Some double integrals are much easier to evaluate in polar form than in rectangular form. This is especially true for regions such as circles and cardioids, and for integrands involving $x^2 + y^2$.

Recall that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by the equations:

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta \qquad \tan\theta = \frac{y}{x}.$$

Polar Rectangle: is a region of the form $R = \{(r, \theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\ \}$.

Change to Polar Coordinates in a Double Integral: If f is continuous on a polar rectangle R given by

$$0 \le a \le r \le b, \quad \alpha \le \theta \le \beta, \quad \text{where} \quad 0 \le \beta - \alpha \le 2\pi$$

then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\ r\ dr\, d\theta.$$

NOTE: this theorem says that we can convert from rectangular to polar coordinates in a double integral by writing $x = r\cos\theta$ and $y = r\sin\theta$, using the appropriate limits of integration for r and θ, and replacing dA by $r\, dr\, d\theta$.

EXAMPLE 1: Evaluate $\displaystyle\iint_R (x + y)\, dA$ where R is the region that lies to the left of the $y-$axis between the circles of $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

EXAMPLE 2: Find the volume of the paraboloid $z = 4 - x^2 - y^2$ above the $xy-$plane.

Theorem: If f is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\ \}$$

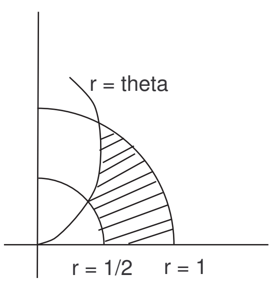
then

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

EXAMPLE 3: Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

EXAMPLE 4: Evaluate $\displaystyle\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} \left(x^2+y^2\right)^{3/2}\,dx\,dy$.

EXAMPLE 5: Find the volume under $z = x^2 + y^2$ and above the region D shaded below:



Homework: pp 1044–1045; 1-21 odd, 29, 31