We will now define a triple integral for functions of three variables. First, we consider the case where f is defined over a rectangular box B for which

$$B = \{(x, y, z) \mid a \le x \le b,\ c \le y \le d,\ r \le z \le s\ \}.$$

Triple Integral: The **triple integral** of f over the box B is

$$\iiint\limits_B f(x, y, z)\, dV = \lim_{t,m,n\to\infty} \sum_{i=1}^{t} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\, \Delta V$$

provided this limit exists.

Theorem: If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint\limits_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz.$$

EXAMPLE 1: Evaluate $\displaystyle\int_0^3 \int_0^2 \int_0^1 (x + y)\, dx\, dy\, dz$

EXAMPLE 2: Evaluate $\int_1^2 \int_1^y \int_1^{x+y} 12x \; dz \; dx \; dy$.

Triple Integrals over a general bounded region E

Type 1 region: A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y. That is,

$$E = \{(x, y, z) \mid (x, y) \in D, \; u_1(x, y) \leq z \leq u_2(x, y) \; \}$$

where D is the projection of E onto the $xy-$plane. (Note that $z = u_1(x, y)$ is the top surface and $z = u_2(x, y)$ is the bottom surface). In this case,

$$\iiint_E f(x, y, z) \, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) \; dz \right] dA.$$

We now turn our attention to the plane region D. If D can be described as

$$D = \{(x, y) \mid a \leq x \leq b, \; g_1(x) \leq y \leq g_2(x) \; \},$$

then the integral described above becomes

$$\iiint_E f(x, y, z) \, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) \; dz \; dy \; dx.$$

Likewise, if D can be described as

$$D = \{(x, y) \mid c \leq y \leq d, \, h_1(y) \leq x \leq h_2(y)\}$$

then the integral above becomes

$$\iiint_E f(x, y, z) \, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) \; dz \; dx \; dy.$$

Type 2 region: A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D,\ u_1(y, z) \leq x \leq u_2(y, z)\ \}$$

where D is the projection of E onto the $yz-$plane. (Note that $x = u_1(y, z)$ is the back surface and $x = u_2(y, z)$ is the front surface). In this case,

$$\iiint\limits_E f(x, y, z)\, dV = \iint\limits_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\ dx \right] dA.$$

Type 3 region: A solid region E is of **type 3** if it is of the form

$$E = \{(x, y, z) \mid (x, z) \in D,\ u_1(x, z) \leq y \leq u_2(x, z)\ \}$$

where D is the projection of E onto the $xz-$plane. (Note that $y = u_1(x, z)$ is the left surface and $y = u_2(x, z)$ is the right surface). In this case,

$$\iiint\limits_E f(x, y, z)\, dV = \iint\limits_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\ dy \right] dA.$$

NOTE: To find the limits for a particular order of integration, you generally first determine the innermost limits, which may be functions of the outer two variables. Then, by projecting the solid E onto the coordinate plane of the two outer two variables, you can then determine their limits of integration using the methods described for double integrals.

EXAMPLE 3: Let E be the solid region in the first octant between the graphs of $z = x^2 + 2y + 1$ and $z = y + 2$. Set up the triple integral used to find the volume of E.

EXAMPLE 4: Evaluate $\iiint\limits_E y\, dV$, where E is bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + 2y + z = 4$.

EXAMPLE 5: Write five other iterated integrals that are equal to $\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z)\, dz\, dy\, dx$.

Homework: pp 1066–1067; 3–11 odd, 15, 17, 25, 31, 33