MATH 22005 | **Triple Integrals in Cylindrical and Spherical Coordinates** | **SECTION 16.8**

Cylindrical Coordinates: Recall that we have shown that the relationships between the rectangular and cylindrical coordinates are given by the formulas: $x = r\cos\theta,\ y = r\sin\theta,\ z = z$. We also observed that cylindrical coordinates are especially useful when the function $f(x, y, z)$ involves the expression $x^2 + y^2$.

Formula for triple integration in cylindrical coordinates: We can convert a triple integral in rectangular coordinates to cylindrical coordinates by using $x = r\cos\theta,\ y = r\sin\theta$, leaving z as it is, using the appropriate limits of integration for z, r, and θ, and replacing dV with $r\ dz\ dr\ d\theta$. Namely, we have

$$\iiint_E f(x, y, z)\ dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\ r\ dz\ dr\ d\theta$$

EXAMPLE 1: Evaluate $\displaystyle\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\ dz\ dy\ dx$.

EXAMPLE 2: Evaluate $\iiint\limits_E 2\,dV$ where E is the solid between the paraboloids $z = 4x^2 + 4y^2$ and $z = 80 - x^2 - y^2$.

Spherical Coordinates: Recall that we have shown that the relationships between the rectangular and spherical coordinates are given by the formulas: $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$. Usually spherical coordinates are used in triple integrals when surfaces such as spheres or cones form the boundary of the region of integration.

In this coordinate system the counterpart to the rectangular box is a **spherical wedge** given by

$$E = \{(\rho,\ \theta,\ \phi) \mid a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\ \}.$$

As a result, we obtain the **formula for triple integration in spherical coordinates** which is given by

$$\boxed{\iiint\limits_E f(x,y,z)\ dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta,\ \rho \sin\phi \sin\theta,\ \rho \cos\phi)\ \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi}$$

where E is the spherical wedge described above.

This formula says that we can convert a triple integral from rectangular to spherical coordinates by using

$$x = \rho \sin\phi \cos\theta,\ y = \rho \sin\phi \sin\theta,\ z = \rho \cos\phi$$

using the appropriate limits of integration for ρ, θ, and ϕ, and replacing dV with $\rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$. Of course, this formula can be extended to include more general spherical regions.

EXAMPLE 3: Evaluate $\displaystyle\iiint\limits_H z\ dV$ where H is the hemispherical region that lies above the $xy-$plane and below the sphere $x^2 + y^2 + z^2 = 1$.

EXAMPLE 4: Evaluate $\displaystyle\iiint_E xyz\, dV$ where E lies between the sphere $\rho = 2$ and $\rho = 4$ and above the cone $\phi = \dfrac{\pi}{3}$.

Homework: pp 1073–1074; 1– 9 odd, 13a, 17, 19, 21, 29 (volume only), 33, 35.