Vector Field on $\mathbb{R}^2$: Let D be a set in $\mathbb{R}^2$, a plane region. A **vector field on** $\mathbb{R}^2$ is a function F that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

NOTES:

- Since $\mathbf{F}(x, y)$ is a two-dimensional vector we can write it in terms of its **component functions** P and Q as $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = \langle P(x, y),\, Q(x, y)\rangle$.

- From this definition you can see that the gradient is an example of a vector field

$$\nabla f = f_x\mathbf{i} + f_y\mathbf{j}.$$

  We know that the graphical interpretation of this field is a family of vectors, each of which points in the direction of maximum increase along the surface given by $z = f(x, y)$.

Vector Field on $\mathbb{R}^3$: Let E be a subset of $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function F that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

Drawing Vector Fields: The best way to picture a vector field is to draw the arrow representing the vector $\mathbf{F}(x, y)$ starting at the point (x, y). Of course, because vector fields consists of infinitely many vectors, it is not possible to create a sketch of the entire field. Instead, when you sketch a vector field, your goal is to sketch representative vectors that help you visualize the field.

EXAMPLE 1: Sketch some vectors in the vector field of $\mathbf{F}(x, y) = \mathbf{i} + x\mathbf{j}$

EXAMPLE 2: Sketch some vectors in the vector field given by $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$.

NOTE: Some computer systems are capable of plotting vector fields in two or three dimensions. They give a better impression of the vector field because the computer can plot a large number of representative vectors.

Some common physical examples of vector fields:

- Velocity Fields: velocity fields describe the motions of systems of particles in the plane or in space. Velocity fields are also determined by the flow of liquid through a container or by the flow of air currents around a moving object.

- Gravitational Fields: gravitational fields are defined by **Newton's Law of Gravitation**, which states that the force of attraction exerted on a particle of mass m_1 located at (x, y, z) by a particle of mass m_2 located at $(0, 0, 0)$ is given by

$$\mathbf{F}(x, y, z) = \frac{-Gm_1m_2}{x^2 + y^2 + z^2}\mathbf{u}$$

  where G is the gravitational constant and $\mathbf{u}$ is the unit vector in the direction from the origin to (x, y, z).

- Electric Force Fields: electric force fields are defined by **Coulomb's Law**, which states that the force exerted on a particle with electric charge q_1 located at (x, y, z) by a particle with electric charge q_2 located at $(0, 0, 0)$ is given by

$$\mathbf{F}(x, y, z) = \frac{cq_1q_2}{\|\mathbf{r}\|^2}\mathbf{u}$$

  where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\mathbf{u} = \dfrac{\mathbf{r}}{\|\mathbf{r}\|}$, and c is a constant that depends on the choice of units for $\|\mathbf{r}\|$, q_1, and q_2.

Conservative Vector Field: A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, there exists a function f such that $\mathbf{F} = \nabla f$. Here, f is called a **potential functions** for $\mathbf{F}$.

NOTE: many important vector fields, including gravitational fields and electric fields, are conservative. The term "conservative" is derived from the classic physical law regarding the conservation of energy. This law states that the sum of the kinetic energy and the potential energy of a particle moving in a conservative force field is constant. The kinetic energy of a particle is the energy due to its motion, and the potential energy is the energy due to its position in the force field.

Homework: pp 1096–1097; 1-9 odd, 11-18 all, 21, 23