Line integral: If f is defined on a smooth curve C given by $x = x(t), \quad y = y(t), \quad a \leq t \leq b$, then the **line integral of f along C** is

$$\int_C f(x, y)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

where (x_i^*, y_i^*) is a point in the ith subarc s_i which has length Δs_i.

If f is a continuous function, then the limit in the above definition always exists and the following formula can be used to evaluate the line integral:

$$\boxed{\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

NOTE: The value of the integral does not depend on the parametrization of the curve.

If C is a **piece-wise smooth curve**; that is, C is the union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$ where the initial point of C_{i+1} is the terminal point of C_i, then the integral of f along C is the sum of the integrals of f along each of the smooth pieces of C:

$$\boxed{\int_C f(x, y)\ ds = \int_{C_1} f(x, y)\ ds + \int_{C_2} f(x, y)\ ds + \cdots + \int_{C_n} f(x, y)\ ds}$$

Line integrals of f along C with respect to x and y: The line integrals with respect to x and y can be evaluated by expressing everything in terms of t: $x = x(t), y = y(t)$, $dx = x'(t)dt, \ dy = y'(t)dt$.

$$\boxed{\int_C f(x, y)dx = \int_a^b f(x(t), y(t))x'(t)dt}$$

$$\boxed{\int_C f(x, y)dy = \int_a^b f(x(t), y(t))y'(t)dt}$$

Sometimes it happens that the line integrals with respect to x and y occur together. Therefore, it is abbreviated

$$\int_C P(x, y)dx + \int_C Q(x, y)dy = \int_C P(x, y)dx + Q(x, y)dy$$

EXAMPLE 1: Evaluate the line integral $\int_C x^2 z ds$ where C is the line segment from $(0,6,-1)$ to $(4,1,5)$.

EXAMPLE 2: Evaluate the line integral $\int_C \sin x \, dx + \cos y \, dy$ where C consists of the top half of the circle $x^2 + y^2 = 1$ from $(1,0)$ to $(-1,0)$ and the line segment from $(-1,0)$ to $(-2,3)$.

EXAMPLE 3: Find the mass of a spring in the shape of the circular helix

$$\mathbf{r}(t) = \frac{1}{\sqrt{2}} \left(\cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k} \right), \quad 0 \le t \le 6\pi$$

where the density of the spring is $\rho(x,y,z) = 1 + z$.

Line integrals of Vector Fields: One of the most important physical applications of line integrals is that of finding the **work** done on an object moving in a force field. The total work done is given by

$$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z)\ ds$$

where $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$. Note that

$$\mathbf{F} \cdot \mathbf{T}\ ds = \mathbf{F} \cdot \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \|\mathbf{r}'(t)\|\, dt = \mathbf{F} \cdot \mathbf{r}'(t)\ dt = \mathbf{F} \cdot d\mathbf{r}.$$

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by the vector function $\mathbf{r}(t),\ a \le t \le b$. Then the **line integral of F along** C is

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t)\ dt = \int_C \mathbf{F} \cdot \mathbf{T}\ ds}$$

Orientation: A given parametrization $x = x(t), y = y(t), a \le t \le b$, determines an orientation of a curve C with a positive direction corresponding to increasing values of the parameter t. If $-C$ denotes the curve consisting of the same points as C but with the opposite orientation then we have

- $\displaystyle\int_{-C} f(x, y)dx = -\int_C f(x, y)dx$
- $\displaystyle\int_{-C} f(x, y)dy = -\int_C f(x, y)dy$
- $\displaystyle\int_{-C} f(x, y)ds = \int_C f(x, y)ds$
- $\displaystyle\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$

EXAMPLE 4: Find the work done by the force field

$$\mathbf{F}(x,y,z) = -\frac{1}{2}x\mathbf{i} - \frac{1}{2}y\mathbf{j} + \frac{1}{4}\mathbf{k}$$

on a particle as it moves along the helix given by

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}$$

from the point $(1,0,0)$ to $(-1,0,3\pi)$.

Homework: pp 1107–1109; #1-15 every other odd, 19, 21, 39