Definition of Definite Integral: If $f$ is continuous function defined for $a \le x \le b$, we divide $[a,b]$ into $n$ subintervals of equal width $\Delta x = \dfrac{b-a}{n}$. We let $x_0 = a,\ x_1,\ x_2,\ \ldots,\ x_n = b$ be the endpoints of these subintervals. Next, we choose sample points $x_1^*,\ x_2^*,\ \ldots,\ x_n^*$ in these subintervals, so $x_i^*$ lies in the $i$-th subinterval $[x_{i-1}, x_i]$. Then the **definite integral of** $f$ from $a$ to $b$ is

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Remarks:

1. $\int$ is called the integral, and $f(x)$ is called the integrand. $a$ is called the lower limit of integration and $b$ is the upper limit of integration. $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called the **Riemann Sum**.
2. If $f$ is positive, then $\int_a^b f(x)\ dx$ is the area under the curve $y = f(x)$ from $x = a$ to $x = b$.

**Double Integrals:** Our goal is to develop a similar notation for double integrals. Here, $f$ would be a function of two variables that is defined on a closed rectangle in the $xy-$plane. For example, suppose $f$ is continuous on $R = [a,b] \times [c,d] = \{(x,y) \mid a \le x \le b,\ c \le y \le d\ \}$. The double integral would give us the volume of the solid whose base is the rectangle $R$ and which is bounded above by the surface $z = f(x,y)$.

The first step is the divide the interval $[a,b]$ into $m$ equal subintervals $[x_{i-1}, x_i]$ each of width $\Delta x$ and to divide the interval $[c,d]$ into $n$ equal subintervals $[y_{j-1}, y_j]$ each of width $\Delta y$. Thus, $1 \le i \le m$ and $1 \le j \le n$. This means that we have divided the rectangle $R$ into subrectangles where the $ij-$th rectangle is given by

$$R_{ij} = \{\ (x,y) \mid x_{i-1} \le x \le x_i,\ \ y_{j-1} \le y \le y_j\ \} = [x_{i-1}, x_i] \times [y_{j-1}, y_j].$$

Next, we select any point $(x_{ij}^*, y_{ij}^*)$ in the rectangle $R_{ij}$. We let $\Delta A$ be the area of each subrectangle; hence, $\Delta A = \Delta x \Delta y$. Imagine each subinterval as a rectangular box with height $f(x_{ij}^*, y_{ij}^*)$ and area of base given by $\Delta A$. The volume of this box is then given by $f(x_{ij}^*, y_{ij}^*)\Delta A$.

If we sum all of these rectangular boxes we obtain the **Riemann Sum** for the function $f$ on the rectangle $R$ is given by

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A = \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta x \Delta y.$$

One can see that the more rectangular boxes we have, the better approximation we have to the volume we are after. Hence, we want to take the limit as both $m \to \infty$ and $n \to \infty$.

**Definition:** The **double integral** of $f$ over the rectangle $R$ is

$$\iint_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

if this limit exists. (If this limit exists, we say that $f$ is **integrable** over $R$).

**Properties of Double Integrals:** Let $f$ and $g$ be continuous over a closed, bounded plane region of $R$, and let $c$ be a constant.

- $\displaystyle\iint_R cf(x,y)\ dA = c\iint_R f(x,y)\ dA$
- $\displaystyle\iint_R [f(x,y) \pm g(x,y)]\ dA = \iint_R f(x,y)\ dA \pm \iint_R g(x,y)\ dA.$
- $\displaystyle\iint_R f(x,y)\ dA \geq 0, \qquad \text{if } f(x,y) \geq 0$
- $\displaystyle\iint_R f(x,y)\ dA \geq \iint_R g(x,y)\ dA, \qquad \text{if } f(x,y) \geq g(x,y)$
- $\displaystyle\iint_R f(x,y)\ dA = \iint_{R_1} f(x,y)\ dA + \iint_{R_2} f(x,y)\ dA$ where $R$ is the union of two nonoverlapping subregions $R_1$ and $R_2$.