0, z_0) and perpendicular to the tangent plane. The direction of the normal line is therefore given by the gradient vector $\nabla F(x_0, y_0, z_0)$ and its symmetric equations are

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}.$$

EXAMPLE 5: Given $x = y^2 + z^2 - 2$. Find

- the tangent plane to the surface at $(-1, 1, 0)$

- the normal line to the surface at $(-1, 1, 0)$

Homework: pp 986–988; 5–25 odd, 29, 33, 39, 41