Some double integrals are much easier to evaluate in polar form than in rectangular form. This is especially true for regions such as circles and cardioids, and for integrands involving \( x^2 + y^2 \).

Recall that the polar coordinates \((r, \theta)\) of a point are related to the rectangular coordinates \((x, y)\) by the equations:

\[
\begin{align*}
    r^2 &= x^2 + y^2 \\
    x &= r \cos \theta \\
    y &= r \sin \theta \\
    \tan \theta &= \frac{y}{x}.
\end{align*}
\]

**Polar Rectangle:** is a region of the form \( R = \{(r, \theta) | a \leq r \leq b, \alpha \leq \theta \leq \beta \} \).

**Change to Polar Coordinates in a Double Integral:** If \( f \) is continuous on a polar rectangle \( R \) given by

\[
0 \leq a \leq r \leq b, \quad \alpha \leq \theta \leq \beta, \quad \text{where} \quad 0 \leq \beta - \alpha \leq 2\pi
\]

then

\[
\int_R f(x, y) \, dA = \int_{\alpha}^{\beta} \int_{a}^{b} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta.
\]

NOTE: this theorem says that we can convert from rectangular to polar coordinates in a double integral by writing \( x = r \cos \theta \) and \( y = r \sin \theta \), using the appropriate limits of integration for \( r \) and \( \theta \), and replacing \( dA \) by \( r \, dr \, d\theta \).

**EXAMPLE 1:** Evaluate \( \int_R (x + y) \, dA \) where \( R \) is the region that lies to the left of the \( y \)-axis between the circles of \( x^2 + y^2 = 1 \) and \( x^2 + y^2 = 4 \).
EXAMPLE 2: Find the volume of the paraboloid $z = 4 - x^2 - y^2$ above the $xy$-plane.

**Theorem:** If $f$ is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, \ h_1(\theta) \leq r \leq h_2(\theta) \}$$

then

$$\iint_D f(x, y) \, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta.$$ 

EXAMPLE 3: Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$. 
EXAMPLE 4: Evaluate \(\int_{-a}^{a} \int_{0}^{\sqrt{a^2 - y^2}} (x^2 + y^2)^{3/2} \, dx \, dy\).

EXAMPLE 5: Find the volume under \(z = x^2 + y^2\) and above the region \(D\) shaded below:

Homework: pp 1044–1045; 1-21 odd, 29, 31