Local Extrema: A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b). The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geq f(a, b)$ for all points (x, y) in some disk with center (a, b), then $f(a, b)$ is a **local minimum value**.

Absolute Extrema: If $f(a, b) \geq f(x, y)$ for all (x, y) in the domain of f, then f has an **absolute maximum** at (a, b). Likewise, if $f(a, b) \leq f(x, y)$ for all (x, y) in the domain of f, then f has an **absolute minimum** at (a, b).

Theorem: If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Critical Point: A point (a, b) is called a **critical point** of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist. Remember that not all critical points will lead to local extrema.

Second Derivative Test: Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that (a, b) is a critical point of f. Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

- If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
- If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- If $D < 0$, then $f(a, b)$ is not a local maximum or minimum. (In this case, (a, b) is called a **saddle point** of f.)

NOTES:

1. If $D = 0$ then the second derivative test gives no information.
2. The formula for D is the determinant of the matrix

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

EXAMPLE 1: Find the local maximum and minimum values and saddle points of $f(x, y) = x^2 + 2xy + 2y^2$.

EXAMPLE 2: Find and classify all critical points of $f(x, y) = (2x - x^2)(2y - y^2)$.

EXAMPLE 3: Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

Closed set: A **closed set** in $\mathbb{R}^2$ is one that contains all its boundary points.

Bounded set: A **bounded set** is $\mathbb{R}^2$ is one that is contained within some disk.

Extreme Value Theorem for functions of two variables: If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ for some points (x_1, y_1) and (x_2, y_2) in D.

Guidelines to find absolute max and absolute min of a continuous function on a closed, bounded set D:

1. Find the critical points of f in D.
2. Evaluate f at each one of the critical points of D.
3. Find the extreme values of f on the boundary of D.
4. The largest of the values in steps 2 and 3 is the absolute maximum value on D; the smallest of the values in steps 2 and 3 is the absolute minimum value of D.

EXAMPLE 4: Find the absolute maximum and minimum values of $f(x, y) = 4x + 6y - x^2 - y^2$ on the set $D = \{ (x, y) \mid 0 \le x \le 4,\ 0 \le y \le 5 \}$.

HOMEWORK: pp 997–998; #5–15 odd, 27–31 odd, 37–41 odd, 47