Chapter 1

Statics

Before reading any of the text in this book, you should read Appendices B and C. The material discussed there (dimensional analysis, checking limiting cases, etc.) is extremely important. It's fairly safe to say that an understanding of these topics is absolutely necessary for an understanding of physics. And they make the subject a lot more fun, too!

For many of you, the material in this first chapter will be mainly review. As such, the text here will be relatively short. This is an "extra" chapter. Its main purpose is that it provides me with an excuse to give you some nice statics problems. Try as many as you like, but don't go overboard; more important and relevant material will soon be at hand.

1.1 Balancing forces

A "static" situation is one where all the objects are motionless. If an object remains motionless, then $F = ma$ tells us that the total force acting on it must be zero. (The converse is not true, of course. The total force on an object is also zero if it moves with constant nonzero velocity. But we'll deal only with statics problems here). The whole goal in a statics problem is to find out what the various forces have to be so that there is zero net force acting on each object (and zero net torque, too, but that's the topic of the next section). Since a force is a vector, this goal involves breaking the force up into its components. You can pick cartesian coordinates, polar coordinates, or another set. It is usually clear from the problem which system will make your calculations easiest. Once you pick a system, you simply have to demand that the total force in each direction is zero.

There are many different types of forces in the world, most of which are large-scale effects of complicated things going on at smaller scales. For example, the tension in a rope comes from the chemical bonds that hold the molecules in the rope together (and these chemical forces are just electrical forces). In doing a mechanics problem involving a rope, there is certainly no need to analyze all the details of the forces taking place at the molecular scale. You simply call the force in the rope a "tension" and get on with the problem. Four types of forces come up repeatedly:

Tension

Tension is the general name for a force that a rope, stick, etc., exerts when it is pulled on. Every piece of the rope feels a tension force in both directions, except the end point, which feels a tension on one side and a force on the other side from whatever object is attached to the end.

In some cases, the tension may vary along the rope. The "Rope wrapped around a pole" example at the end of this section is a good illustration of this. In other cases, the tension must be the same everywhere. For example, in a hanging massless rope, or in a massless rope hanging over a frictionless pulley, the tension must be the same at all points, because otherwise there would be a net force on at least one tiny piece, and then $F = ma$ would yield an infinite acceleration for this tiny piece.

Normal force

This is the force perpendicular to a surface that the surface applies to an object. The total force applied by a surface is usually a combination of the normal force and the friction force (see below). But for frictionless surfaces such as greasy ones or ice, only the normal force exists. The normal force comes about because the surface actually compresses a tiny bit and acts like a very rigid spring. The surface gets squashed until the restoring force equals the force the object applies.

Remark: For the most part, the only difference between a "tension" and a "normal force" is the direction of the force. Both situations can be modeled by a spring. In the case of a tension, the spring (a rope, a stick, or whatever) is stretched, and the force on the given object is directed toward the spring. In the case of a normal force, the spring is compressed, and the force on the given object is directed away from the spring. Things like sticks can provide both normal forces and tensions. But a rope, for example, has a hard time providing a normal force.

In practice, in the case of elongated objects such as sticks, a compressive force is usually called a "compressive tension," or a "negative tension," instead of a normal force. So by these definitions, a tension can point either way. At any rate, it's just semantics. If you use any of these descriptions for a compressed stick, people will know what you mean. ♣

Friction

Friction is the force parallel to a surface that a surface applies to an object. Some surfaces, such as sandpaper, have a great deal of friction. Some, such as greasy ones, have essentially no friction. There are two types of friction, called "kinetic" friction and "static" friction.

Kinetic friction (which we won't cover in this chapter) deals with two objects moving relative to each other. It is usually a good approximation to say that the kinetic friction between two objects is proportional to the normal force between them. The constant of proportionality is called μ_k (the "coefficient of kinetic friction"), where μ_k depends on the two surfaces involved. Thus, $F = \mu_k N$, where N

is the normal force. The direction of the force is opposite to the motion.

Static friction deals with two objects at rest relative to each other. In the static case, we have $F \leq \mu_s N$ (where μ_s is the "coefficient of static friction"). Note the inequality sign. All we can say prior to solving a problem is that the static friction force has a *maximum* value equal to $F_{\text{max}} = \mu_s N$. In a given problem, it is most likely less than this. For example, if a block of large mass M sits on a surface with coefficient of friction μ_s, and you give the block a tiny push to the right (tiny enough so that it doesn't move), then the friction force is of course not equal to $\mu_s N = \mu_s Mg$ to the left. Such a force would send the block sailing off to the left. The true friction force is simply equal and opposite to the tiny force you apply. What the coefficient μ_s tells us is that if you apply a force larger than $\mu_s Mg$ (the maximum friction force on a horizontal table), then the block will end up moving to the right.

Gravity

Consider two point objects, with masses M and m, separated by a distance R. Newton's gravitational force law says that the force between these objects is attractive and has magnitude $F = GMm/R^2$, where $G = 6.67 \cdot 10^{-11}\ \text{m}^3/(\text{kg} \cdot \text{s}^2)$. As we will show in Chapter 4, the same law applies to spheres. That is, a sphere may be treated like a point mass located at its center. Therefore, an object on the surface of the earth feels a gravitational force equal to

$$F = m\left(\frac{GM}{R^2}\right) \equiv mg, \tag{1.1}$$

where M is the mass of the earth, and R is its radius. This equation defines g. Plugging in the numerical values, we obtain (as you can check) $g \approx 9.8\ \text{m/s}^2$. Every object on the surface of the earth feels a force of mg downward. If the object is not accelerating, then there must also be other forces present (normal forces, etc.) to make the total force equal to zero.

Example (Block on a plane): A block of mass M rests on a fixed plane inclined at angle θ. You apply a horizontal force of Mg on the block, as shown in Fig. 1.1.

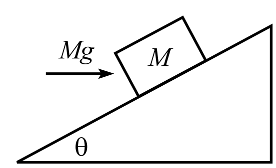

Figure 1.1

(a) Assume that the friction force between the block and the plane is large enough to keep the block at rest. What are the normal and friction forces (call them N and F_f) that the plane exerts on the block?

(b) Let the coefficient of static friction be μ. For what range of angles θ will the block remain still?

Solution:

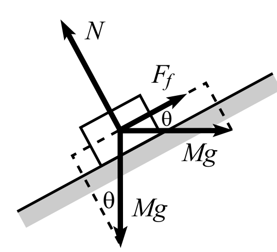

Figure 1.2

(a) We will break the forces up into components parallel and perpendicular to the plane. (The horizontal and vertical components would also work, but the calculation would be a little longer.) The forces are N, F_f, the applied Mg, and the weight Mg, as shown in Fig. 1.2. Balancing the forces parallel and perpendicular to the plane gives, respectively (with upward along the plane taken to be positive),

$$\begin{aligned} F_f &= Mg\sin\theta - Mg\cos\theta, \qquad \text{and} \\ N &= Mg\cos\theta + Mg\sin\theta. \end{aligned} \tag{1.2}$$

Remarks: Note that if $\tan\theta > 1$, then F_f is positive (that is, it points up the plane). And if $\tan\theta < 1$, then F_f is negative (that is, it points down the plane). There is no need to worry about which way it points when drawing the diagram. Just pick a direction to be positive, and if F_f comes out to be negative (as it does in the above figure because $\theta < 45°$), so be it.

F_f ranges from $-Mg$ to Mg, as θ ranges from 0 to $\pi/2$ (convince yourself that these limiting values make sense). As an exercise, you can show that N is maximum when $\tan\theta = 1$, in which case $N = \sqrt{2}Mg$ and $F_f = 0$. ♣

(b) The coefficient μ tells us that $|F_f| \leq \mu N$. Using eqs. (1.2), this inequality becomes

$$Mg|\sin\theta - \cos\theta| \leq \mu Mg(\cos\theta + \sin\theta). \tag{1.3}$$

The absolute value here signifies that we must consider two cases:

- If $\tan\theta \geq 1$, then eq. (1.3) becomes

$$\sin\theta - \cos\theta \leq \mu(\cos\theta + \sin\theta) \qquad \Longrightarrow \qquad \tan\theta \leq \frac{1+\mu}{1-\mu}. \tag{1.4}$$

- If $\tan\theta \leq 1$, then eq. (1.3) becomes

$$-\sin\theta + \cos\theta \leq \mu(\cos\theta + \sin\theta) \qquad \Longrightarrow \qquad \tan\theta \geq \frac{1-\mu}{1+\mu}. \tag{1.5}$$

Putting these two ranges for θ together, we have

$$\frac{1-\mu}{1+\mu} \leq \tan\theta \leq \frac{1+\mu}{1-\mu}. \tag{1.6}$$

Remarks: For very small μ, these bounds both approach 1, which means that θ must be very close to 45°. This makes sense. If there is very little friction, then the components along the plane of the horizontal and vertical Mg forces must nearly cancel; hence, $\theta \approx 45°$. A special value for μ is 1, because from eq. (1.6), we see that $\mu = 1$ is the cutoff value that allows θ to reach 0 and $\pi/2$. If $\mu \geq 1$, then any tilt of the plane is allowed. ♣

Let's now do an example involving a rope in which the tension varies with position. We'll need to consider differential pieces of the rope to solve this problem.

Example (Rope wrapped around a pole): A rope wraps an angle θ around a pole. You grab one end and pull with a tension T_0. The other end is attached to a large object, say, a boat. If the coefficient of static friction between the rope and the pole is μ, what is the largest force the rope can exert on the boat, if the rope is not to slip around the pole?

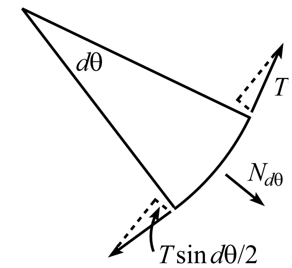

Figure 1.3

Solution: Consider a small piece of the rope that subtends an angle $d\theta$. Let the tension in this piece be T (which will vary slightly over the small length). As shown in Fig. 1.3, the pole exerts a small outward normal force, $N_{d\theta}$, on the piece. This normal force exists to balance the inward components of the tensions at the ends. These inward components have magnitude $T\sin(d\theta/2)$. Therefore, $N_{d\theta} = 2T\sin(d\theta/2)$. The small-angle approximation, $\sin x \approx x$, then allows us to write this as $N_{d\theta} = T\,d\theta$.

The friction force on the little piece of rope satisfies $F_{d\theta} \leq \mu N_{d\theta} = \mu T\,d\theta$. This friction force is what gives rise to the difference in tension between the two ends of the piece. In other words, the tension, as a function of θ, satisfies

$$\begin{aligned} T(\theta + d\theta) &\leq T(\theta) + \mu T\,d\theta \\ \Longrightarrow \quad dT &\leq \mu T\,d\theta \\ \Longrightarrow \quad \int \frac{dT}{T} &\leq \int \mu\,d\theta \\ \Longrightarrow \quad \ln T &\leq \mu\theta + C \\ \Longrightarrow \quad T &\leq T_0 e^{\mu\theta}, \end{aligned} \tag{1.7}$$

where we have used the fact that $T = T_0$ when $\theta = 0$.

The exponential behavior here is quite strong (as exponential behaviors tend to be). If we let $\mu = 1$, then just a quarter turn around the pole produces a factor of $e^{\pi/2} \approx 5$. One full revolution yields a factor of $e^{2\pi} \approx 530$, and two full revolutions yield a factor of $e^{4\pi} \approx 300,000$. Needless to say, the limiting factor in such a case is not your strength, but rather the structural integrity of the pole around which the rope winds.

1.2 Balancing torques

In addition to balancing forces in a statics problem, we must also balance torques. We'll have much more to say about torque in Chapters 7 and 8, but we'll need one important fact here.

Consider the situation in Fig. 1.4, where three forces are applied perpendicularly to a stick, which is assumed to remain motionless. F_1 and F_2 are the forces at the ends, and F_3 is the force in the interior. We have, of course, $F_3 = F_1 + F_2$, because the stick is at rest.

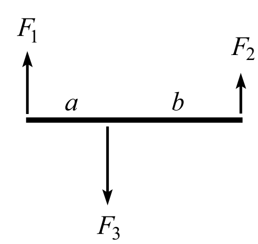

Figure 1.4

Claim 1.1 *If the system is motionless, then $F_3 a = F_2(a + b)$. In other words, the torques (force times distance) around the left end cancel. And you can show that they cancel around any other point, too.*

We'll prove this claim in Chapter 7 by using angular momentum, but let's give a short proof here.

Proof: We'll make one reasonable assumption, namely, that the correct relationship between the forces and distances is of the form,

$$F_3 f(a) = F_2 f(a + b), \tag{1.8}$$

where $f(x)$ is a function to be determined.[1] Applying this assumption with the roles of "left" and "right" reversed in Fig. 1.4, we have

$$F_3 f(b) = F_1 f(a+b) \tag{1.9}$$

Adding the two preceding equations, and using $F_3 = F_1 + F_2$, gives

$$f(a) + f(b) = f(a+b). \tag{1.10}$$

This equation implies that $f(nx) = nf(x)$ for any x and for any rational number n, as you can show. Therefore, assuming $f(x)$ is continuous, it must be the linear function, $f(x) = Ax$, as we wanted to show. The constant A is irrelevant, because it cancels in eq. (1.8).[2] ■

Note that dividing eq. (1.8) by eq. (1.9) gives $F_1 f(a) = F_2 f(b)$, and hence $F_1 a = F_2 b$, which says that the torques cancel around the point where F_3 is applied. You can show that the torques cancel around any arbitrary pivot point.

When adding up all the torques in a given physical setup, it is of course required that you use the same pivot point when calculating each torque.

In the case where the forces aren't perpendicular to the stick, the claim applies to the components of the forces perpendicular to the stick. This makes sense, because the components parallel to the stick have no effect on the rotation of the stick around the pivot point. Therefore, referring to the figures shown below, the equality of the torques can be written as

$$F_a a \sin\theta_a = F_b b \sin\theta_b. \tag{1.11}$$

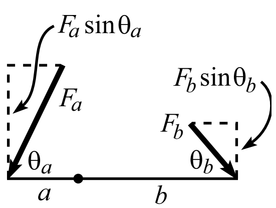

Figure 1.5

This equation can be viewed in two ways:

- $(F_a \sin\theta_a)a = (F_b \sin\theta_b)b$. In other words, we effectively have smaller forces acting on the given "lever-arms" (see Fig. 1.5).

- $F_a(a\sin\theta_a) = F_b(b\sin\theta_b)$. In other words, we effectively have the given forces acting on smaller "lever-arms" (see Fig. 1.6).

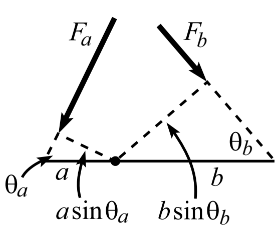

Figure 1.6

Claim 1.1 shows that even if you apply just a tiny force, you can balance the torque due to a very large force, provided that you make your lever-arm sufficiently long. This fact led a well-known mathematician of long ago to claim that he could move the earth if given a long enough lever-arm.

One morning while eating my Wheaties,
I felt the earth move 'neath my feeties.
The cause for alarm
Was a long lever-arm,
At the end of which grinned Archimedes!

[1] What we're doing here is simply assuming linearity in F. That is, two forces of F applied at a point should be the same as a force of $2F$ applied at that point. You can't really argue with that.

[2] Another proof of this claim is given in Problem 12.

One handy fact that comes up often is that the gravitational torque on a stick of mass M is the same as the gravitational torque due to a point-mass M located at the center of the stick. The truth of this statement relies on the fact that torque is a linear function of the distance to the pivot point (see Exercise 7). More generally, the gravitational torque on an object of mass M may be treated simply as the gravitational torque due to a force Mg located at the center of mass.

We'll have much more to say about torque in Chapters 7 and 8, but for now we'll simply use the fact that in a statics problem, the torques around any given point must balance.

Example (Leaning ladder): A ladder leans against a frictionless wall. If the coefficient of friction with the ground is μ, what is the smallest angle the ladder can make with the ground and not slip?

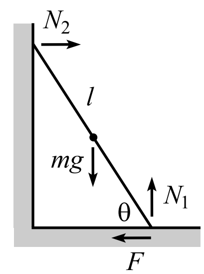

Figure 1.7

Solution: Let the ladder have mass m and length ℓ. As shown in Fig. 1.7, we have three unknown forces: the friction force, F, and the normal forces, N_1 and N_2. And we fortunately have three equations that will allow us to solve for these three forces: $\Sigma F_{\text{vert}} = 0$, $\Sigma F_{\text{horiz}} = 0$, and $\Sigma\tau = 0$.

Looking at the vertical forces, we see that $N_1 = mg$. And then looking at the horizontal forces, we see that $N_2 = F$. So we have quickly reduced the unknowns from three to one.

We will now use $\Sigma\tau = 0$ to find N_2 (or F). But first we must pick the "pivot" point around which we will calculate the torques. Any stationary point will work fine, but certain choices make the calculations easier than others. The best choice for the pivot is generally the point at which the most forces act, because then the $\Sigma\tau = 0$ equation will have the smallest number of terms in it (because a force provides no torque around the point where it acts, since the lever-arm is zero).

In this problem, there are two forces acting at the bottom end of the ladder, so this is the best choice for the pivot.[3] Balancing the torques due to gravity and N_2, we have

$$N_2\ell\sin\theta = mg(\ell/2)\cos\theta \qquad \Longrightarrow \qquad N_2 = \frac{mg}{2\tan\theta}\,. \tag{1.12}$$

This is also the value of the friction force F. The condition $F \le \mu N_1 = \mu mg$ therefore becomes

$$\frac{mg}{2\tan\theta} \le \mu mg \qquad \Longrightarrow \qquad \tan\theta \ge \frac{1}{2\mu}\,. \tag{1.13}$$

Remarks: The factor of 1/2 in this answer comes from the fact that the ladder behaves like a point mass located halfway up. As an exercise, you can show that the answer for the analogous problem, but now with a massless ladder and a person standing a fraction f of the way up, is $\tan\theta \ge f/\mu$.

Note that the total force exerted on the ladder by the floor points up at an angle given by $\tan\beta = N_1/F = (mg)/(mg/2\tan\theta) = 2\tan\theta$. We see that this force does *not* point along the ladder. There is simply no reason why it should. But there *is* a nice reason why it should point upward with twice the slope of the ladder. This is the direction that causes the lines of the three forces on the ladder to be concurrent, as shown in Fig. 1.8.

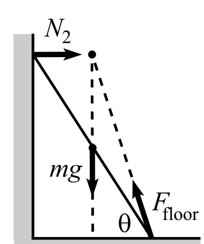

Figure 1.8

[3] But you should verify that other choices for the pivot, for example, the middle or top of the ladder, give the same result.

This concurrency is a neat little theorem for statics problems involving three forces. The proof is simple. If the three lines weren't concurrent, then one force would produce a nonzero torque around the intersection point of the other two lines of force.[4] ♣

Statics problems often involve a number of decisions. If there are various parts to the system, then you must decide which subsystems you want to balance the forces and torques on. And furthermore, you must decide which point to use as the origin for calculating the torques. There are invariably many choices that will give you the information you need, but some will make your calculations much cleaner than others (Exercise 11 is a good example of this). The only way to know how to choose wisely is to start solving problems, so you may as well tackle some. . .

[4]The one exception to this reasoning is where no two of the lines intersect; that is, where all three lines are parallel. Equilibrium is certainly possible in such a scenario, as we saw in Claim 1.1. Of course, you can hang onto the concurrency theorem in this case if you consider the parallel lines to meet at infinity.

1.3 Exercises

Section 1.1 Balancing forces

1. **Pulling a block** $*$

   A person pulls on a block with a force F, at an angle θ with respect to the horizontal. The coefficient of friction between the block and the ground is μ. For what θ is the F required to make the block slip a minimum?

2. **Bridges** $**$

   (a) Consider the first bridge in Fig. 1.9, made of three equilateral triangles of beams. Assume that the seven beams are massless and that the connection between any two of them is a hinge. If a car of mass m is located at the middle of the bridge, find the forces (and specify tension or compression) in the beams. Assume that the supports provide no horizontal forces on the bridge.

   (b) Same question, but now with the second bridge in Fig. 1.9, made of seven equilateral triangles.

   (c) Same question, but now with the general case of $4n-1$ equilateral triangles.

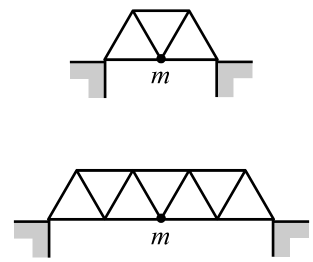

Figure 1.9

3. **Keeping the book up** $*$

   The task of Problem 4 is to find the minimum force required to keep a book up. What is the maximum allowable force? Is there a special angle that arises? Given μ, make a rough plot of the allowed values of F for $-\pi/2 < \theta < \pi/2$.

4. **Rope between inclines** $**$

   A rope rests on two platforms that are both inclined at an angle θ (which you are free to pick), as shown in Fig. 1.10. The rope has uniform mass density, and its coefficient of friction with the platforms is 1. The system has left-right symmetry. What is the largest possible fraction of the rope that does not touch the platforms? What angle θ allows this maximum value?

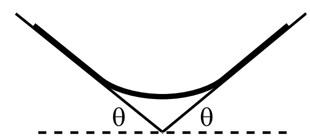

Figure 1.10

5. **Hanging chain** $**$

   A chain of mass M hangs between two walls, with its ends at the same height. The chain makes an angle of θ with each wall, as shown in Fig. 1.11. Find the tension in the chain at the lowest point. Solve this by:

   (a) Considering the forces on half of the chain. (This is the quick way.)

   (b) Using the fact that the height of a hanging chain is given by $y(x) = (1/\alpha)\cosh(\alpha x)$, and considering the vertical forces on an infinitesimal piece at the bottom. (This is the long way.)

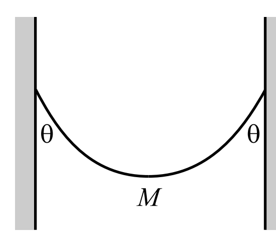

Figure 1.11

Section 1.2: Balancing torques

6. **Direction of the force** $*$

   A stick is connected to other parts of a system by hinges at its ends. Show that if the stick is massless, then the forces it feels at the hinges are directed along the stick; but if the stick has mass, then the forces need not point along the stick.

7. **Gravitational torque** $*$

   A horizontal stick of mass M and length L is pivoted at one end. Integrate the gravitational torque along the stick (relative to the pivot), and show that the result is the same as the torque due to a mass M located at the center of the stick.

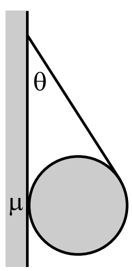

Figure 1.12

8. **Tetherball** $*$

   A ball is held up by a string, as shown in Fig. 1.12, with the string tangent to the ball. If the angle between the string and the wall is θ, what is the minimum coefficient of static friction between the ball and the wall, if the ball is not to fall?

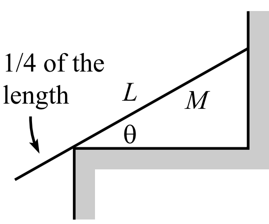

Figure 1.13

9. **Ladder on a corner** $*$

   A ladder of mass M and length L leans against a frictionless wall, with a quarter of its length hanging over a corner, as shown in Fig. 1.13. Assuming that there is sufficient friction at the corner to keep the ladder at rest, what is the total force that the corner exerts on the ladder?

10. **Stick on a corner** $*$

    You hold one end of a stick of mass M and length L. A quarter of the way up the stick, it rests on a frictionless corner of a table, as shown in Fig. 1.14. The stick makes an angle θ with the horizontal. What is the magnitude of the force your hand must apply, to keep the stick in this position? For what angle is the vertical component of your force equal to zero?

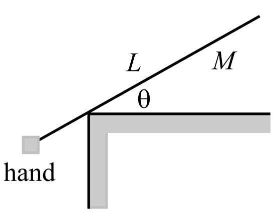

Figure 1.14

11. **Two sticks** $**$

    Two sticks, each of mass m and length ℓ, are connected by a hinge at their top ends. They each make an angle θ with the vertical. A massless string connects the bottom of the left stick to the right stick, perpendicularly, as shown in Fig. 1.15. The whole setup stands on a frictionless table.

    (a) What is the tension in the string?

    (b) What force does the left stick exert on the right stick at the hinge? *Hint:* No messy calculations required!

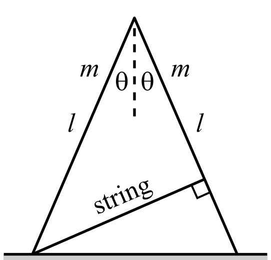

Figure 1.15

12. **Two sticks and a wall** $**$

Two sticks are connected, with hinges, to each other and to a wall. The bottom stick is horizontal and has length L, and the sticks make an angle of θ with each other, as shown in Fig. 1.16. If both sticks have the same mass per unit length, ρ, find the horizontal and vertical components of the force that the wall exerts on the top hinge, and show that the magnitude goes to infinity for both $\theta \to 0$ and $\theta \to \pi/2$.[5]

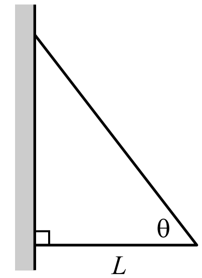

Figure 1.16

13. **Stick on a circle** $**$

Using the result from Problem 16 for the setup shown in Fig. 1.17, show that if the system is to remain at rest, then the coefficient of friction:

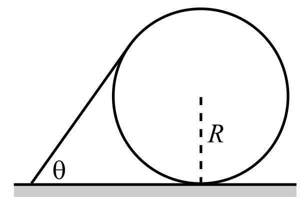

Figure 1.17

(a) between the stick and the circle must satisfy

$$\mu \geq \frac{\sin\theta}{(1+\cos\theta)} . \tag{1.14}$$

(b) between the stick and the ground must satisfy[6]

$$\mu \geq \frac{\sin\theta\cos\theta}{(1+\cos\theta)(2-\cos\theta)} . \tag{1.15}$$

[5]The force must therefore achieve a minimum at some intermediate angle. If you want to go through the algebra, you can show that this minimum occurs when $\cos\theta = \sqrt{3} - 1$, which gives $\theta \approx 43°$.

[6]If you want to go through the algebra, you can show that the maximum of the right-hand side occurs when $\cos\theta = \sqrt{3} - 1$, which gives $\theta \approx 43°$. (Yes, I did just cut and paste this from the previous footnote. But it's still correct!) This is the angle for which the stick is most likely to slip on the ground.

1.4 Problems

Section 1.1: Balancing forces

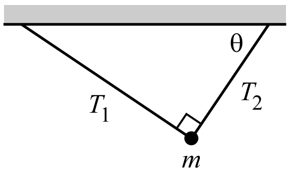

Figure 1.18

1. **Hanging mass**

   A mass m, held up by two strings, hangs from a ceiling, as shown in Fig. 1.18. The strings form a right angle. In terms of the angle θ shown, what is the tension in each string?

2. **Block on a plane**

   A block sits on a plane that is inclined at an angle θ. Assume that the friction force is large enough to keep the block at rest. What are the horizontal components of the friction and normal forces acting on the block? For what θ are these horizontal components maximum?

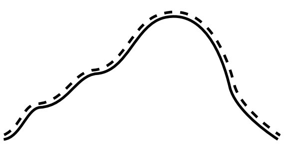

Figure 1.19

3. **Motionless chain** $*$

   A frictionless planar curve is in the shape of a function which has its endpoints at the same height but is otherwise arbitrary. A chain of uniform mass per unit length rests on the curve from end to end, as shown in Fig. 1.19. Show, by considering the net force of gravity along the curve, that the chain will not move.

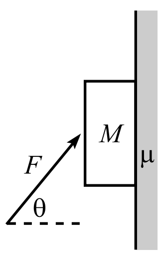

Figure 1.20

4. **Keeping the book up** $*$

   A book of mass M is positioned against a vertical wall. The coefficient of friction between the book and the wall is μ. You wish to keep the book from falling by pushing on it with a force F applied at an angle θ with respect to the horizontal $(-\pi/2 < \theta < \pi/2)$, as shown in Fig. 1.20. For a given θ, what is the minimum F required? What is the limiting value of θ, below which there does not exist an F that will keep the book up?

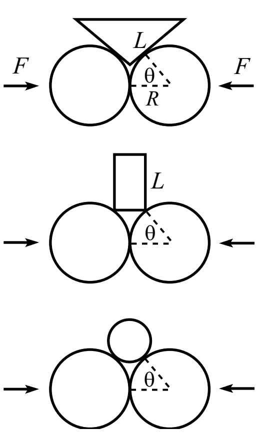

Figure 1.21

5. **Objects between circles** $**$

   Each of the following planar objects is placed, as shown in Fig. 1.21, between two frictionless circles of radius R. The mass density of each object is σ, and the radii to the points of contact make an angle θ with the horizontal. For each case, find the horizontal force that must be applied to the circles to keep them together. For what θ is this force maximum or minimum?

   (a) An isosceles triangle with common side length L.

   (b) A rectangle with height L.

   (c) A circle.

6. **Hanging rope**

   A rope with length L and mass density ρ per unit length is suspended vertically from one end. Find the tension as a function of height along the rope.

7. **Rope on a plane** $*$

   A rope with length L and mass density ρ per unit length lies on a plane inclined at angle θ (see Fig. 1.22). The top end is nailed to the plane, and the coefficient of friction between the rope and plane is μ. What are the possible values for the tension at the top of the rope?

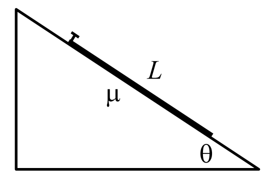

Figure 1.22

8. **Supporting a disk** $**$

   (a) A disk of mass M and radius R is held up by a massless string, as shown in Fig. 1.23. The surface of the disk is frictionless. What is the tension in the string? What is the normal force per unit length the string applies to the disk?

   (b) Let there now be friction between the disk and the string, with coefficient μ. What is the smallest possible tension in the string at its lowest point?

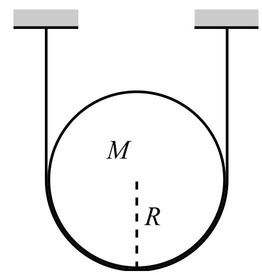

Figure 1.23

9. **Hanging chain** $****$

   (a) A chain with uniform mass density per unit length hangs between two given points on two walls. Find the shape of the chain. Aside from an arbitrary additive constant, the function describing the shape should contain one unknown constant.

   (b) The unknown constant in your answer depends on the horizontal distance d between the walls, the vertical distance λ between the support points, and the length ℓ of the chain (see Fig. 1.24). Find an equation involving these given quantities that determines the unknown constant.

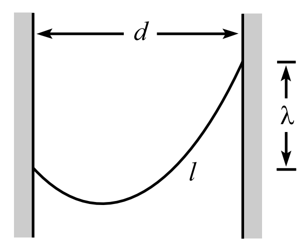

Figure 1.24

10. **Hanging gently** $**$

    A chain with uniform mass density per unit length hangs between two supports located at the same height, a distance $2d$ apart (see Fig. 1.25). What should the length of the chain be so that the magnitude of the force at the supports is minimized? You may use the fact that a hanging chain takes the form, $y(x) = (1/\alpha)\cosh(\alpha x)$. You will eventually need to solve an equation numerically.

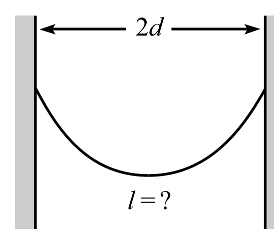

Figure 1.25

11. **Mountain Climber** $****$

    A mountain climber wishes to climb up a frictionless conical mountain. He wants to do this by throwing a lasso (a rope with a loop) over the top and climbing up along the rope. Assume that the climber is of negligible height, so that the rope lies along the mountain, as shown in Fig. 1.26.

    At the bottom of the mountain are two stores. One sells "cheap" lassos (made of a segment of rope tied to a loop of *fixed* length). The other sells "deluxe" lassos (made of one piece of rope with a loop of *variable* length; the loop's

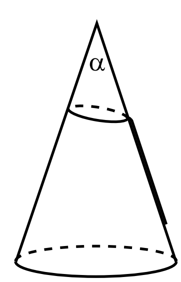

Figure 1.26

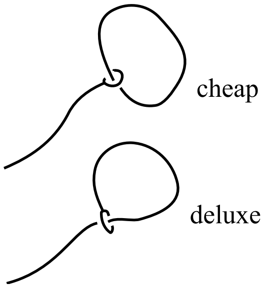

Figure 1.27

length may change without any friction of the rope with itself). See Fig. 1.27.

When viewed from the side, the conical mountain has an angle α at its peak. For what angles α can the climber climb up along the mountain if he uses:

(a) a "cheap" lasso?

(b) a "deluxe" lasso?

Section 1.2: Balancing torques

12. **Equality of torques** $**$

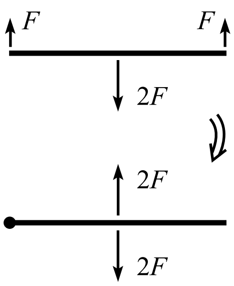

Figure 1.28

This problem gives another way of demonstrating Claim 1.1, using an inductive argument. We'll get you started, and then you can do the general case.

Consider the situation where forces F are applied upward at the ends of a stick of length ℓ, and a force $2F$ is applied downward at the midpoint (see Fig. 1.28). The stick will not rotate (by symmetry), and it will not translate (because the net force is zero). If we wish, we may consider the stick to have a pivot at the left end. If we then erase the force F on the right end and replace it with a force $2F$ at the middle, then the two $2F$ forces in the middle will cancel, so the stick will remain at rest.[7] Therefore, we see that a force F applied at a distance ℓ from a pivot is equivalent to a force $2F$ applied at a distance $\ell/2$ from the pivot, in the sense that they both have the same effect in cancelling out the rotational effect of the downwards $2F$ force.

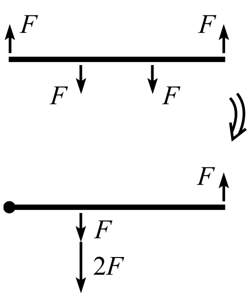

Figure 1.29

Now consider the situation where forces F are applied upward at the ends, and forces F are applied downward at the $\ell/3$ and $2\ell/3$ marks (see Fig. 1.29). The stick will not rotate (by symmetry), and it will not translate (because the net force is zero). Consider the stick to have a pivot at the left end. From the above paragraph, the force F at $2\ell/3$ is equivalent to a force $2F$ at $\ell/3$. Making this replacement, we now have a total force of $3F$ at the $\ell/3$ mark. Therefore, we see that a force F applied at a distance ℓ is equivalent to a force $3F$ applied at a distance $\ell/3$.

Your task is to now use induction to show that a force F applied at a distance ℓ is equivalent to a force nF applied at a distance ℓ/n, and to then argue why this demonstrates Claim 1.1.

13. **Find the force** $*$

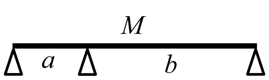

Figure 1.30

A stick of mass M is held up by supports at each end, with each support providing a force of $Mg/2$. Now put another support somewhere in the middle, say, at a distance a from one support and b from the other; see Fig. 1.30. What forces do the three supports now provide? Can you solve this?

[7]There will now be a different force applied at the pivot, namely zero, but the purpose of the pivot is to simply apply whatever force is necessary to keep the left end motionless.

14. **Leaning sticks** $*$

One stick leans on another as shown in Fig. 1.31. A right angle is formed where they meet, and the right stick makes an angle θ with the horizontal. The left stick extends infinitesimally beyond the end of the right stick. The coefficient of friction between the two sticks is μ. The sticks have the same mass density per unit length and are both hinged at the ground. What is the minimum angle θ for which the sticks do not fall?

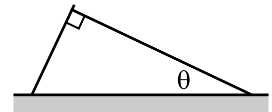

Figure 1.31

15. **Supporting a ladder** $*$

A ladder of length L and mass M has its bottom end attached to the ground by a pivot. It makes an angle θ with the horizontal, and is held up by a massless stick of length ℓ which is also attached to the ground by a pivot (see Fig. 1.32). The ladder and the stick are perpendicular to each other. Find the force that the stick exerts on the ladder.

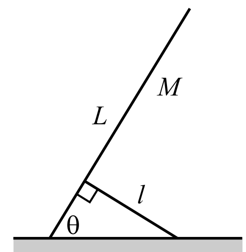

Figure 1.32

16. **Stick on a circle** $**$

A stick of mass density ρ per unit length rests on a circle of radius R (see Fig. 1.33). The stick makes an angle θ with the horizontal and is tangent to the circle at its upper end. Friction exists at all points of contact, and assume that it is large enough to keep the system at rest. Find the friction force between the ground and the circle.

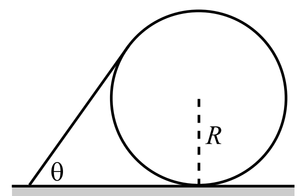

Figure 1.33

17. **Leaning sticks and circles** $***$

A large number of sticks (with mass density ρ per unit length) and circles (with radius R) lean on each other, as shown in Fig. 1.34. Each stick makes an angle θ with the horizontal and is tangent to a circle at its upper end. The sticks are hinged to the ground, and every other surface is *frictionless* (unlike in the previous problem). In the limit of a very large number of sticks and circles, what is the normal force between a stick and the circle it rests on, very far to the right? (Assume that the last circle leans against a wall, to keep it from moving.)

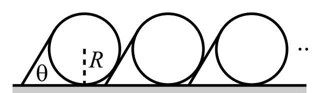

Figure 1.34

18. **Balancing the stick** $**$

Given a semi-infinite stick (that is, one that goes off to infinity in one direction), determine how its density should depend on position so that it has the following property: If the stick is cut at an arbitrary location, the remaining semi-infinite piece will balance on a support that is located a distance ℓ from the end (see Fig. 1.35).

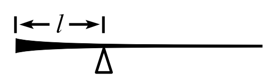

Figure 1.35

19. **The spool** $**$

A spool consists of an axle of radius r and an outside circle of radius R which rolls on the ground. A thread is wrapped around the axle and is pulled with tension T, at an angle θ with the horizontal (see Fig. 1.36).

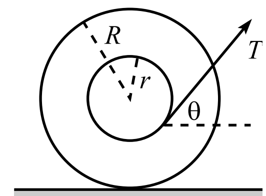

Figure 1.36

(a) Given R and r, what should θ be so that the spool does not move? Assume that the friction between the spool and the ground is large enough so that the spool doesn't slip.

(b) Given R, r, and the coefficient of friction μ between the spool and the ground, what is the largest value of T for which the spool remains at rest?

(c) Given R and μ, what should r be so that you can make the spool slip with as small a T as possible? That is, what should r be so that the upper bound on T from part (b) is as small as possible? What is the resulting value of T?

1.5 Solutions

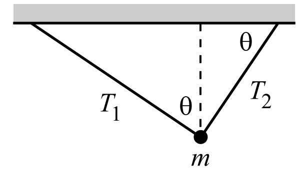

Figure 1.37

1. **Hanging mass**

   Balancing the horizontal and vertical force components on the mass gives, respectively (see Fig. 1.37),

$$\begin{aligned} T_1 \sin\theta &= T_2 \cos\theta, \\ T_1 \cos\theta + T_2 \sin\theta &= mg. \end{aligned} \tag{1.16}$$

   Solving for T_1 in the first equation, and substituting into the second equation, gives

$$T_1 = mg\cos\theta, \qquad \text{and} \qquad T_2 = mg\sin\theta. \tag{1.17}$$

   As a double-check, these have the correct limits when $\theta \to 0$ or $\theta \to \pi/2$.

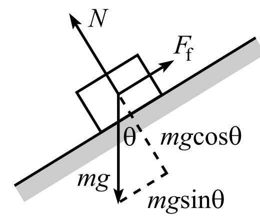

Figure 1.38

2. **Block on a plane**

   Balancing the forces shown in Fig. 1.38, wee see that $F = mg\sin\theta$ and $N = mg\cos\theta$. The horizontal components of these are $F\cos\theta = mg\sin\theta\cos\theta$ (to the right), and $N\sin\theta = mg\cos\theta\sin\theta$ (to the left). These are equal, as they must be, because the net horizontal force on the block is zero. To maximize the value of $mg\sin\theta\cos\theta$, we can either take the derivative, or we can write it as $(mg/2)\sin 2\theta$, from which it is clear that the maximum occurs at $\theta = \pi/4$. The maximum value is $mg/2$.

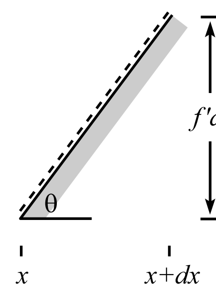

Figure 1.39

3. **Motionless chain**

   Let the curve be described by the function $f(x)$, and let it run from $x = a$ to $x = b$. Consider a little piece of the chain between x and $x + dx$ (see Fig. 1.39). The length of this piece is $\sqrt{1+f'^2}\,dx$, and so its mass is $\rho\sqrt{1+f'^2}\,dx$, where ρ is the mass per unit length. The component of the gravitational acceleration along the curve is $-g\sin\theta = -gf'/\sqrt{1+f'^2}$, with positive corresponding to moving along the curve from a to b. The total force along the curve is therefore

$$\begin{aligned} F &= \int_a^b (-g\sin\theta)\,dm \\ &= \int_a^b \left(\frac{-gf'}{\sqrt{1+f'^2}}\right)\left(\rho\sqrt{1+f'^2}\,dx\right) \\ &= -\rho g \int_a^b f'\,dx \\ &= -g\rho\big(f(a) - f(b)\big) \\ &= 0. \end{aligned} \tag{1.18}$$

4. **Keeping the book up**

   The normal force from the wall is $F\cos\theta$, so the friction force holding the book up is at most $\mu F\cos\theta$. The other vertical forces on the book are the gravitational force, which is $-Mg$, and the vertical component of F, which is $F\sin\theta$. If the book is to stay up, we must have

$$\mu F\cos\theta + F\sin\theta - Mg \geq 0. \tag{1.19}$$

   Therefore, F must satisfy

$$F \geq \frac{Mg}{\mu\cos\theta + \sin\theta}\,. \tag{1.20}$$

There is no possible F that satisfies this condition if the right-hand side is infinite. This occurs when

$$\tan\theta = -\mu. \tag{1.21}$$

If θ is more negative than this, then it is impossible to keep the book up, no matter how hard you push.

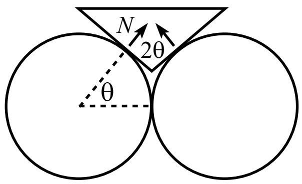

Figure 1.40

5. **Objects between circles**

(a) Let N be the normal force between the circles and the triangle. The goal in this problem is to find the horizontal component of N, that is, $N\cos\theta$.

From Fig. 1.40, we see that the upward force on the triangle from the normal forces is $2N\sin\theta$. This must equal the weight of the triangle, which is $g\sigma$ times the area. Since the bottom angle of the isosceles triangle is 2θ, the top side has length $2L\sin\theta$, and the altitude to this side is $L\cos\theta$. So the area of the triangle is $L^2\sin\theta\cos\theta$. The mass is therefore $\sigma L^2\sin\theta\cos\theta$. Equating the weight with the upward component of the normal forces gives $N = (g\sigma L^2/2)\cos\theta$. The horizontal component of N is therefore

$$N\cos\theta = \frac{g\sigma L^2\cos^2\theta}{2}. \tag{1.22}$$

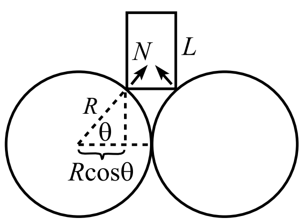

Figure 1.41

This equals zero when $\theta = \pi/2$, and it increases as θ decreases, even though the triangle is getting smaller. It has the interesting property of approaching the finite number $g\sigma L^2/2$, as $\theta \to 0$.

(b) In Fig. 1.41, the base of the rectangle has length $2R(1-\cos\theta)$. Its mass is therefore $\sigma 2RL(1-\cos\theta)$. Equating the weight with the upward component of the normal forces, $2N\sin\theta$, gives $N = g\sigma RL(1-\cos\theta)/\sin\theta$. The horizontal component of N is therefore

$$N\cos\theta = \frac{g\sigma RL(1-\cos\theta)\cos\theta}{\sin\theta}. \tag{1.23}$$

This equals zero for both $\theta = \pi/2$ and $\theta = 0$ (because $1-\cos\theta \approx \theta^2/2$ goes to zero faster than $\sin\theta \approx \theta$, for small θ). Taking the derivative to find where it reaches a maximum, we obtain (using $\sin^2\theta = 1-\cos^2\theta$),

$$\cos^3\theta - 2\cos\theta + 1 = 0. \tag{1.24}$$

Fortunately, there is an easy root of this cubic equation, namely $\cos\theta = 1$, which we know is not the maximum. Dividing through by the factor $(\cos\theta - 1)$ gives

$$\cos^2\theta + \cos\theta - 1 = 0. \tag{1.25}$$

The roots of this quadratic equation are

$$\cos\theta = \frac{-1\pm\sqrt{5}}{2}. \tag{1.26}$$

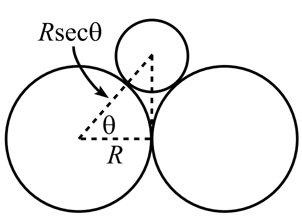

Figure 1.42

We must choose the plus sign, because we need $|\cos\theta| \le 1$. So our answer is $\cos\theta = 0.618$, which interestingly is the golden ratio. The angle θ is $\approx 51.8°$.

(c) In Fig. 1.42, the length of the hypotenuse shown is $R\sec\theta$, so the radius of the top circle is $R(\sec\theta - 1)$. Its mass is therefore $\sigma\pi R^2(\sec\theta-1)^2$. Equating the weight with the upward component of the normal forces, $2N\sin\theta$, gives $N = g\sigma\pi R^2(\sec\theta - 1)^2/(2\sin\theta)$. The horizontal component of N is therefore

$$N\cos\theta = \frac{g\sigma\pi R^2\cos\theta}{2\sin\theta}\left(\frac{1}{\cos\theta} - 1\right)^2. \tag{1.27}$$

This equals zero when $\theta = 0$ (using $\cos\theta \approx 1 - \theta^2/2$ and $\sin\theta \approx \theta$, for small θ). For $\theta \to \pi/2$, it behaves like $1/\cos\theta$, which goes to infinity. In this limit, N points almost vertically, but its magnitude is so large that the horizontal component still approaches infinity.

6. **Hanging rope**

   Let $T(y)$ be the tension as a function of height. Consider a small piece of the rope between y and $y + dy$ ($0 \le y \le L$). The forces on this piece are $T(y + dy)$ upward, $T(y)$ downward, and the weight $\rho g\,dy$ downward. Since the rope is at rest, we have $T(y + dy) = T(y) + \rho g\,dy$. Expanding this to first order in dy gives $T'(y) = \rho g$. The tension in the bottom of the rope is zero, so integrating from $y = 0$ up to a position y gives

$$T(y) = \rho g y. \tag{1.28}$$

   As a double-check, at the top end we have $T(L) = \rho g L$, which is the weight of the entire rope, as it should be.

   Alternatively, you can simply write down the answer, $T(y) = \rho g y$, by noting that the tension at a given point in the rope is what supports the weight of all the rope below it.

7. **Rope on a plane**

   The component of the gravitational force along the plane is $(\rho L)g\sin\theta$, and the maximum value of the friction force is $\mu N = \mu(\rho L)g\cos\theta$. Therefore, you might think that the tension at the top of the rope is $\rho Lg\sin\theta - \mu\rho Lg\cos\theta$. However, this is not necessarily the value. The tension at the top depends on how the rope is placed on the plane.

   If, for example, the rope is placed on the plane without being stretched, the friction force will point upwards, and the tension at the top will indeed equal $\rho Lg\sin\theta - \mu\rho Lg\cos\theta$. Or it will equal zero if $\mu\rho Lg\cos\theta > \rho Lg\sin\theta$, in which case the friction force need not achieve its maximum value.

   If, on the other hand, the rope is placed on the plane after being stretched (or equivalently, it is dragged up along the plane and then nailed down), then the friction force will point downwards, and the tension at the top will equal $\rho Lg\sin\theta + \mu\rho Lg\cos\theta$.

   Another special case occurs when the rope is placed on a frictionless plane, and then the coefficient of friction is "turned on" to μ. The friction force will still be zero. Changing the plane from ice to sandpaper (somehow without moving the rope) won't suddenly cause there to be a friction force. Therefore, the tension at the top will equal $\rho Lg\sin\theta$.

   In general, depending on how the rope is placed on the plane, the tension at the top can take any value from a maximum of $\rho Lg\sin\theta + \mu\rho Lg\cos\theta$, down to a minimum of $\rho Lg\sin\theta - \mu\rho Lg\cos\theta$ (or zero, whichever is larger). If the rope were replaced by a stick (which could support a compressive force), then the tension could achieve negative values down to $\rho Lg\sin\theta - \mu\rho Lg\cos\theta$, if this happens to be negative.

8. **Supporting a disk**

(a) The gravitational force downward on the disk is Mg, and the force upward is $2T$. These forces must balance, so

$$T = \frac{Mg}{2}. \tag{1.29}$$

We can find the normal force per unit length that the string applies to the disk in two ways.

First method: Let $N\,d\theta$ be the normal force on an arc of the disk that subtends an angle $d\theta$. Such an arc has length $R\,d\theta$, so N/R is the desired normal force per unit arclength. The tension in the string is constant because the string is massless, so N is constant, independent of θ. The upward component of the normal force is $N\,d\theta\cos\theta$, where θ is measured from the vertical (that is, $-\pi/2 \le \theta \le \pi/2$ here). Since the total upward force is Mg, we must have

$$\int_{-\pi/2}^{\pi/2} N\cos\theta\,d\theta = Mg. \tag{1.30}$$

The integral equals $2N$, so we find $N = Mg/2$. The normal force per unit length, N/R, is then $Mg/2R$.

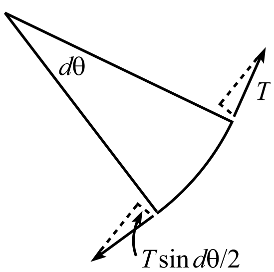

Figure 1.43

Second method: Consider the normal force, $N\,d\theta$, on a small arc of the disk that subtends and angle $d\theta$. The tension forces on each end of the corresponding small piece of string almost cancel, but they don't exactly, because they point in slightly different directions. Their non-zero sum is what produces the normal force on the disk. From Fig. 1.43, we see that the two forces have a sum of $2T\sin(d\theta/2)$, directed inward. Since $d\theta$ is small, we can use $\sin x \approx x$ to approximate this as $T\,d\theta$. Therefore, $N\,d\theta = T\,d\theta$, and so $N = T$. The normal force per unit arclength, N/R, then equals T/R. Using $T = Mg/2$ from eq. (1.29), we arrive at $N/R = Mg/2R$.

(b) Let $T(\theta)$ be the tension, as a function of θ, for $-\pi/2 \le \theta \le \pi/2$. T will depend on θ now, because there is a tangential friction force. Most of the work for this problem was already done in the example at the end of Section 1.1. We will simply invoke the second line of eq. (1.7), which says that[8]

$$dT \le \mu T\,d\theta. \tag{1.31}$$

Separating variables and integrating from the bottom of the rope up to an angle θ gives $\ln\big((T(\theta)/T(0)\big) \le \mu\theta$. Exponentiating this gives

$$T(\theta) \le T(0)e^{\mu\theta}. \tag{1.32}$$

Letting $\theta = \pi/2$, and using $T(\pi/2) = Mg/2$, we have $Mg/2 \le T(0)e^{\mu\pi/2}$. We therefore see that the tension at the bottom point must satisfy

$$T(0) \ge \frac{Mg}{2}e^{-\mu\pi/2}. \tag{1.33}$$

[8]This holds for $\theta > 0$. There would be a minus sign on the right-hand side if $\theta < 0$. But since the tension is symmetric around $\theta = 0$ in the case we're concerned with, we'll just deal with $\theta > 0$.

This minimum value of $T(0)$ goes to $Mg/2$ as $\mu \to 0$, as it should. And it goes to zero as $\mu \to \infty$, as it should (imagine a very sticky surface, so that the friction force from the rope near $\theta = \pi/2$ accounts for essentially all the weight). But interestingly, it doesn't exactly equal zero, no matter now large μ is.

9. **Hanging chain**

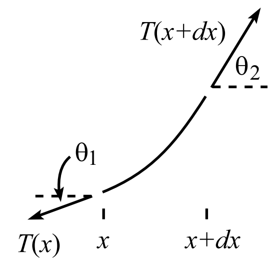

Figure 1.44

(a) Let the chain be described by the function $y(x)$, and let the tension be described by the function $T(x)$. Consider a small piece of the chain, with endpoints at x and $x + dx$, as shown in Fig. 1.44. Let the tension at x pull downward at an angle θ_1 with respect to the horizontal, and let the tension at $x + dx$ pull upward at an angle θ_2 with respect to the horizontal. Balancing the horizontal and vertical forces on the small piece of chain gives

$$\begin{aligned} T(x+dx)\cos\theta_2 &= T(x)\cos\theta_1, \\ T(x+dx)\sin\theta_2 &= T(x)\sin\theta_1 + \frac{g\rho\, dx}{\cos\theta_1}, \end{aligned} \tag{1.34}$$

where ρ is the mass per unit length. The second term on the right-hand side is the weight of the small piece, because $dx/\cos\theta_1$ (or $dx/\cos\theta_2$, which is essentially the same) is its length. We must now somehow solve these two differential equations for the two unknown functions, $y(x)$ and $T(x)$. There are various ways to do this. Here is one method, broken down into three steps.

First step: Squaring and adding eqs. (1.34) gives

$$\big(T(x+dx)\big)^2 = \big(T(x)\big)^2 + 2T(x)g\rho\tan\theta_1\, dx + \mathcal{O}(dx^2). \tag{1.35}$$

Writing $T(x + dx) \approx T(x) + T'(x)\, dx$, and using $\tan\theta_1 = dy/dx \equiv y'$, we can simplify eq. (1.35) to (neglecting second-order terms in dx)

$$T' = g\rho y'. \tag{1.36}$$

Therefore,

$$T = g\rho y + c_1, \tag{1.37}$$

where c_1 is a constant of integration.

Second step: Let's see what we can extract from the first equation in eqs. (1.34). Using

$$\cos\theta_1 = \frac{1}{\sqrt{1+(y'(x))^2}}, \qquad \text{and} \qquad \cos\theta_2 = \frac{1}{\sqrt{1+(y'(x+dx))^2}}, \tag{1.38}$$

and expanding things to first order in dx, the first of eqs. (1.34) becomes

$$\frac{T+T'dx}{\sqrt{1+(y'+y''dx)^2}} = \frac{T}{\sqrt{1+y'^2}}\,. \tag{1.39}$$

All of the functions here are evaluated at x, which we won't bother writing. Expanding the first square root gives (to first order in dx)

$$\frac{T+T'dx}{\sqrt{1+y'^2}}\left(1-\frac{y'y''dx}{1+y'^2}\right) = \frac{T}{\sqrt{1+y'^2}}\,. \tag{1.40}$$

To first order in dx this yields

$$\frac{T'}{T} = \frac{y'y''}{1+y'^2}\,. \tag{1.41}$$

Integrating both sides gives

$$\ln T + c_2 = \frac{1}{2}\ln(1+y'^2), \tag{1.42}$$

where c_2 is a constant of integration. Exponentiating then gives

$$c_3^2T^2 = 1+y'^2, \tag{1.43}$$

where $c_3 \equiv e^{c_2}$.

Third step: We will now combine eq. (1.43) with eq. (1.37) to solve for $y(x)$. Eliminating T gives $c_3^2(g\rho y+c_1)^2 = 1+y'^2$. We can rewrite this is the somewhat nicer form,

$$1+y'^2 = \alpha^2(y+h)^2, \tag{1.44}$$

where $\alpha \equiv c_3 g\rho$, and $h = c_1/g\rho$. At this point we can cleverly guess (motivated by the fact that $1+\sinh^2 z = \cosh^2 z$) that the solution for y is given by

$$y(x)+h = \frac{1}{\alpha}\cosh\alpha(x+a). \tag{1.45}$$

Or, we can separate variables to obtain

$$dx = \frac{dy}{\sqrt{\alpha^2(y+h)^2-1}}\,, \tag{1.46}$$

and then use the fact that the integral of $1/\sqrt{z^2-1}$ is $\cosh^{-1} z$, to obtain the same result.

The shape of the chain is therefore a hyperbolic cosine function. The constant h isn't too important, because it simply depends on where we pick the $y=0$ height. Furthermore, we can eliminate the need for the constant a if we pick $x=0$ to be where the lowest point of the chain is (or where it would be, in the case where the slope is always nonzero). In this case, using eq. (1.45), we see that $y'(0)=0$ implies $a=0$, as desired. We then have (ignoring the constant h) the nice simple result,

$$y(x) = \frac{1}{\alpha}\cosh(\alpha x). \tag{1.47}$$

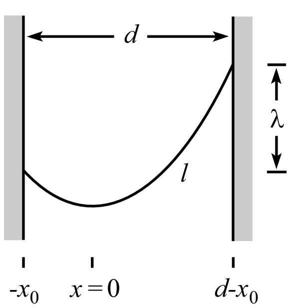

Figure 1.45

(b) The constant α can be determined from the locations of the endpoints and the length of the chain. As stated in the problem, the position of the chain may be described by giving (1) the horizontal distance d between the two endpoints, (2) the vertical distance λ between the two endpoints, and (3) the length ℓ of the chain, as shown in Fig. 1.45. Note that it is not obvious what the horizontal distances between the ends and the minimum point (which we have chosen as the $x=0$ point) are. If $\lambda=0$, then these distances are simply $d/2$. But otherwise, they are not so clear.

If we let the left endpoint be located at $x=-x_0$, then the right endpoint is located at $x=d-x_0$. We now have two unknowns, x_0 and α. Our two conditions are[9]

$$y(d-x_0) - y(-x_0) = \lambda, \tag{1.48}$$

[9]We will take the right end to be higher than the left end, without loss of generality.

along with the condition that the length equals ℓ, which takes the form (using eq. (1.47))

$$\begin{aligned} \ell &= \int_{-x_0}^{d-x_0} \sqrt{1+y'^2}\,dx \\ &= \frac{1}{\alpha}\sinh(\alpha x)\Big|_{-x_0}^{d-x_0}, \end{aligned} \tag{1.49}$$

where we have used $(d/dz)\cosh z = \sinh z$, and $1+\sinh^2 z = \cosh^2 z$. Writing out eqs. (1.48) and (1.49) explicitly, we have

$$\begin{aligned} \cosh\big(\alpha(d-x_0)\big) - \cosh(-\alpha x_0) &= \alpha\lambda, \\ \sinh\big(\alpha(d-x_0)\big) - \sinh(-\alpha x_0) &= \alpha\ell. \end{aligned} \tag{1.50}$$

If we take the difference of the squares of these two equations, and use the hyperbolic identities $\cosh^2 x - \sinh^2 x = 1$ and $\cosh x\cosh y - \sinh x \sinh y = \cosh(x-y)$, we obtain

$$2 - 2\cosh(\alpha d) = \alpha^2(\lambda^2 - \ell^2). \tag{1.51}$$

This is the desired equation that determines α. Given d, λ, and ℓ, we can numerically solve for α. Using a "half-angle" formula, you can show that eq. (1.51) may also be written as

$$2\sinh(\alpha d/2) = \alpha\sqrt{\ell^2 - \lambda^2}. \tag{1.52}$$

REMARK: Let's check a couple limits. If $\lambda = 0$ and $\ell = d$ (that is, the chain forms a horizontal straight line), then eq. (1.52) becomes $2\sinh(\alpha d/2) = \alpha d$. The solution to this is $\alpha = 0$, which does indeed correspond to a horizontal straight line, because for small α, eq. (1.47) behaves like $\alpha x^2/2$ (up to an additive constant), which varies slowly with x for small α. Another limit is where ℓ is much larger than both d and λ. In this case, eq. (1.52) becomes $2\sinh(\alpha d/2) \approx \alpha\ell$. The solution to this is a very large α, which corresponds to a "droopy" chain, because eq. (1.47) varies rapidly with x for large α. ♣

10. **Hanging gently**

We must first find the mass of the chain by calculating its length. Then we must determine the slope of the chain at the supports, so we can find the components of the force there.

Using the given information, $y(x) = (1/\alpha)\cosh(\alpha x)$, the slope of the chain as a function of x is

$$y' = \frac{d}{dx}\left(\frac{1}{\alpha}\cosh(\alpha x)\right) = \sinh(\alpha x). \tag{1.53}$$

The total length is therefore (using $1+\sinh^2 z = \cosh^2 z$)

$$\begin{aligned} \ell &= \int_{-d}^{d} \sqrt{1+y'^2}\,dx \\ &= \int_{-d}^{d} \cosh(\alpha x) \\ &= \frac{2}{\alpha}\sinh(\alpha d). \end{aligned} \tag{1.54}$$

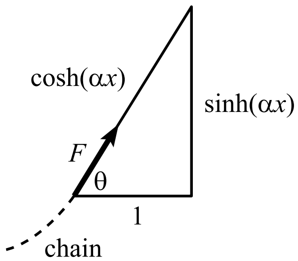

Figure 1.46

The weight of the rope is $W = \rho \ell g$, where ρ is the mass per unit length. Each support applies a vertical force of $W/2$. This must equal $F \sin\theta$, where F is the total force at each support, and θ is the angle it makes with the horizontal. Since $\tan\theta = y'(d) = \sinh(\alpha d)$, we see from Fig. 1.46 that $\sin\theta = \tanh(\alpha d)$. Therefore,

$$\begin{aligned} F &= \frac{1}{\sin\theta}\left(\frac{W}{2}\right) \\ &= \frac{1}{\tanh(\alpha d)}\left(\frac{\rho g \sinh(\alpha d)}{\alpha}\right) \\ &= \frac{\rho g}{\alpha}\cosh(\alpha d). \end{aligned} \tag{1.55}$$

Taking the derivative of this (as a function of α), and setting the result equal to zero to find the minimum, gives

$$\tanh(\alpha d) = \frac{1}{\alpha d}\,. \tag{1.56}$$

This must be solved numerically. The result is

$$\alpha d \approx 1.1997 \equiv \eta. \tag{1.57}$$

We therefore have $\alpha = \eta/d$, and so the shape of the chain that requires the minimum F is

$$y(x) \approx \frac{d}{\eta}\cosh\left(\frac{\eta x}{d}\right). \tag{1.58}$$

From eqs. (1.54) and (1.57), the length of the chain is

$$\ell = \frac{2d}{\eta}\sinh(\eta) \approx (2.52)d. \tag{1.59}$$

To get an idea of what the chain looks like, we can calculate the ratio of the height, h, to the width, $2d$.

$$\begin{aligned} \frac{h}{2d} &= \frac{y(d) - y(0)}{2d} \\ &= \frac{\cosh(\eta) - 1}{2\eta} \\ &\approx 0.338. \end{aligned} \tag{1.60}$$

We can also calculate the angle of the rope at the supports, using $\tan\theta = \sinh(\alpha d)$. This gives $\tan\theta = \sinh\eta$, and so $\theta \approx 56.5°$.

Remark: We can also ask what shape the chain should take in order to minimize the horizontal or vertical component of F.

The vertical component, F_y, is simply half the weight, so we want the shortest possible chain, namely a horizontal one (which requires an infinite F.) This corresponds to $\alpha = 0$.

The horizontal component, F_x, equals $F\cos\theta$. From Fig. 1.46, we see that $\cos\theta = 1/\cosh(\alpha d)$. Therefore, eq. (1.55) gives $F_x = \rho g/\alpha$. This goes to zero as $\alpha \to \infty$, which corresponds to a chain of infinite length, that is, a very "droopy" chain. ♣

11. **Mountain Climber**

(a) We will take advantage of the fact that a cone is "flat", in the sense that we can make one out of a piece of paper, without crumpling the paper.

Cut the cone along a straight line emanating from the peak and passing through the knot of the lasso, and roll the cone flat onto a plane. Call the resulting figure, which is a sector of a circle, S (see Fig. 1.47). If the cone is very sharp, then S will look like a thin "pie piece". If the cone is very wide, with a shallow slope, then S will look like a pie with a piece taken out of it.

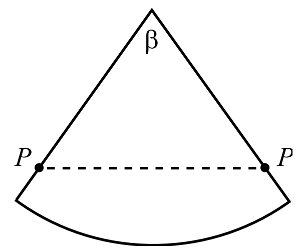

Figure 1.47

Points on the straight-line boundaries of the sector S are identified with each other. Let P be the location of the lasso's knot. Then P appears on each straight-line boundary, at equal distances from the tip of S. Let β be the angle of the sector S.

The key to this problem is to realize that the path of the lasso's loop must be a straight line on S, as shown by the dotted line in Fig. 1.47. (The rope will take the shortest distance between two points because there is no friction. And rolling the cone onto a plane does not change distances.) A straight line between the two identified points P is possible if and only if the sector S is smaller than a semicircle. The condition for a climbable mountain is therefore $\beta < 180°$.

What is this condition, in terms of the angle of the peak, α? Let C denote a cross-sectional circle of the mountain, a distance d (measured along the cone) from the top.[10] A semicircular S implies that the circumference of C equals πd. This then implies that the radius of C equals $d/2$. Therefore,

$$\sin(\alpha/2) < \frac{d/2}{d} = \frac{1}{2} \qquad \Longrightarrow \qquad \alpha < 60°. \tag{1.61}$$

This is the condition under which the mountain is climbable. In short, having $\alpha < 60°$ guarantees that there is a loop around the cone with shorter length than the distance straight to the peak and back.

REMARK: When viewed from the side, the rope will appear perpendicular to the side of the mountain at the point opposite the lasso's knot. A common mistake is to assume that this implies that the climbable condition is $\alpha < 90°$. This is not the case, because the loop does not lie in a plane. Lying in a plane, after all, would imply an elliptical loop. But the loop must certainly have a kink in it where the knot is, because there must exist a vertical component to the tension there, to hold the climber up. If we had posed the problem with a planar, triangular mountain, then the condition would have been $\alpha < 90°$.

(b) Use the same strategy as in part (a). Roll the cone onto a plane. If the mountain is very steep, then the climber's position can fall by means of the loop growing larger. If the mountain has a shallow slope, the climber's position can fall by means of the loop growing smaller. The only situation in which the climber will not fall is the one where the change in position of the knot along the mountain is exactly compensated by the change in length of the loop.

In terms of the sector S in a plane, this condition requires that if we move P a distance ℓ up (or down) along the mountain, the distance between the identified points P must decrease (or increase) by ℓ. In Fig. 1.47, we must therefore have an equilateral triangle, so $\beta = 60°$.

[10] We are considering such a circle for geometrical convenience. It is *not* the path of the lasso; see the remark below.

What peak-angle α does this correspond to? As in part (a), let C be a cross-sectional circle of the mountain, a distance d (measured along the cone) from the top. Then $\beta = 60°$ implies that the circumference of C equals $(\pi/3)d$. This then implies that the radius of C equals $d/6$. Therefore,

$$\sin(\alpha/2) = \frac{d/6}{d} = \frac{1}{6} \qquad \Longrightarrow \qquad \alpha \approx 19°. \tag{1.62}$$

This is the condition under which the mountain is climbable. We see that there is exactly one angle for which the climber can climb up along the mountain. The cheap lasso is therefore much more useful than the fancy deluxe lasso (assuming, of course, that you want to use it for climbing mountains, and not, say, for rounding up cattle).

Remark: Another way to see the $\beta = 60°$ result is to note that the three directions of rope emanating from the knot must all have the same tension, because the deluxe lasso is one continuous piece of rope. They must therefore have $120°$ angles between themselves (to provide zero net force on the massless knot). This implies that $\beta = 60°$ in Fig. 1.47.

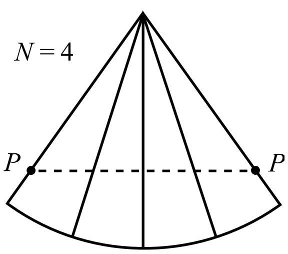

Figure 1.48

Further remarks: For each type of lasso, we can also ask the question: For what angles can the mountain be climbed if the lasso is looped N times around the top of the mountain? The solution here is similar to that above.

For the "cheap" lasso of part (a), roll the cone N times onto a plane, as shown in Fig. 1.48 for $N = 4$. The resulting figure, S_N, is a sector of a circle divided into N equal sectors, each representing a copy of the cone. As above, S_N must be smaller than a semicircle. The circumference of the circle C (defined above) must therefore be less than $\pi d/N$. Hence, the radius of C must be less than $d/2N$. Thus,

$$\sin(\alpha/2) < \frac{d/2N}{d} = \frac{1}{2N} \qquad \Longrightarrow \qquad \alpha < 2\sin^{-1}\left(\frac{1}{2N}\right). \tag{1.63}$$

For the "deluxe" lasso of part (b), again roll the cone N times onto a plane. From the reasoning in part (b), we must have $N\beta = 60°$. The circumference of C must therefore be $\pi d/3N$, and so its radius must be $d/6N$. Therefore,

$$\sin(\alpha/2) = \frac{d/6N}{d} = \frac{1}{6N} \qquad \Longrightarrow \qquad \alpha = 2\sin^{-1}\left(\frac{1}{6N}\right). \tag{1.64}$$

12. **Equality of torques**

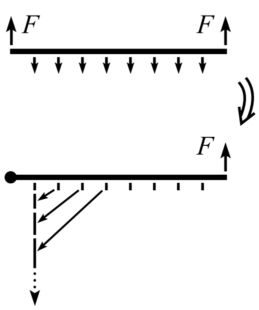

Figure 1.49

The proof by induction is as follows. Assume that we have shown that a force F applied at a distance d is equivalent to a force kF applied at a distance d/k, for all integers k up to $n-1$. We now want to show that the statement holds for $k = n$.

Consider the situation in Fig. 1.49. Forces F are applied at the ends of a stick, and forces $2F/(n-1)$ are applied at the $j\ell/n$ marks (for $1 \le j \le n-1$). The stick will not rotate (by symmetry), and it will not translate (because the net force is zero). Consider the stick to have a pivot at the left end. Replacing the interior forces by their equivalent ones at the ℓ/n mark (see Fig. 1.49) gives a total force there equal to

$$\frac{2F}{n-1}\Big(1+2+3+\cdots+(n-1)\Big) = \frac{2F}{n-1}\left(\frac{n(n-1)}{2}\right) = nF. \tag{1.65}$$

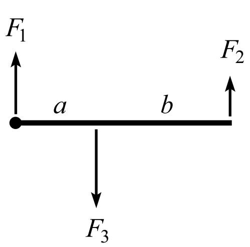

Figure 1.50

We therefore see that a force F applied at a distance ℓ is equivalent to a force nF applied at a distance ℓ/n, as was to be shown.

We can now show that Claim 1.1 holds, for arbitrary distances a and b (see Fig. 1.50).

Consider the stick to be pivoted at its left end, and let ϵ be a tiny distance (small compared to a). Then a force F_3 at a distance a is equivalent to a force $F_3(a/\epsilon)$ at a distance ϵ.[11] But a force $F_3(a/\epsilon)$ at a distance ϵ is equivalent to a force $F_3(a/\epsilon)(\epsilon/(a+b)) = F_3a/(a+b)$ at a distance $(a+b)$. This equivalent force at the distance $(a+b)$ must cancel the force F_2 there, because the stick is motionless. Therefore, we have $F_3a/(a+b) = F_2$, which proves the claim.

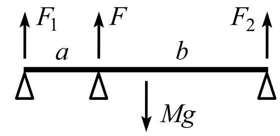

Figure 1.51

13. **Find the force**

In Fig. 1.51, let the supports at the ends exert forces F_1 and F_2, and let the support in the interior exert a force F. Then

$$F_1 + F_2 + F = Mg. \tag{1.66}$$

Balancing torques around the left and right ends gives, respectively,

$$\begin{aligned} Fa + F_2(a+b) &= Mg\frac{a+b}{2}, \\ Fb + F_1(a+b) &= Mg\frac{a+b}{2}, \end{aligned} \tag{1.67}$$

where we have used the fact that the stick can be treated as a point mass at its center. Note that the equation for balancing the torques around the center of mass is redundant; it is obtained by taking the difference of the two previous equations and then dividing by 2. And balancing torques around the middle pivot also takes the form of a linear combination of these equations, as you can show.

It appears as though we have three equations and three unknowns, but we really have only two equations, because the sum of eqs. (1.67) gives eq. (1.66). Therefore, since we have two equations and three unknowns, the system is underdetermined. Solving eqs. (1.67) for F_1 and F_2 in terms of F, we see that any forces of the form

$$(F_1, F, F_2) = \left(\frac{Mg}{2} - \frac{Fb}{a+b},\ F,\ \frac{Mg}{2} - \frac{Fa}{a+b}\right) \tag{1.68}$$

are possible. In retrospect, it makes sense that the forces are not determined. By changing the height of the new support an infinitesimal distance, we can make F be anything from 0 up to $Mg(a+b)/2b$, which is when the stick comes off the left support (assuming $b \geq a$).

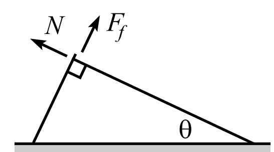

Figure 1.52

14. **Leaning sticks**

Let M_l be the mass of the left stick, and let M_r be the mass of the right stick. Then $M_l/M_r = \tan\theta$ (see Fig. 1.52). Let N and F_f be the normal and friction forces between the sticks. F_f has a maximum value of μN. Balancing the torques on the left stick (around the contact point with the ground) gives

$$N = \frac{M_l g}{2}\sin\theta. \tag{1.69}$$

Balancing the torques on the right stick (around the contact point with the ground) gives

$$F_f = \frac{M_r g}{2}\cos\theta. \tag{1.70}$$

[11]Technically, we can use the reasoning in the previous paragraph to say this only if a/ϵ is an integer, but since a/ϵ is very large, we can simply pick the closest integer to it, and there will be negligible error.

The condition $F_f \le \mu N$ becomes

$$M_r \cos\theta \le \mu M_l \sin\theta. \tag{1.71}$$

Using $M_l/M_r = \tan\theta$, this becomes

$$\tan^2\theta \ge \frac{1}{\mu}\,. \tag{1.72}$$

This is the condition for the sticks not to fall. This answer checks in the two extremes: In the limit $\mu \to 0$, we see that θ must be very close to $\pi/2$, which makes sense. And in the limit $\mu \to \infty$ (that is, very sticky sticks), we see that θ can be very small, which also makes sense.

15. **Supporting a ladder**

Let F be the desired force. Note that F must be directed along the stick, because otherwise there would be a net torque on the (massless) stick relative to the pivot at its right end. This would contradict the fact that it is at rest.

Look at torques on the ladder around the pivot at its bottom. The gravitational force provides a torque of $Mg(L/2)\cos\theta$, tending to turn it clockwise; and the force F from the stick provides a torque of $F(\ell/\tan\theta)$, tending to turn it counterclockwise. Equating these two torques gives

$$F = \frac{MgL}{2\ell}\sin\theta. \tag{1.73}$$

Remarks: F goes to zero as $\theta \to 0$, as it should.[12] And F increases to $MgL/2\ell$, as $\theta \to \pi/2$, which isn't so obvious (the required torque from the stick is very small, but its lever arm is also very small). However, in the special case where the ladder is exactly vertical, no force is required. You can see that our calculations above are not valid in this case, because we divided by $\cos\theta$, which is zero when $\theta = \pi/2$.

The normal force at the pivot of the stick (which equals the vertical component of F, because the stick is massless) is equal to $MgL\sin\theta\cos\theta/2\ell$. This has a maximum value of $MgL/4\ell$ at $\theta = \pi/4$. ♣

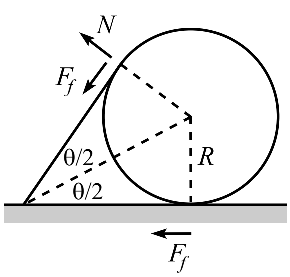

Figure 1.53

16. **Stick on a circle**

Let N be the normal force between the stick and the circle, and let F_f be the friction force between the ground and the circle (see Fig. 1.53). Then we immediately see that the friction force between the stick and the circle is also F_f, because the torques from the two friction forces on the circle must cancel.

Looking at torques on the stick around the point of contact with the ground, we have $Mg\cos\theta(L/2) = NL$, where M is the mass of the stick and L is its length. Therefore, $N = (Mg/2)\cos\theta$. Balancing the horizontal forces on the circle then gives $N\sin\theta = F_f + F_f\cos\theta$. So we have

$$F_f = \frac{N\sin\theta}{1+\cos\theta} = \frac{Mg\sin\theta\cos\theta}{2(1+\cos\theta)}\,. \tag{1.74}$$

But $M = \rho L$, and from Fig. 1.53 we have $L = R/\tan(\theta/2)$. Using the identity $\tan(\theta/2) = \sin\theta/(1+\cos\theta)$, we finally obtain

$$F_f = \frac{1}{2}\rho g R\cos\theta. \tag{1.75}$$

[12] For $\theta \to 0$, we would need to lengthen the ladder with a massless extension, because the stick would have to be very far to the right to remain perpendicular to the ladder.

In the limit $\theta \to \pi/2$, F_f approaches zero, which makes sense. In the limit $\theta \to 0$ (which corresponds to a very long stick), the friction force approaches $\rho g R/2$, which isn't so obvious.

17. **Leaning sticks and circles**

Let S_i be the ith stick, and let C_i be the ith circle. The normal forces C_i feels from S_i and S_{i+1} are equal in magnitude, because these two forces provide the only horizontal forces on the frictionless circle, so they must cancel. Let N_i be this normal force.

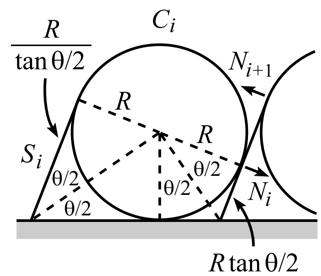

Figure 1.54

Look at the torques on S_{i+1}, relative to the hinge on the ground. The torques come from N_i, N_{i+1}, and the weight of S_{i+1}. From Fig. 1.54, we see that N_i acts at a point which is a distance $R\tan(\theta/2)$ away from the hinge. Since the stick has a length $R/\tan(\theta/2)$, this point is a fraction $\tan^2(\theta/2)$ up along the stick. Therefore, balancing the torques on S_{i+1} gives

$$\frac{1}{2}Mg\cos\theta + N_i\tan^2\frac{\theta}{2} = N_{i+1}. \tag{1.76}$$

N_0 is by definition 0, so we have $N_1 = (Mg/2)\cos\theta$ (as in the previous problem). If we successively use eq. (1.76), we see that N_2 equals $(Mg/2)\cos\theta\big(1+\tan^2(\theta/2)\big)$, and N_3 equals $(Mg/2)\cos\theta\big(1+\tan^2(\theta/2)+\tan^4(\theta/2)\big)$, and so on. In general,

$$N_i = \frac{Mg\cos\theta}{2}\left(1+\tan^2\frac{\theta}{2}+\tan^4\frac{\theta}{2}+\cdots+\tan^{2(i-1)}\frac{\theta}{2}\right). \tag{1.77}$$

In the limit $i\to\infty$, we may write this infinite geometric sum in closed form as

$$\lim_{i\to\infty} N_i \equiv N_\infty = \frac{Mg\cos\theta}{2}\left(\frac{1}{1-\tan^2(\theta/2)}\right). \tag{1.78}$$

Note that this is the solution to eq. (1.76), with $N_i = N_{i+1}$. So if a limit exists, it must equal this. Using $M = \rho L = \rho R/\tan(\theta/2)$, we can rewrite N_∞ as

$$N_\infty = \frac{\rho R g\cos\theta}{2\tan(\theta/2)}\left(\frac{1}{1-\tan^2(\theta/2)}\right). \tag{1.79}$$

The identity $\cos\theta = \cos^2(\theta/2)-\sin^2(\theta/2)$ may then be used to write this as

$$N_\infty = \frac{\rho R g\cos^3(\theta/2)}{2\sin(\theta/2)}\,. \tag{1.80}$$

Remarks: N_∞ goes to infinity for $\theta\to 0$, which makes sense, because the sticks are very long. All of the N_i are essentially equal to half the weight of a stick (in order to cancel the torque from the weight relative to the pivot). For $\theta\to\pi/2$, we see from eq. (1.80) that N_∞ approaches $\rho R g/4$, which is not at all obvious; the N_i start off at $N_1 = (Mg/2)\cos\theta \approx 0$, but gradually increase to $\rho R g/4$, which is a quarter of the weight of a stick.

Note that the horizontal force that must be applied to the last circle far to the right is $N_\infty\sin\theta = \rho R g\cos^4(\theta/2)$. This ranges from $\rho R g$ for $\theta\to 0$, to $\rho R g/4$ for $\theta\to\pi/2$. ♣

18. **Balancing the stick**

Let the stick go off to infinity in the positive x direction, and let it be cut at $x = x_0$. Then the pivot point is located at $x = x_0+\ell$ (see Fig. 1.55). Let the density be $\rho(x)$. The condition that the total gravitational torque relative to $x_0+\ell$ equal zero is

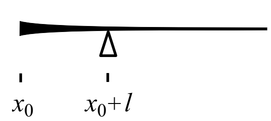

Figure 1.55

$$\tau = \int_{x_0}^{\infty}\rho(x)\big(x-(x_0+\ell)\big)g\,dx = 0. \tag{1.81}$$

We want this to equal zero for all x_0, so the derivative of τ with respect to x_0 must be zero. τ depends on x_0 through both the limits of integration and the integrand. In taking the derivative, the former dependence requires finding the value of the integrand at the limits, while the latter dependence requires taking the derivative of the integrand with respect to x_0, and then integrating. We obtain, using the fact that $\rho(\infty) = 0$,

$$0 = \frac{d\tau}{dx_0} = \ell\rho(x_0) - \int_{x_0}^{\infty} \rho(x)\,dx. \tag{1.82}$$

Taking the derivative of this equation with respect to x_0 gives

$$\ell\rho'(x_0) = -\rho(x_0). \tag{1.83}$$

The solution to this is (rewriting the arbitrary x_0 as x)

$$\rho(x) = Ae^{-x/\ell}. \tag{1.84}$$

We therefore see that the density decreases exponentially with x. The smaller ℓ is, the quicker it falls off. Note that the density at the pivot is $1/e$ times the density at the left end. And you can show that $1 - 1/e \approx 63\,\%$ of the mass is contained between the left end and the pivot.

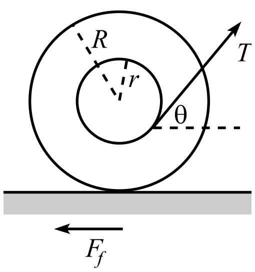

Figure 1.56

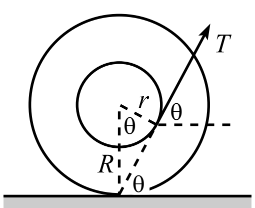

Figure 1.57

19. **The spool**

(a) Let F_f be the friction force the ground provides. Balancing the horizontal forces on the spool gives (see Fig. 1.56)

$$T\cos\theta = F_f. \tag{1.85}$$

Balancing torques around the center of the spool gives

$$Tr = F_f R. \tag{1.86}$$

These two equations imply

$$\cos\theta = \frac{r}{R}. \tag{1.87}$$

The niceness of this result suggests that there is a quicker way to obtain it. And indeed, we see from Fig. 1.57 that $\cos\theta = r/R$ is the angle that causes the line of the tension to pass through the contact point on the ground. Since gravity and friction provide no torque around this point, the total torque around it is therefore zero, and the spool remains at rest.

(b) The normal force from the ground is

$$N = Mg - T\sin\theta. \tag{1.88}$$

Using eq. (1.85), the statement $F_f \le \mu N$ becomes $T\cos\theta \le \mu(Mg - T\sin\theta)$. Hence,

$$T \le \frac{\mu Mg}{\cos\theta + \mu\sin\theta}, \tag{1.89}$$

where θ is given in eq. (1.87).

(c) The maximum value of T is given in (1.89). This depends on θ, which in turn depends on r. We want to find the r which minimizes this maximum T.

Taking the derivative with respect to θ, we find that the θ that maximizes the denominator in eq. (1.89) is given by $\tan\theta_0 = \mu$. You can then show that the value of T for this θ_0 is

$$T_0 = \frac{\mu M g}{\sqrt{1+\mu^2}} = Mg\sin\theta_0. \tag{1.90}$$

To find the corresponding r, we can use eq. (1.87) to write $\tan\theta = \sqrt{R^2 - r^2}/r$. The relation $\tan\theta_0 = \mu$ then yields

$$r_0 = \frac{R}{\sqrt{1+\mu^2}}\,. \tag{1.91}$$

This is the r that yields the smallest upper bound on T. In the limit $\mu = 0$, we have $\theta_0 = 0$, $T_0 = 0$, and $r_0 = R$. And in the limit $\mu = \infty$, we have $\theta_0 = \pi/2$, $T_0 = Mg$, and $r_0 = 0$.