Chapter 11

Relativity (Dynamics)

In the previous chapter, we dealt only with abstract particles flying through space and time. We didn't concern ourselves with the nature of the particles, how they got to be moving the way they were moving, or what would happen if various particles interacted. In this chapter we will deal with these issues. That is, we will discuss masses, forces, energy, momentum, etc.

The two main results of this chapter are that the momentum and energy of a particle are given by

$$\mathbf{p} = \gamma m \mathbf{v}, \qquad \text{and} \qquad E = \gamma m c^2, \tag{11.1}$$

where $\gamma \equiv 1/\sqrt{1 - v^2/c^2}$, and m is the mass of the particle.[1] When $v \ll c$, the expression for $\mathbf{p}$ reduces to $\mathbf{p} = m\mathbf{v}$, as it should for a non-relativistic particle. When $v = 0$, the expression for E reduces to the well-known $E = mc^2$.

11.1 Energy and momentum

In this section, we'll give some justification for eqs. (11.1). The reasoning here should convince you of their truth. An alternative, and perhaps more convincing, motivation comes from the 4-vector formalism in Chapter 12. In the end, however, the justification for eqs. (11.1) is obtained through experiments. Every day, experiments in high-energy accelerators are verifying the truth of these expressions. (More precisely, they are verifying that these energy and momenta are *conserved* in any type of collision.) We therefore conclude, with reasonable certainty, that eqs. (11.1) are the correct expressions for energy and momentum.

But actual experiments aside, let's consider a few thought-experiments that motivate the above expressions.

[1] People use the word "mass" in different ways in relativity. They talk about "rest mass" and "relativistic mass". These terms, however, are misleading. There is only one thing that can reasonably be called "mass" in relativity. It is the same thing that we call "mass" in Newtonian physics, and it is what some people would call "rest mass", although the qualifier "rest" is redundant. See Section 11.8 for a discussion of this issue.

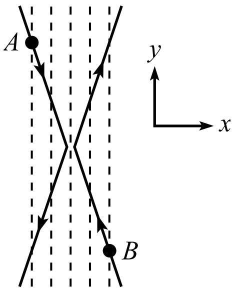

Figure 11.1

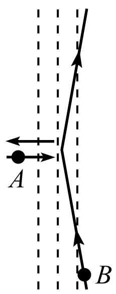

Figure 11.2

11.1.1 Momentum

Consider the following system. In the lab frame, identical particles A and B move as shown in Fig. 11.1. They move with equal and opposite small speeds in the x-direction, and with equal and opposite large speeds in the y-direction. Their paths are arranged so that they glance off each other and reverse their motion in the x-direction.

For clarity, imagine a series of equally spaced vertical lines for reference. Assume that both A and B have identical clocks that tick every time they cross one of the lines.

Consider now the reference frame that moves in the y-direction, with the same v_y as A. In this frame, the situation looks like Fig. 11.2. The collision simply changes the sign of the x-velocities of the particles. Therefore, the x-momenta of the two particles must be the same.[2]

However, the x-speeds of the two particles are *not* the same in this frame. A is essentially at rest in this frame, and B is moving with a very large speed, v. Therefore, B's clock is running slower than A's, by a factor essentially equal to $1/\gamma \equiv \sqrt{1-v^2/c^2}$. And since B's clock ticks once for every vertical line it crosses (this fact is independent of the frame), B must therefore be moving slower in the x-direction, by a factor of $1/\gamma$.

Therefore, the Newtonian expression, $p_x = mv_x$, cannot be the correct one for momentum, because B's momentum would be smaller than A's (by a factor of $1/\gamma$), due to their different v_x's. But the γ factor in

$$p_x = \gamma m v_x \equiv \frac{mv_x}{\sqrt{1-v^2/c^2}} \tag{11.2}$$

precisely takes care of this problem, because $\gamma \approx 1$ for A, and $\gamma = 1/\sqrt{1-v^2/c^2}$ for B, which precisely cancels the effect of B's smaller v_x.

To obtain the three-dimensional form for $\mathbf{p}$, we now note that the vector $\mathbf{p}$ must point in the same direction as the vector $\mathbf{v}$ points.[3] Therefore, eq. (11.2) implies that the momentum vector must be

$$\mathbf{p} = \gamma m \mathbf{v} \equiv \frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}}\,, \tag{11.3}$$

in agreement with eq. (11.1). Note that that all the components of $\mathbf{p}$ have the same denominator, which involves the whole speed, $v^2 = v_x^2 + v_y^2 + v_z^2$. The denominator of, say, p_x, is *not* $\sqrt{1-v_x^2/c^2}$.

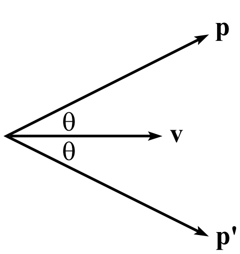

Figure 11.3

REMARK: The above setup is only one specific type of collision, among an infinite number of possible types of collisions. What we've shown with this setup is that the only

[2] This is true because if, say, A's p_x were larger than B's p_x, then the total p_x would point to the right before the collision, and to the left after the collision. Since momentum is something we want to be conserved, this cannot be the case.

[3] This is true because any other direction for $\mathbf{p}$ would violate rotation invariance. If someone claims that $\mathbf{p}$ points in the direction shown in Fig. 11.3, then he would be hard-pressed to explain why it doesn't instead point along the direction $\mathbf{p}'$ shown. In short, the direction of $\mathbf{v}$ is the only preferred direction in space.

possible vector of the form $f(v)m\mathbf{v}$ (where f is some function) that has any chance at being conserved in all collisions is $\gamma m\mathbf{v}$ (or some constant multiple of this). We haven't proved that it actually *is* conserved in all collisions. This is where the gathering of data from experiments comes in. But we've shown above that it would be a waste of time to consider, for example, the vector $\gamma^5 m\mathbf{v}$. ♣

11.1.2 Energy

Having given some justification for the momentum expression, $\mathbf{p} = \gamma m\mathbf{v}$, let us now try to justify the energy expression,

$$E = \gamma mc^2. \tag{11.4}$$

More precisely, we will show that γmc^2 is *conserved* in interactions (or at least in the specific interaction below). There are various ways to do this. The best way, perhaps, is to use the 4-vector formalism in Chapter 12. But we'll study one simple setup here that should do the job.

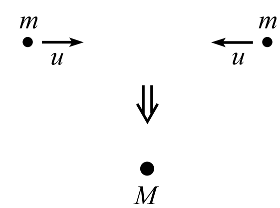

Figure 11.4

Consider the following system. Two identical particles of mass m head toward each other, both with speed u, as shown in Fig. 11.4. They stick together and form a particle of mass M. M is at rest, due to the symmetry of the situation. At the moment we cannot assume anything about the size of M. We will find below that it does *not* equal the naive value of $2m$.

This is a fairly uninteresting setup (conservation of momentum gives $0 = 0$), but now consider the less trivial view from a frame moving to the left at speed u. This situation is shown in Fig. 11.5. The right mass is at rest, M moves to the right at speed u, and the left mass moves to the right at speed $v = 2u/(1+u^2)$, from the velocity addition formula.[4] Note that the γ-factor associated with this speed v is

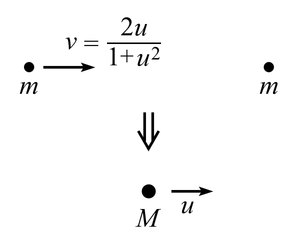

Figure 11.5

$$\gamma_v \equiv \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-\left(\frac{2u}{1+u^2}\right)^2}} = \frac{1+u^2}{1-u^2}\,. \tag{11.5}$$

Conservation of momentum in this collision then gives

$$\begin{aligned} \gamma_v mv + 0 &= \gamma_u Mu \\ \Longrightarrow \quad m\left(\frac{1+u^2}{1-u^2}\right)\left(\frac{2u}{1+u^2}\right) &= \frac{Mu}{\sqrt{1-u^2}} \\ \Longrightarrow \quad M = \frac{2m}{\sqrt{1-u^2}}\,. & \end{aligned} \tag{11.6}$$

Conservation of momentum therefore tells us that M does *not* equal $2m$. But if u is very small, then M is approximately equal to $2m$, as we know from everyday experience.

Using the value of M from eq. (11.6), let's now check that our candidate for energy, $E = \gamma mc^2$, is conserved in this collision. There is no freedom left in any of

[4] We're going to set $c = 1$ for a little while here, because this calculation would get a bit messy if we kept in the c's. We'll discuss the issue of setting $c = 1$ in more detail later in this section.

the parameters, so γmc^2 is either conserved or it isn't. In the original frame where M is at rest, E is conserved if

$$\gamma_0 Mc^2 = 2(\gamma_u mc^2) \qquad \Longleftrightarrow \qquad \frac{2m}{\sqrt{1-u^2}} = 2\left(\frac{1}{\sqrt{1-u^2}}\right)m\,, \tag{11.7}$$

which is indeed true.

Let's also check that E is conserved in the frame where the right mass is at rest. E is conserved if

$$\begin{aligned} \gamma_v mc^2 + \gamma_0 mc^2 &= \gamma_u Mc^2, \qquad \text{or} \\ \left(\frac{1+u^2}{1-u^2}\right)m + m &= \frac{M}{\sqrt{1-u^2}}\,, \qquad \text{or} \\ \frac{2m}{1-u^2} &= \left(\frac{2m}{\sqrt{1-u^2}}\right)\frac{1}{\sqrt{1-u^2}}\,, \end{aligned} \tag{11.8}$$

which is indeed true. So E is also conserved in this frame.

Hopefully at this point you're convinced that γmc^2 is a believable expression for the energy of a particle. But just as in the case of momentum, we haven't proved that γmc^2 actually *is* conserved in all collisions. This is the duty of experiments. But we've shown that it would be a waste of time to consider, for example, the quantity $\gamma^4 mc^2$.

One thing that we certainly need to check is that if E and p are conserved in one reference frame, then they are conserved in any other. We'll demonstrate this in Section 11.2. A conservation law shouldn't depend on what frame you're in, after all.

Remarks:

1. To be precise, we should say that technically we're not trying to justify eqs. (11.1) here. These two equations by themselves are devoid of any meaning. All they do is define the letters $\mathbf{p}$ and E. Our goal is to make a meaningful physical statement, not just a definition.

   The meaningful physical statement we want to make is that the quantities $\gamma m\mathbf{v}$ and γmc^2 are *conserved* in an interaction among particles (and this is what we tried to justify above). This fact then makes these quantities worthy of special attention, because conserved quantities are very helpful in understanding what is happening in a given physical situation. And anything worthy of special attention certainly deserves a label, so we may then attach the names "momentum" and "energy" to $\gamma m\mathbf{v}$ and γmc^2. Any other names would work just as well, of course, but we choose these because in the limit of small speeds, $\gamma m\mathbf{v}$ and γmc^2 reduce (as we will soon show) to some other nicely conserved quantities, which someone already tagged with the labels "momentum" and "energy" long ago.

2. As mentioned above, the fact of the matter is that we can't *prove* that $\gamma m\mathbf{v}$ and γmc^2 are conserved. In Newtonian physics, conservation of $\mathbf{p} \equiv m\mathbf{v}$ is basically postulated by Newton's third law, and we're not going to be able to do any better than that here. All we can hope to do as physicists is provide some motivation for considering $\gamma m\mathbf{v}$ and γmc^2, show that it is consistent for $\gamma m\mathbf{v}$ and γmc^2 to be conserved during an interaction, and gather a large amount of experimental evidence, all of which is

consistent with $\gamma m\mathbf{v}$ and γmc^2 being conserved. That's how physics works. You can't prove anything. So you learn to settle for the things you can't disprove.

Consider, when seeking gestalts,
The theories that physics exalts.
It's not that they're known
To be written in stone.
It's just that we can't say they're false.

As far as the experimental evidence goes, suffice it to say that high-energy accelerators, cosmological observations, and many other forums are continually verifying everything that we think is true about relativistic dynamics. If the theory is not correct, then we know that it must be the limiting theory of a more complete one (just as Newtonian physics is a limiting theory of relativity). But all this experimental induction has to count for something...

"To three, five, and seven, assign
A name," the prof said, "We'll define."
But he botched the instruction
With woeful induction
And told us the next prime was nine.

3. Conservation of energy in relativistic mechanics is actually a much simpler concept than it is in nonrelativistic mechanics, because $E = \gamma m$ is conserved, period. We don't have to worry about the generation of heat, which ruins conservation of the nonrelativistic $E = mv^2/2$. The heat is simply built into the energy. In the example above, the two m's collide and generate heat in the resulting mass M. This heat shows up as an increase in mass, which makes M larger than $2m$. The energy that corresponds to the increase in mass is due to the initial kinetic energy of the two m's.

4. Problem 1 gives an alternate derivation of the energy and momentum expressions in eq. (11.1). This derivation uses additional facts, namely that the energy and momentum of a photon are given by $E = h\nu$ and $p = h\nu/c$, where ν is the frequency of the light wave, and h is Planck's constant. ♣

Any multiple of γmc^2 is also conserved, of course. Why did we pick γmc^2 to label as "E" instead of, say, $5\gamma mc^3$? Consider the approximate form γmc^2 takes in the Newtonian limit, that is, in the limit $v \ll c$. We have, using the Taylor series expansion for $(1-x)^{-1/2}$,

$$\begin{aligned} E \equiv \gamma mc^2 &= \frac{mc^2}{\sqrt{1-v^2/c^2}} \\ &= mc^2\left(1 + \frac{v^2}{2c^2} + \frac{3v^4}{8c^4} + \cdots\right) \\ &= mc^2 + \frac{1}{2}mv^2 + \cdots. \end{aligned} \tag{11.9}$$

The dots represent higher-order terms in v^2/c^2, which may be neglected if $v \ll c$. In an elastic collision in Newtonian physics, no heat is generated, so mass is conserved. That is, the quantity mc^2 has a fixed value. We therefore see that conservation of

$E \equiv \gamma mc^2$ reduces to the familiar conservation of Newtonian kinetic energy, $mv^2/2$, for elastic collisions in the limit of slow speeds.

Likewise, we picked $\mathbf{p} \equiv \gamma m\mathbf{v}$, instead of, say, $6\gamma mc^4\mathbf{v}$, because the former reduces to the familiar Newtonian momentum, $m\mathbf{v}$, in the limit of slow speeds.

Whether abstract, profound, or just mystic,
Or boring, or somewhat simplistic,
A theory must lead
To results that we need
In limits, nonrelativistic.

Whenever we use the term "energy", we will mean the total energy, γmc^2. If we use the term "kinetic energy", we will mean a particle's excess energy over its energy when it is motionless, that is, $\gamma mc^2 - mc^2$. Note that kinetic energy is *not* necessarily conserved in a collision, because mass is not necessarily conserved, as we saw in eq. (11.6) in the above scenario, where $M = 2m/\sqrt{1-u^2}$. In the CM frame, there was kinetic energy before the collision, but none after. Kinetic energy is a rather artificial concept in relativity. You virtually always want to use the total energy, γmc^2, when solving a problem.

Note the following extremely important relation,

$$\begin{aligned} E^2 - |\mathbf{p}|^2c^2 &= \gamma^2m^2c^4 - \gamma^2m^2|\mathbf{v}|^2c^2 \\ &= \gamma^2m^2c^4\left(1-\frac{v^2}{c^2}\right) \\ &= m^2c^4. \end{aligned} \tag{11.10}$$

This is a primary ingredient in solving relativistic collision problems, as we will soon see. It replaces the KE $= p^2/2m$ relation between kinetic energy and momentum in Newtonian physics. It can be derived in more profound ways, as we will see in Chapter 12. Let's put it in a box, since it's so important,

$$\boxed{E^2 = p^2c^2 + m^2c^4}\,. \tag{11.11}$$

In the case where $m = 0$ (as with photons), eq. (11.11) says that $E = pc$. This is the key equation for massless objects. For photons, the two equations, $\mathbf{p} = \gamma m\mathbf{v}$ and $E = \gamma mc^2$, don't tell us much, because $m = 0$ and $\gamma = \infty$, so their product is undetermined. But $E^2 - |\mathbf{p}|^2c^2 = m^2c^4$ still holds, and we conclude that $E = pc$.

Note that any massless particle must have $\gamma = \infty$. That is, it must travel at speed c. If this weren't the case, then $E = \gamma mc^2$ would equal zero, in which case the particle isn't much of a particle. We'd have a hard time observing something with no energy.

Another nice relation, which holds for particles of any mass, is

$$\frac{\mathbf{p}}{E} = \frac{\mathbf{v}}{c^2}\,. \tag{11.12}$$

Given p and E, this is definitely the quickest way to get v.

Setting $c = 1$

For the remainder of our treatment of relativity, we will invariably work in units where $c = 1$. For example, instead of one meter being the unit of distance, we can make $3 \cdot 10^8$ meters equal to one unit. Or, we can keep the meter as is, and make $1/(3 \cdot 10^8)$ seconds the unit of time. In such units, our various expressions become

$$\mathbf{p} = \gamma m \mathbf{v}, \qquad E = \gamma m, \qquad E^2 = p^2 + m^2, \qquad \frac{\mathbf{p}}{E} = \mathbf{v}. \tag{11.13}$$

Said in another way, you can simply ignore all the c's in your calculations (which will generally save you a lot of strife), and then put them back into your final answer to make the units correct. For example, let's say the goal of a certain problem is to find the time of some event. If your answer comes out to be ℓ, where ℓ is a given length, then you know that the correct answer (in terms of the usual mks units) has to be ℓ/c, because this has units of time. In order for this procedure to work, there must be only one way to put the c's back in at the end. This is always the case, because if there were two ways, then we would have $c^a = c^b$, for some numbers $a \neq b$. But this is impossible, because c has units.

The general size of mc^2

What is the general size of mc^2? If we let $m = 1\,\text{kg}$, then we have $mc^2 = (1\,\text{kg})(3 \cdot 10^8\,\text{m/s})^2 \approx 10^{17}\,\text{J}$. How big is this? A typical household electric bill might around \$50 per month, or \$600 per year. At about 10 cents per kilowatt-hour, this translates to 6000 kilowatt-hours per year. Since there are 3600 seconds in an hour, this converts to $(6000)(10^3)(3600) \approx 2 \cdot 10^{10}$ watt-seconds. That is, $2 \cdot 10^{10}$ Joules per year. We therefore see that if one kilogram were converted completely into usable energy, it would be enough to provide electricity to $10^{17}/(2 \cdot 10^{10})$, or 5 million, homes for a year. That's a lot.

In a nuclear reactor, only a small fraction of the mass energy is converted into usable energy. Most of the mass remains in the final products, which doesn't help in lighting up your home. If a particle were to combine with its antiparticle, then it would be possible for all of the mass energy to be converted into usable energy. But we're still a few years away from this.

However, even a small fraction of the very large quantity, $E = mc^2$, can still be large, as evidenced by the use of nuclear power and nuclear weapons. Any quantity with a few factors of c is bound to change the face of the world.

11.2 Transformations of E and $\vec{p}$

Consider the following one-dimensional situation, where all the motion is along the x-axis. A particle has energy E' and momentum p' in frame S'. Frame S' moves at speed v with respect to frame S, in the positive x-direction (see Fig. 11.6). What are E and p in S?

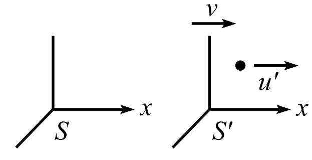

Figure 11.6

Let u' be the particle's speed in S'. From the velocity addition formula, the particle's speed in S is (dropping the factors of c)

$$u = \frac{u'+v}{1+u'v}\,. \tag{11.14}$$

This is all we need to know, because a particle's velocity determines its energy and momentum. But we'll need to go through a little algebra to make things look pretty. The γ-factor associated with the speed u is

$$\gamma_u = \frac{1}{\sqrt{1-\left(\frac{u'+v}{1+u'v}\right)^2}} = \frac{1+u'v}{\sqrt{(1-u'^2)(1-v^2)}} \equiv \gamma_{u'}\gamma_v(1+u'v). \tag{11.15}$$

The energy and momentum in S' are

$$E' = \gamma_{u'} m, \qquad \text{and} \qquad p' = \gamma_{u'} m u', \tag{11.16}$$

while the energy and momentum in S are, using eq. (11.15),

$$\begin{aligned} E &= \gamma_u m = \gamma_{u'}\gamma_v(1+u'v)m, \\ p &= \gamma_u m u = \gamma_{u'}\gamma_v(1+u'v)m\left(\frac{u'+v}{1+u'v}\right) = \gamma_{u'}\gamma_v(u'+v)m. \end{aligned} \tag{11.17}$$

Using the E' and p' from eq. (11.16), we can rewrite E and p as (with $\gamma \equiv \gamma_v$)

$$\begin{aligned} E &= \gamma(E' + vp'), \\ p &= \gamma(p' + vE'). \end{aligned} \tag{11.18}$$

These are transformations for E and p between frames. If you want to put the factors of c back in, then the vE' term becomes vE'/c^2. These equations are easy to remember, because they look *exactly* like the Lorentz transformations for the coordinates t and x in eq. (10.17). This is no coincidence, as we will see in Chapter 12.

REMARK: We can perform a few checks on eqs. (11.18). If $u' = 0$ (so that $p' = 0$ and $E' = m$), then $E = \gamma m$ and $p = \gamma m v$, as they should. Also, if $u' = -v$ (so that $p' = -\gamma m v$ and $E' = \gamma m$), then $E = m$ and $p = 0$, as they should. ♣

Note that since the transformations in eq. (11.18) are linear, they also hold if E and p represent the total energy and momentum of a collection of particles. That is,

$$\begin{aligned} \sum E &= \gamma\left(\sum E' + v\sum p'\right), \\ \sum p &= \gamma\left(\sum p' + v\sum E'\right). \end{aligned} \tag{11.19}$$

Indeed, any (corresponding) linear combinations of the energies and momenta are valid here, in place of the sums. For example, we can use the combinations $(E_1^b +$

$3E_2^a - 7E_5^b$) and $(p_1^b + 3p_2^a - 7p_5^b)$ in eq. (11.18), where the subscripts indicate which particle, and the superscripts indicate before or after a collision. You can verify this by simply taking the appropriate linear combination of eqs. (11.18) for the various particles. This consequence of linearity is a very important and useful result, as will become clear in the remarks below.

You can use eq. (11.18) to show that

$$E^2 - p^2 = E'^2 - p'^2, \tag{11.20}$$

just as we did to obtain the $t^2 - x^2 = t'^2 - x'^2$ result in eq. (10.37). The E's and p's here can represent any (corresponding) linear combinations of the E's and p's of the various particles, due to the linearity of eq. (11.18). For one particle, we already know that eq. (11.20) is true, because both sides are equal to m^2, from eq. (11.10). For many particles, the invariant $E_{\text{total}}^2 - p_{\text{total}}^2$ is equal to the square of the total energy in the CM frame (which reduces to m^2 for one particle), because $p_{\text{total}} = 0$ in the CM frame, by definition.

Remarks:

1. In the previous section, we said that we needed to show that if E and p are conserved in one reference frame, then they are conserved in any other frame (because a conservation law shouldn't depend on what frame you're in). Eq. (11.18) quickly gives us this result, because the E and p in one frame are linear functions of the E' and p' in another frame. If the total $\Delta E'$ and $\Delta p'$ in S' are zero, then eq. (11.18) says that the total ΔE and Δp in S must also be zero. We have used the fact that ΔE is a linear combination of the E's, and that Δp is a linear combination of the p's, so eq. (11.18) applies to these linear combinations.

2. Eq. (11.18) makes it clear that if you accept the fact that $p = \gamma m v$ is conserved in all frames, then you must also accept the fact that $E = \gamma m$ is conserved in all frames (and vice versa). This is true because the second of eqs. (11.18) says that if Δp and $\Delta p'$ are both zero, then $\Delta E'$ must also be zero (again, we have used linearity). E and p have no choice but to go hand in hand. ♣

Eq. (11.18) applies to the x-component of the momentum. How do the transverse components, p_y and p_z, transform? Just as with the y and z coordinates in the Lorentz transformations, p_y and p_z do not change between frames. The analysis in Chapter 12 makes this obvious, so for now we'll simply state that

$$\begin{aligned} p_y &= p'_y, \\ p_z &= p'_z, \end{aligned} \tag{11.21}$$

if the relative velocity between the frames is in the x-direction. If you really want to show explicitly that the transverse components do not change between frames, or if you are worried that a nonzero speed in the y direction will mess up the relationship between p_x and E that we calculated in eq. (11.18), then Exercise 4 is for you. But it's a bit tedious, so feel free to settle for the much cleaner reasoning in Chapter 12.

11.3 Collisions and decays

The strategy for studying relativistic collisions is the same as that for studying nonrelativistic ones. You simply have to write down all the conservation of energy and momentum equations, and then solve for whatever variables you want to solve for. The conservation principles are the same as they've always been. It's just that now the energy and momentum take the new forms in eq. (11.1).

In writing down the conservation of energy and momentum equations, it proves extremely useful to put E and $\mathbf{p}$ together into one four-component vector,

$$P \equiv (E, \mathbf{p}) \equiv (E, p_x, p_y, p_z). \tag{11.22}$$

This is called the *energy-momentum 4-vector*, or the *4-momentum*, for short.[5] Our notation in this chapter will be to use an uppercase P to denote a 4-momentum and a lowercase $\mathbf{p}$ or p to denote a spatial momentum. The components of a 4-momentum are generally indexed from 0 to 3, so that $P_0 \equiv E$, and $(P_1, P_2, P_3) \equiv \mathbf{p}$. For one particle, we have

$$P = (\gamma m, \gamma m v_x, \gamma m v_y, \gamma m v_z). \tag{11.23}$$

The 4-momentum for a collection of particles simply consists of the total E and total $\mathbf{p}$ of all the particles.

There are deep reasons for considering this four-component vector (as we will see in Chapter 12), but for now we will view it as simply a matter of convenience. If nothing else, it helps with the bookkeeping. Conservation of energy and momentum in a collision reduce to the concise statement,

$$P_{\text{before}} = P_{\text{after}}, \tag{11.24}$$

where these are the total 4-momenta of all the particles.

If we define the *inner product* between two 4-momenta, $A \equiv (A_0, A_1, A_2, A_3)$ and $B \equiv (B_0, B_1, B_2, B_3)$, to be

$$A \cdot B \equiv A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3, \tag{11.25}$$

then the relation $E^2 - p^2 = m^2$ (which is true for one particle) may be concisely written as

$$P^2 \equiv P \cdot P = m^2. \tag{11.26}$$

In other words, the square of a particle's 4-momentum equals the square of its mass. This relation will prove to be very useful in collision problems. Note that it is frame-independent, as we saw in eq. (11.20).

This inner product is different from the one we're used to in three-dimensional space. It has one positive sign and three negative signs, in contrast with the usual three positive signs. But we are free to define it however we wish, and we did indeed pick a good definition, because our inner product is invariant under Lorentz-transformations, just as the usual 3-D inner product is invariant under rotations.[6]

[5] If we were keeping in the factors of c, then the first term would be E/c, although some people instead multiply the $\mathbf{p}$ by c. Either convention is fine.

[6] For the inner product of a 4-momentum with itself (which could be any linear combination of

Example (Relativistic billiards): A particle with mass m and energy E approaches an identical particle at rest. They collide (elastically) in such a way that they both scatter at an angle θ relative to the incident direction (see Fig. 11.7). What is θ in terms of E and m? What is θ in the relativistic and non-relativistic limits?

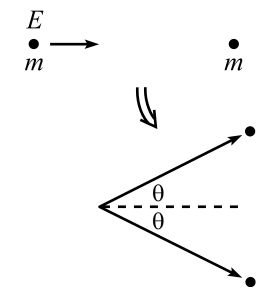

Figure 11.7

Solution: The first thing we should always do is write down the 4-momenta. The 4-momenta before the collision are

$$P_1 = (E, p, 0, 0), \qquad P_2 = (m, 0, 0, 0), \tag{11.27}$$

where $p = \sqrt{E^2 - m^2}$. The 4-momenta after the collision are (primes now denote "after")

$$P_1' = (E', p'\cos\theta, p'\sin\theta, 0), \qquad P_2' = (E', p'\cos\theta, -p'\sin\theta, 0), \tag{11.28}$$

where $p' = \sqrt{E'^2 - m^2}$. Conservation of energy gives $E' = (E+m)/2$, and conservation of p_x gives $p'\cos\theta = p/2$. Therefore, the 4-momenta after the collision are

$$P_{1,2}' = \left(\frac{E+m}{2}\,,\,\frac{p}{2}\,,\,\pm\frac{p}{2}\tan\theta,\,0\right). \tag{11.29}$$

From eq. (11.26), the squares of these 4-momenta must be m^2. Therefore,

$$\begin{aligned} m^2 &= \left(\frac{E+m}{2}\right)^2 - \left(\frac{p}{2}\right)^2 (1 + \tan^2\theta) \\ \Longrightarrow \quad 4m^2 &= (E+m)^2 - \frac{(E^2 - m^2)}{\cos^2\theta} \\ \Longrightarrow \quad \cos^2\theta &= \frac{E^2 - m^2}{E^2 + 2Em - 3m^2} = \frac{E+m}{E+3m}\,. \end{aligned} \tag{11.30}$$

The relativistic limit is $E \gg m$, which yields $\cos\theta \approx 1$. Therefore, both particles scatter almost directly forward.

The nonrelativistic limit is $E \approx m$ (it's *not* $E \approx 0$), which yields $\cos\theta \approx 1/\sqrt{2}$. Therefore, $\theta \approx 45°$, and the particles scatter with a $90°$ angle between them. This agrees with the result from the example in Section 4.7.2, a result which pool players are very familiar with.

Decays are basically the same as collisions. All you have to do is conserve energy and momentum, as the following example shows.

Example (Decay at an angle): A particle with mass M and energy E decays into two identical particles. In the lab frame, they are emitted at angles $90°$ and θ, as shown in Fig. 11.8. What are the energies of the created particles?

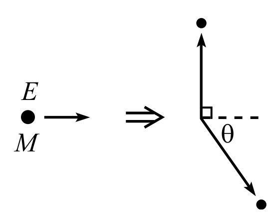

Figure 11.8

We'll give two solutions. The second one shows how 4-momenta can be used in a very clever and time-saving way.

4-momenta of various particles), this invariance is simply the statement in eq. (11.20). For the inner product of two different 4-momenta, we'll prove the invariance in Section 12.3.

First Solution: The 4-momentum before the decay is

$$P = (E, p, 0, 0), \tag{11.31}$$

where $p = \sqrt{E^2 - M^2}$. Let the created particles have mass m. The 4-momenta after the collision are

$$P_1 = (E_1, 0, p_1, 0), \qquad P_2 = (E_2, p_2 \cos\theta, -p_2 \sin\theta, 0). \tag{11.32}$$

Conservation of p_x immediately gives $p_2 \cos\theta = p$, which then implies that $p_2 \sin\theta = p \tan\theta$. Conservation of p_y says that the final p_y's are opposites. Therefore, the 4-momenta after the collision are

$$P_1 = (E_1, 0, p \tan\theta, 0), \qquad P_2 = (E_2, p, -p \tan\theta, 0). \tag{11.33}$$

Conservation of energy gives $E = E_1 + E_2$. Writing this in terms of the momenta and masses gives

$$E = \sqrt{p^2 \tan^2\theta + m^2} + \sqrt{p^2(1 + \tan^2\theta) + m^2}. \tag{11.34}$$

Putting the first radical on the left side, squaring, and solving for that radical (which is E_1) gives

$$E_1 = \frac{E^2 - p^2}{2E} = \frac{M^2}{2E}. \tag{11.35}$$

In a similar manner, we find that E_2 equals

$$E_2 = \frac{E^2 + p^2}{2E} = \frac{2E^2 - M^2}{2E}. \tag{11.36}$$

These add up to E, as they should.

Second Solution: With the 4-momenta defined as in eqs. (11.31) and (11.32), conservation of energy and momentum can be combined into the statement, $P = P_1 + P_2$. Therefore,

$$\begin{aligned} P - P_1 &= P_2, \\ \Longrightarrow \quad (P - P_1) \cdot (P - P_1) &= P_2 \cdot P_2, \\ \Longrightarrow \quad P^2 - 2P \cdot P_1 + P_1^2 &= P_2^2, \\ \Longrightarrow \quad M^2 - 2EE_1 + m^2 &= m^2, \\ \Longrightarrow \quad E_1 &= \frac{M^2}{2E}. \end{aligned} \tag{11.37}$$

And then $E_2 = E - E_1 = (2E^2 - M^2)/2E$.

This solution should convince you that 4-momenta can save you a lot of work. What happened here was that the expression for P_2 was fairly messy, but we arranged things so that it only appeared in the form of P_2^2, which is simply m^2. 4-momenta provide a remarkably organized method for sweeping unwanted garbage under the rug.

11.4 Particle-physics units

A branch of physics that uses relativity as one of its main ingredient is Elementary-Particle Physics, which is the study of the building blocks of matter (electrons, quarks, neutrinos, etc.). It is unfortunately the case that most of the elementary particles we want to study don't exist naturally in the world. We therefore have to create them in particle accelerators by colliding other particles together at very high energies. The high speeds involved require the use of relativistic dynamics. Newtonian physics is essentially useless.

What is a typical size of a rest energy, mc^2, of an elementary particle? The rest energy of a proton (which isn't really elementary; it's made up of quarks, but never mind) is

$$E_{\rm p} = m_{\rm p}c^2 = (1.67\cdot 10^{-27}\,{\rm kg})(3\cdot 10^8\,{\rm m/s})^2 = 1.5\cdot 10^{-10}\text{ joules.} \tag{11.38}$$

This is very small, of course. So a joule is probably not the best unit to work with. We would get very tired of writing the negative exponents over and over.

We could perhaps work with "nanojoules", but particle-physicists like to work instead with the "eV", the *electron-volt*. This is the amount of energy gained by an electron when it passes through a potential of one volt. The electron charge is $e = 1.6022\cdot 10^{-19}\,{\rm C}$, and a volt is defined as $1\,{\rm V} = 1\,{\rm J/C}$. So the conversion from eV to joules is[7]

$$1\text{ eV} = (1.6022\cdot 10^{-19}\,{\rm C})(1\,{\rm J/C}) = 1.6022\cdot 10^{-19}\,{\rm J}. \tag{11.39}$$

Therefore, in terms of eV, the rest-energy of a proton is $938\cdot 10^6\,$eV. We now have the opposite problem of having a large exponent hanging around. But this is easily remedied by the prefix "M", which stands for "mega", or "million". So we finally have a proton rest energy of

$$E_{\rm p} = 938\text{ MeV}. \tag{11.40}$$

You can work out for yourself that the electron has a rest-energy of $E_{\rm e} = 0.511$ MeV. The rest energies of various particles are listed in the table below. The ones preceded by a "$\approx$" are the averages of differently charged particles, whose energies differ by a few MeV. These (and the many other) elementary particles have specific properties (spin, charge, etc.), but for the present purposes they need only be thought of as point objects having a definite mass.

[7] This is getting a little picky, but "eV" should actually be written as "eV", because "eV" stands for two things that are multiplied together (in contrast with, for example, the "kg" symbol for "kilogram"), one of which is the electron charge, which is usually denoted by e.

particle	rest energy (MeV)
electron (e)	0.511
muon (μ)	105.7
tau (τ)	1784
proton (p)	938.3
neutron (n)	939.6
lambda (Λ)	1115.6
sigma (Σ)	$\approx$ 1193
delta (Δ)	$\approx$ 1232
pion (π)	$\approx$ 137
kaon (K)	$\approx$ 496

We now come to a slight abuse of language. When particle-physicists talk about masses, they say things like, "The mass of a proton is 938 MeV." This, of course, makes no sense, because the units are wrong; a mass can't equal an energy. But what they mean is that if you take this energy and divide it by c^2, then you get the mass. It would truly be a pain to keep saying, "The mass is such-and-such an energy, divided by c^2." For a quick conversion back to kilograms, you can show that

$$1 \text{ MeV}/c^2 = 1.783 \cdot 10^{-30} \text{ kg}. \tag{11.41}$$

11.5 Force

11.5.1 Force in one dimension

"Force" is a fairly intuitive concept. It is how hard you push or pull on something. We were told long ago that $\mathbf{F}$ equals $m\mathbf{a}$, and this makes sense. If you push an object in a certain direction, then it accelerates in that direction. But, alas, we've now outgrown the $\mathbf{F} = m\mathbf{a}$ definition. It's time to look at things a different way.

The force on an object is hereby *defined* to be the rate of change of momentum (we'll just deal with one-dimensional motion for now),

$$F = \frac{dp}{dt}\,. \tag{11.42}$$

This is actually the definition in nonrelativistic physics too, but in that case, where $p = mv$, we obtain $F = ma$ anyway. So it doesn't matter if we define F to be dp/dt or ma. But in the relativistic case, it does matter, because $p = \gamma mv$, and γ can change with time. This will complicate things, and it will turn out that F does *not* equal ma. Why do we define F to be dp/dt instead of ma? One reason is given in the first remark below. Another arises from the general 4-vector formalism in Chapter 12.

To see what form the F in eq. (11.42) takes in terms of the acceleration, a, note that

$$\frac{d\gamma}{dt} \equiv \frac{d}{dt}\left(\frac{1}{\sqrt{1-v^2}}\right) = \frac{v\dot{v}}{(1-v^2)^{3/2}} \equiv \gamma^3 va. \tag{11.43}$$

Therefore, assuming that m is constant, we have

$$\begin{aligned} F = \frac{d(\gamma m v)}{dt} &= m(\dot{\gamma}v + \gamma\dot{v}) \\ &= ma\gamma(\gamma^2 v^2 + 1) \\ &= \gamma^3 ma. \end{aligned} \tag{11.44}$$

This doesn't look as nice as $F = ma$, but that's the way it goes.

> They *said*, "F is ma, bar none."
> What they *meant* wasn't quite as much fun.
> It's dp by dt,
> Which just happens to be
> Good ol' "ma" when γ is 1.

Consider now the quantity dE/dx, where E is the energy, $E = \gamma m$. We have

$$\begin{aligned} \frac{dE}{dx} = \frac{d(\gamma m)}{dx} &= m\frac{d(1/\sqrt{1-v^2})}{dx} \\ &= \gamma^3 mv\frac{dv}{dx}\,. \end{aligned} \tag{11.45}$$

But $v(dv/dx) = dv/dt \equiv a$. Therefore, $dE/dx = \gamma^3 ma$, and eq. (11.44) gives

$$F = \frac{dE}{dx}\,. \tag{11.46}$$

Note that eqs. (11.42) and (11.46) take exactly the same form as in the nonrelativistic case. The only new thing in the relativistic case is that the expressions for p and E are modified.

REMARKS:

1. Eq. (11.42) is devoid of any physical content, because all it does is define F. If F were instead defined through eq. (11.46), then eq. (11.46) would be devoid of any content. The whole point of this section, and the only thing of any substance, is that (with the definitions $p = \gamma mv$ and $E = \gamma m$)

$$\frac{dp}{dt} = \frac{dE}{dx}\,. \tag{11.47}$$

This is the physically meaningful statement. If we then want to label both sides of the equation with the letter F for "force," so be it. But "force" is simply a name.

2. The result in eq. (11.46) suggests another way to arrive at the $E = \gamma m$ relation. The reasoning is exactly the same as in the nonrelativistic derivation of energy conservation in Section 4.1. Define F, as we have done, through eq. (11.42). Then integrate eq. (11.44) from x_1 to x_2 to obtain

$$\int_{x_1}^{x_2} F\,dx \quad = \quad \int_{x_1}^{x_2} (\gamma^3 ma)\,dx$$

$$
\begin{aligned}
&= \int_{x_1}^{x_2} \left(\gamma^3 mv \frac{dv}{dx}\right) dx \\
&= \int_{v_1}^{v_2} \gamma^3 mv\, dv \\
&= \gamma m \Big|_{v_1}^{v_2}, \qquad (11.48)
\end{aligned}
$$

where we have used eq. (11.46). If we then define the "potential energy" as

$$
V(x) \equiv -\int_{x_0}^{x} F(x)\, dx, \qquad (11.49)
$$

where x_0 is an arbitrary reference point, we obtain

$$
V(x_1) + \gamma m\Big|_{v_1} = V(x_2) + \gamma m\Big|_{v_2}. \qquad (11.50)
$$

We see that the quantity $V + \gamma m$ is independent of x. It is therefore worthy of a name, and we use the name "energy" due to the similarity with the Newtonian result.[8]

The work-energy theorem (that is, $\int F\, dx = \Delta E$) holds in relativistic physics, just as it does in the nonrelativistic case. The only difference is that E is γm instead of $mv^2/2$. ♣

11.5.2 Force in two dimensions

In two dimensions, the concept of force becomes a little strange. In particular, as we will see, the acceleration of an object need not point in the same direction as the force. We start with the definition,

$$
\mathbf{F} = \frac{d\mathbf{p}}{dt}. \qquad (11.51)
$$

This is a vector equation. Without loss of generality, let us deal with only two spatial dimensions. Consider a particle moving in the x-direction, and let us apply a force, $\mathbf{F} = (F_x, F_y)$. The particle's momentum is

$$
\mathbf{p} = \frac{m(v_x, v_y)}{\sqrt{1 - v_x^2 - v_y^2}}. \qquad (11.52)
$$

Taking the derivative of this, and using the fact that v_y is initially zero, we obtain

$$
\begin{aligned}
\mathbf{F} &= \frac{d\mathbf{p}}{dt}\Big|_{v_y=0} \\
&= m\left(\frac{\dot{v}_x}{\sqrt{1-v^2}} + \frac{v_x(v_x\dot{v}_x + v_y\dot{v}_y)}{(\sqrt{1-v^2})^3}, \frac{\dot{v}_y}{\sqrt{1-v^2}} + \frac{v_y(v_x\dot{v}_x + v_y\dot{v}_y)}{(\sqrt{1-v^2})^3}\right)\Bigg|_{v_y=0}
\end{aligned}
$$

[8] Actually, this derivation only suggests that E is given by γm up to an additive constant. For all we know, E might take the form, $E = \gamma m - m$, which would make the energy of a motionless particle equal to zero. An argument along the lines of Section 11.1.2 is required to show that the additive constant is zero.

$$
\begin{aligned}
&= m\left(\frac{\dot{v}_x}{\sqrt{1-v^2}}\left(1+\frac{v^2}{1-v^2}\right), \frac{\dot{v}_y}{\sqrt{1-v^2}}\right) \\
&= m\left(\frac{\dot{v}_x}{(\sqrt{1-v^2})^3}, \frac{\dot{v}_y}{\sqrt{1-v^2}}\right) \\
&\equiv m(\gamma^3 a_x, \gamma a_y). \qquad (11.53)
\end{aligned}
$$

We see that this is *not* proportional to (a_x, a_y). The first component agrees with eq. (11.44), but the second component has only one factor of γ. The difference comes from the fact that γ has a first-order change if v_x changes, but not if v_y changes, assuming that v_y is initially zero. The particle therefore responds differently to forces in the x- and y-directions. It is easier to accelerate something in the transverse direction.

11.5.3 Transformation of forces

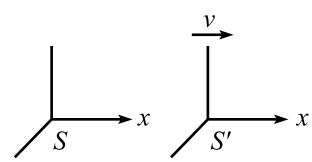

Figure 11.9

Let a force act on a particle. How are the components of the force in the particle's frame, S', related to the components of the force in another frame, S?[9] Let the relative motion be along the x- and x'-axes, as in Fig. 11.9. In frame S, eq. (11.53) says

$$(F_x, F_y) = m(\gamma^3 a_x, \gamma a_y). \qquad (11.54)$$

And in frame S', the γ factor for the particle equals 1, so eq. (11.53) reduces to the usual expression,

$$(F'_x, F'_y) = m(a'_x, a'_y). \qquad (11.55)$$

Let's now try to relate these two forces, by writing the primed accelerations on the right-hand side of eq. (11.55) in terms of the unprimed accelerations.

First, we have $a'_y = \gamma^2 a_y$. This is true because transverse distances are the same in the two frames, but times are shorter in S' by a factor γ. That is, $dt' = dt/\gamma$. We have indeed put the γ in the right place here, because the particle is essentially at rest in S', so the usual time dilation holds. Therefore, $a'_y \equiv d^2y'/dt'^2 = d^2y/(dt/\gamma)^2 \equiv \gamma^2 a_y$.

Second, we have $a'_x = \gamma^3 a_x$. In short, this is true because time dilation brings in two factors of γ (as in the a_y case), and length contraction brings in one. In a little more detail: Let the particle move from one point to another in frame S', as it accelerates from rest in S'. Mark these two points, which are a distance $a'_x(dt')^2/2$ apart, in S'. As S' flies past S, the distance between the two marks will be length contracted by a factor γ, as viewed by S. This distance (which is the excess distance the particle has over what it would have had if there were no acceleration) is what S calls $a_x(dt)^2/2$. Therefore,

$$\frac{1}{2}a_x\,dt^2 = \frac{1}{\gamma}\left(\frac{1}{2}a'_x\,dt'^2\right) \qquad \Longrightarrow \qquad a'_x = \gamma a_x\left(\frac{dt}{dt'}\right)^2 = \gamma^3 a_x. \qquad (11.56)$$

[9]To be more precise, S' is the instantaneous inertial frame of the particle. Once the force is applied, the particle's frame will no longer be S'. But for a very small elapsed time, the particle will still essentially be in S'.

Eq. (11.55) may now be written as

$$(F'_x, F'_y) = m(\gamma^3 a_x, \gamma^2 a_y). \tag{11.57}$$

Finally, comparing eqs. (11.54) and (11.57), we find

$$F_x = F'_x, \qquad \text{and} \qquad F_y = \frac{F'_y}{\gamma}. \tag{11.58}$$

We see that the longitudinal force is the same in the two frames, but the transverse force is larger by a factor of γ in the particle's frame.

REMARKS:

1. What if someone comes along and relabels the primed and unprimed frames in eq. (11.58), and concludes that the transverse force is *smaller* in the particle's frame? He certainly can't be correct, given that eq. (11.58) is true, but where is the error?

   The error lies in the fact that we (correctly) used $dt' = dt/\gamma$ above, because this is the relevant expression concerning two events along the particle's worldline. We are interested in two such events, because we want to see how the particle moves. The inverted expression, $dt = dt'/\gamma$, deals with two events located at the same position in S, and therefore has nothing to do with the situation at hand. Similar reasoning holds for the relation between dx and dx'. There is indeed one frame here that is special among all the possible frames, namely the particle's instantaneous inertial frame.

2. If you want to compare forces in two frames, neither of which is the particle's rest frame, then just use eq. (11.58) twice and relate each of the forces to the rest-frame forces. It quickly follows that for another frame S'', we have $F''_x = F_x$, and $\gamma'' F''_y = \gamma F_y$, where the γ's are measured relative to the rest fame, S'. ♣

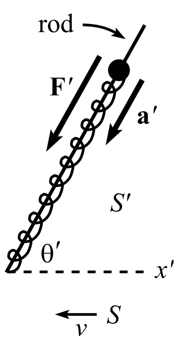

Figure 11.10

Example (Bead on a rod): A spring with a tension has one end attached to the end of a rod, and the other end attached to a bead which is constrained to move along the rod. The rod makes an angle θ' with respect to the x'-axis, and is fixed at rest in the S' frame (see Fig. 11.10). The bead is released and is pulled along the rod.

When the bead is released, what does the situation look like in the frame, S, of someone moving to the left at speed v? In answering this, draw the directions of

(a) the rod,

(b) the acceleration of the bead, and

(c) the force on the bead.

In frame S, does the wire exert a force of constraint?

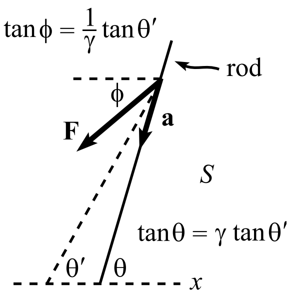

Figure 11.11

Solution: In frame S:

(a) The horizontal span of the rod is decreased by a factor γ, due to length contraction, and the vertical span is unchanged, so we have $\tan\theta = \gamma\tan\theta'$, as shown in Fig. 11.11.

(b) The acceleration must point along the rod, because the bead always lies on the rod. Quantitatively, the position of the bead in frame S takes the form of $(x, y) = (vt - a_x t^2/2, -a_y t^2/2)$, by the definition of acceleration. The position relative to the starting point on the rod, which has coordinates $(vt, 0)$, is then $(\Delta x, \Delta y) = (-a_x t^2/2, -a_y t^2/2)$. The condition for the bead to stay on the rod is that the ratio of these coordinates be equal the slope of the rod in Frame S. Therefore, $a_y/a_x = \tan\theta$, so the acceleration points along the rod.

(c) The y-component of the force on the bead is decreased by a factor γ, by eq. (11.58), so we have $\tan\phi = (1/\gamma)\tan\theta'$, as shown in the figure.

As a double-check that **a** does indeed point along the rod, we can use eq. (11.53) to write $a_y/a_x = \gamma^2 F_y/F_x$. Then eq. (11.58) gives $a_y/a_x = \gamma F'_y/F'_x = \gamma\tan\theta' = \tan\theta$, which is the direction of the wire.

The wire does *not* exert a force of constraint. The bead need not touch the wire in S', so it need not touch it in S. Basically, there is no need to have an extra force to combine with **F** to make the result point along **a**, because **F** simply does not have to be collinear with **a**.

11.6 Rocket motion

Up to this point, we have dealt with situations where the masses of our particles are constant, or where they change abruptly (as in a decay, where the sum of the masses of the products is less than the mass of the initial particle). But in many problems, the mass of an object changes continuously. A rocket is the classic example of this type of situation. Hence, we will use the term “rocket motion” to describe the general class of problems where the mass changes continuously.

The relativistic rocket itself encompasses all of the important ideas, so let’s study that example here. Many more examples are left for the problems. We’ll present three solutions to the rocket problem, the last of which is rather slick. In the end, the solutions are all basically the same, but it should be helpful to see the various ways of looking at the problem.

Example (Relativistic rocket): Assume that a rocket propels itself by continually converting mass into photons and firing them out the back. Let m be the instantaneous mass of the rocket, and let v be the instantaneous speed with respect to the ground. Show that

$$\frac{dm}{m} + \frac{dv}{1 - v^2} = 0. \tag{11.59}$$

If the initial mass is M, and the initial v is zero, integrate eq.(11.59) to obtain

$$m = M\sqrt{\frac{1 - v}{1 + v}}\,. \tag{11.60}$$

First solution: The strategy of this solution will be to use conservation of momentum in the ground frame.

Consider the effect of a small mass being converted into photons. The mass of the rocket goes from m to $m+dm$ (where dm is negative). So in the frame of the rocket, photons with total energy $E_r = -dm$ (which is positive) are fired out the back. In the frame of the rocket, these photons have momentum $p_r = dm$ (which is negative). Let the rocket move with speed v with respect to the ground. Then the momentum of the photons in the ground frame, p_g, may be found via the Lorentz transformation,

$$p_g = \gamma(p_r + vE_r) = \gamma\big(dm + v(-dm)\big) = \gamma(1-v)\,dm. \qquad (11.61)$$

This is still negative, of course.

REMARK: A common error is to say that the converted mass $(-dm)$ takes the form of photons of energy $(-dm)$ in the ground frame. This is incorrect, because although the photons have energy $(-dm)$ in the rocket frame, they are redshifted (due to the Doppler effect) in the ground frame. From eq. (10.48), we see that the frequency (and hence the energy) of the photons decreases by a factor of $\sqrt{(1-v)/(1+v)}$ when going from the rocket frame to the ground frame. This factor equals the $\gamma(1-v)$ factor in eq. (11.61). ♣

We may now use conservation of momentum in the ground frame to say that

$$(m\gamma v)_{\text{old}} = \gamma(1-v)\,dm + (m\gamma v)_{\text{new}} \qquad \Longrightarrow \qquad \gamma(1-v)\,dm + d(m\gamma v) = 0. \qquad (11.62)$$

The $d(m\gamma v)$ term may be expanded to give

$$\begin{aligned} d(m\gamma v) &= (dm)\gamma v + m(d\gamma)v + m\gamma(dv) \\ &= \gamma v\,dm + m(\gamma^3 v\,dv)v + m\gamma\,dv \\ &= \gamma v\,dm + m\gamma(\gamma^2v^2+1)\,dv \\ &= \gamma v\,dm + m\gamma^3\,dv. \end{aligned} \qquad (11.63)$$

Therefore, eq. (11.62) gives

$$\begin{aligned} 0 &= \gamma(1-v)\,dm + \gamma v\,dm + m\gamma^3 dv \\ &= \gamma\,dm + m\gamma^3\,dv. \end{aligned} \qquad (11.64)$$

Hence,

$$\frac{dm}{m} + \frac{dv}{1-v^2} = 0, \qquad (11.65)$$

in agreement with eq. (11.59). We must now integrate this. With the given initial values, we have

$$\int_M^m \frac{dm}{m} + \int_0^v \frac{dv}{1-v^2} = 0. \qquad (11.66)$$

We could simply look up the dv integral in a table, but let's do it from scratch.[10] Writing $1/(1-v^2)$ as the sum of two fractions gives

$$\begin{aligned} \int_0^v \frac{dv}{1-v^2} &= \frac{1}{2}\int_0^v \left(\frac{1}{1+v} + \frac{1}{1-v}\right) dv \\ &= \frac{1}{2}\Big(\ln(1+v) - \ln(1-v)\Big)\Big|_0^v \\ &= \frac{1}{2}\ln\left(\frac{1+v}{1-v}\right). \end{aligned} \qquad (11.67)$$

[10]Tables often list the integral of $1/(1-v^2)$ as $\tanh^{-1}(v)$, which you can show is equivalent to the result in eq. (11.67).

Eq. (11.66) therefore gives

$$\begin{aligned} \ln\left(\frac{m}{M}\right) &= -\frac{1}{2}\ln\left(\frac{1+v}{1-v}\right) \\ \Longrightarrow \qquad m &= M\sqrt{\frac{1-v}{1+v}}\,, \end{aligned} \tag{11.68}$$

in agreement with eq. (11.60). This result is independent of the rate at which the mass is converted into photons. It is also independent of the frequency of the emitted photons. Only the total mass expelled matters.

Note that eq. (11.68) quickly tells us that the energy of the rocket, as a function of velocity, is

$$E = \gamma m = \gamma M\sqrt{\frac{1-v}{1+v}} = \frac{M}{1+v}\,. \tag{11.69}$$

This has the interesting property of approaching $M/2$ as $v \to c$. In other words, half of the initial energy remains with the rocket, and half ends up as photons (see Exercise 18).

Remark: From eq. (11.61), or from the previous remark, we see that the ratio of the energy of the photons in the ground frame to that in the rocket frame is $\sqrt{(1-v)/(1+v)}$. This factor is the same as the factor in eq. (11.68). In other words, the photons' energy decreases in exactly the same manner as the mass of the rocket (assuming that the photons are ejected with the same frequency in the rocket frame throughout the process). Therefore, in the ground frame, the ratio of the photons' energy to the mass of the rocket doesn't change with time. There must be a nice intuitive explanation for this, but it eludes me. ♣

Second solution: The strategy of this solution will be to use $F = dp/dt$ in the ground frame.

Let τ denote the time in the rocket frame. Then in the rocket frame, $dm/d\tau$ is the rate at which the mass of the rocket decreases and is converted into photons (dm is negative). The photons therefore acquire momentum at the rate $dp/d\tau = dm/d\tau$ in the rocket frame. Since force is the rate of change of momentum, we see that a force of $dm/d\tau$ pushes the photons backward, and an equal and opposite force of $F = -dm/d\tau$ pushes the rocket forward in the rocket frame.

Now go to the ground frame. We know from eq. (11.58) that the longitudinal force is the same in both frames, so $F = -dm/d\tau$ is also the force on the rocket in the ground frame. And since $t = \gamma\tau$, where t is the time on the ground (the photon emissions occur at the same place in the rocket frame, so we have indeed put the time-dilation factor of γ in the right place), we have

$$F = -\gamma\frac{dm}{dt}\,. \tag{11.70}$$

Remark: We can also calculate the force on the rocket by working entirely in the ground frame. Consider a mass $(-dm)$ that is converted into photons. Initially, this mass is traveling along with the rocket, so it has momentum $(-dm)\gamma v$. After it is converted into photons, it has momentum $\gamma(1-v)\,dm$ (from the first solution above). The change in momentum is therefore $\gamma(1-v)\,dm - (-dm)\gamma v = \gamma\,dm$. Since force is the rate of change of momentum, a force of $\gamma\,dm/dt$ pushes the photons backwards, and an equal and opposite force of $F = -\gamma\,dm/dt$ therefore pushes the rocket forwards. ♣

Now things get a little tricky. It is tempting to write down $F = dp/dt = d(m\gamma v)/dt = (dm/dt)\gamma v + m\, d(\gamma v)/dt$. This, however, is not correct, because the dm/dt term is not relevant here. When the force is applied to the rocket at an instant when the rocket has mass m, the only thing the force cares about is that the mass of the rocket at the given instant is m. It doesn't care that m is changing.[11] Therefore, the correct expression we want is

$$F = m\frac{d(\gamma v)}{dt} . \tag{11.71}$$

As in the first solution above, or in eq. (11.44), we have $d(\gamma v)/dt = \gamma^3\, dv/dt$. Using the F from eq. (11.70), we arrive at

$$-\gamma\frac{dm}{dt} = m\gamma^3\frac{dv}{dt} , \tag{11.72}$$

which is equivalent to eq. (11.64). The solution proceeds as above.

Third solution: The strategy of this solution will be to use conservation of energy and momentum in the ground frame, in a slick way.

Consider a clump of photons fired out the back. The energy and momentum of these photons are equal in magnitude and opposite in sign (with the convention that the photons are fired in the negative direction). By conservation of energy and momentum, the same statement must be true about the changes in energy and momentum of the rocket. That is,

$$d(\gamma m) = -d(\gamma m v) \qquad \Longrightarrow \qquad d(\gamma m + \gamma m v) = 0. \tag{11.73}$$

Therefore, $\gamma m(1 + v)$ is a constant. We are given that $m = M$ when $v = 0$. Hence, the constant must be M. Therefore,

$$\gamma m(1 + v) = M \qquad \Longrightarrow \qquad m = M\sqrt{\frac{1-v}{1+v}} . \tag{11.74}$$

Now, *that's* a quick solution, if there ever was one!

11.7 Relativistic strings

Consider a "massless" string with a tension that is constant (that is, independent of length).[12] We will call such objects *relativistic strings*, and we will study them for two reasons. First, these strings, or reasonable approximations thereof, actually do occur in nature. For example, the gluon force which holds quarks together is approximately constant over distance. And second, they open the door to a whole new supply of problems we can solve, like the following one.

[11] Said in a different way, the momentum associated with the missing mass still exists. It's just that it's not part of the rocket anymore. This issue is expanded on in Appendix E.

[12] By "massless," we mean that the string has no mass in its unstretched (that is, zero-length) state. Once it is stretched, it will have energy, and hence mass.

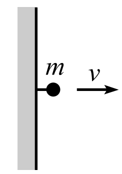

Figure 11.12

Example (Mass connected to a wall): A mass m is connected to a wall by a relativistic string with tension T. The mass starts next to the wall and has initial speed v away from it (see Fig. 11.12). What is the maximum distance from the wall the mass achieves? How much time does it take to reach this point?

Solution: Let ℓ be the maximum distance from the wall. The initial energy of the mass is $E = \gamma m$. The final energy at $x = \ell$ is simply m, because the mass is instantaneously at rest there. Integrating $F = dE/dx$, and using the fact that the force always equals $-T$, gives

$$F\Delta x = \Delta E \qquad \Longrightarrow \qquad (-T)\ell = m - \gamma m \qquad \Longrightarrow \qquad \ell = \frac{m(\gamma - 1)}{T}\,. \tag{11.75}$$

Let t be the time it takes to reach this point. The initial momentum of the mass is $p = \gamma mv$. Integrating $F = dp/dt$, and using the fact that the force always equals $-T$, gives

$$F\Delta t = \Delta p \qquad \Longrightarrow \qquad (-T)t = 0 - \gamma mv \qquad \Longrightarrow \qquad t = \frac{\gamma mv}{T}\,. \tag{11.76}$$

Note that we *cannot* use $F = ma$ to do this problem. F does not equal ma. It equals dp/dt (and also dE/dx).

Relativistic strings may seem a bit strange, but there is nothing more to solving a one-dimensional problem than the two equations,

$$F = \frac{dp}{dt}\,, \qquad \text{and} \qquad F = \frac{dE}{dx}\,. \tag{11.77}$$

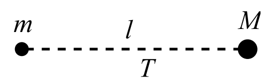

Figure 11.13

Example (Where the masses meet): A relativistic string of length ℓ and tension T connects a mass m and a mass M (see Fig. 11.13). The masses are released from rest. Where do they meet?

Solution: Let the masses meet at a distance x from the initial position of m. At this meeting point, $F = dE/dx$ tells us that the energy of m is $m + Tx$, and the energy of M is $M + T(\ell - x)$. Using $p = \sqrt{E^2 - m^2}$ we see that the magnitudes of the momenta at the meeting point are

$$p_m = \sqrt{(m + Tx)^2 - m^2} \qquad \text{and} \qquad p_M = \sqrt{(M + T(\ell - x))^2 - M^2}\,. \tag{11.78}$$

But $F = dp/dt$ then tells us that these must be equal, because the same force (in magnitude, but opposite in direction) acts on the two masses for the same time. Equating the above p's gives

$$x = \frac{\ell\big(T(\ell/2) + M\big)}{M + m + T\ell}\,. \tag{11.79}$$

This is reassuring, because the answer is simply the location of the initial center of mass, with the string being treated (quite correctly) like a stick of length ℓ and mass $T\ell$ (divided by c^2).

REMARK: Let's check a few limits. In the limit of large T or ℓ (more precisely, in the limit $T\ell \gg Mc^2$ and $T\ell \gg mc^2$), we have $x = \ell/2$. This makes sense, because in this case the masses are negligible and therefore both move at essentially speed c, and hence meet in the middle. In the limit of small T or ℓ (more precisely, in the limit $T\ell \ll Mc^2$ and $T\ell \ll mc^2$), we have $x = M\ell/(M+m)$, which is simply the Newtonian result for an everyday-strength spring. ♣

11.8 Mass

Some treatments of relativity refer to the mass of a motionless particle as the "rest-mass" m_0, and the mass of moving particle as the "relativistic mass" $m_{\text{rel}} = \gamma m_0$. This terminology is misleading and should be avoided. There is no such thing as "relativistic mass." There is only one "mass" associated with an object. This mass is what the above treatments would call the "rest mass."[13] And since there is only one type of mass, there is no need to use the qualifier "rest" or the subscript "0." We therefore simply use the notation "m." In this section, we will explain why "relativistic mass" is not a good concept to use.[14]

Why might someone want to call $m_{\text{rel}} \equiv \gamma m$ the mass of a moving particle? The basic reason is that the momentum takes the nice Newtonian form of $\mathbf{p} = m_{\text{rel}}\mathbf{v}$. The tacit assumption here is that the goal is to assign a mass to the particle such that all the Newtonian expressions continue to hold, with the only change being a modified mass. That is, we want our particle to act in exactly the same way that a particle of mass γm would, according to our everyday intuition.[15]

If we insist on hanging onto our Newtonian rules, let's see what they imply. If we want our particle to act as a mass γm does, then we must have $\mathbf{F} = (\gamma m)\mathbf{a}$. However, we saw in Section 11.5.2 that although this equation is true for transverse forces, it is *not* true for longitudinal forces. The γm would have to be replaced by $\gamma^3 m$ for a longitudinal force. As far as acceleration goes, a mass reacts differently to forces that point in different directions. We therefore see that it is impossible to assign a unique mass to a moving particle, such that it behaves in a Newtonian way under all circumstances. Not only is the goal of thinking of things in a Newtonian way ill-advised, it is doomed to failure.

[13] For example, the mass of an electron is $9.11 \cdot 10^{-31}$ kg, and the mass of a liter of water is 1 kg, independent of the speed.

[14] Of course, you can *define* the quantity γm with any name you want. You can call it "relativistic mass," or you can call it "pumpkin pie." The point is that the connotations associated with these definitions will mislead you into thinking certain things are true when they are not. The quantity γm does *not* behave as you might want a mass to behave (as we will show). And it also doesn't make for a good dessert.

[15] This goal should send up a red flag. It is similar to trying to think about quantum mechanics in terms of classical mechanics. It simply cannot be done. All analogies will eventually break down and lead to incorrect conclusions. It is quite silly to try to think about a (more) correct theory (relativity or quantum mechanics) in terms of an incorrect theory (classical mechanics), simply because our intuition (which is limited and incorrect) is based on the latter.

> "Force is my a times my 'mass',"
> Said the driver, when starting to pass.
> But from what we've just learned,
> He was right when he turned,
> But wrong when he stepped on the gas.

The above argument closes the case on this subject, but there are a few other arguments that show why it is not good to think of γm as a mass.

The word "mass" is used to describe what is on the right-hand side of the equation, $E^2 - |\mathbf{p}|^2 = m^2$. The m^2 here is an *invariant*, that is, it is something that is independent of the frame of reference. E and the components of $\mathbf{p}$, on the other hand, are components of a 4-vector. They depend on the frame. If "mass" is to be used in this definite way to describe an invariant, then it doesn't make sense to also use it to describe the quantity γm, which is frame-dependent. And besides, there is certainly no need to give γm another name. It already goes by the name "E," up to factors of c.

It is often claimed that γm is the "mass" that appears in the expression for gravitational force. If this were true, then it might be reasonable to use "mass" as a label for the quantity γm. But, in fact, it is not true. The gravitational force depends in a somewhat complicated way on the motion of the particle. For example, the force depends on whether the particle is moving longitudinally or transversely to the source. We cannot demonstrate this fact here, but suffice it to say that if one insists on using the naive force law, $F = Gm_1m_2/r^2$, then it is impossible to label the particle with a unique mass.

11.9 Exercises

Section 11.2: Transformations of E and $\vec{p}$

1. **Energy of two masses** $*$

   Two masses M move at speed V, one to the east and one to the west. What is the total energy of the system?

   Now consider the setup as viewed from a frame moving to the west at speed u. Find the energy of each mass in this frame. Is the total energy larger or smaller than the total energy in the lab frame?

2. **System of particles** $*$

   Given p_{total} and E_{total} for a system of particles, use a Lorentz transformation to find the velocity of the CM. More precisely, find the speed of the frame in which the total momentum is zero.

3. **CM frame** $**$

   A mass m travels at speed $3c/5$, and another mass m sits at rest.

   (a) Find the energy and momentum of the two particles in the lab frame.
   (b) Find the speed of the CM of the system, by using a velocity-addition argument.
   (c) Find the energy and momentum of the two particles in the CM frame.
   (d) Verify that the E's and p's are related by the relevant Lorentz transformations.
   (e) Verify that $E_{\text{total}}^2 - p_{\text{total}}^2$ is the same in both frames.

4. **Transformation for 2-D motion** $**$

   A particle has velocity (u'_x, u'_y) in frame S', which travels at speed v in the x-direction relative to frame S. Use the velocity addition formulas in Section 10.3.3 (eqs. (10.33) and (10.35)) to show that E and p_x transform according to eq. (11.18), and also that $p_y = p'_y$.

   Hint: This gets a bit messy, but the main thing you need to show is

   $$\gamma_u = \gamma_{u'}\gamma_v(1 + u'_x v), \quad \text{where} \quad u = \sqrt{u_x^2 + u_y^2} \quad \text{and} \quad u' = \sqrt{u_x'^2 + u_y'^2} \tag{11.80}$$

   are the speeds in the two frames.

Section 11.3: Collisions and decays

5. **Photon, mass collision** $*$

   A photon with energy E collides with a stationary mass m. The combine to form one particle. What is the mass of this particle? What is its speed?

6. **A decay** $*$

   A mass M decays into a mass m and a photon. If the speed of m is v, find m and also the energy of the photon (in terms of M and v).

7. **Three photons** $*$

   A mass m travels with speed v. It decays into three photons, one of which travels in the forward direction, and the other two of which move at angles of 120° (in the lab frame) as shown in Fig. 11.14. What are the energies of these three photons?

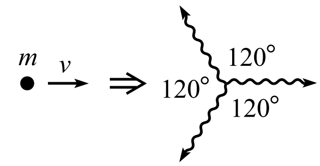

Figure 11.14

8. **Perpendicular photon** $*$

   A photon with energy E collides with a mass M. The mass M scatters off at an angle. If the resulting photon moves perpendicularly to the incident photon's direction, as shown in Fig. 11.15, what is its energy?

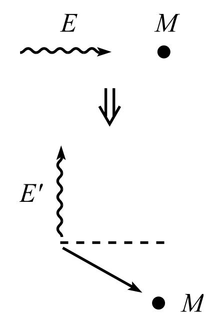

Figure 11.15

9. **Another perpendicular photon** $**$

   A mass m moving with speed $4c/5$ collides with another mass m at rest. The collision produces a photon with energy E traveling perpendicularly to the original direction, and a mass M traveling in another direction, as shown in Fig. 11.16. In terms of E and m, what is M? What is the largest value of E (in terms of m) for which this setup is possible?

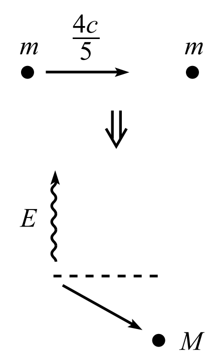

Figure 11.16

10. **Colliding diagonally** $*$

    A mass m moving northeastward at speed $4c/5$ collides with a photon moving southeastward. The result of the collision is one particle of mass M moving eastward, as shown in Fig. 11.17 Find the energy of the photon, the mass M, and the speed of M. (Give the first two of these answers in terms of m.)

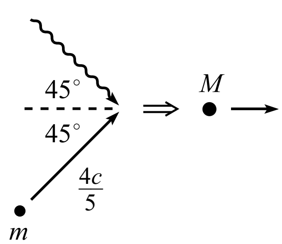

Figure 11.17

11. **Decay into photons** $*$

    A mass M traveling at $3c/5$ decays into a mass $M/4$ and two photons. One photon moves perpendicularly to the original direction, the other photon moves off at an angle θ, and the mass $M/4$ is at rest, as shown in Fig. 11.18. What is θ?

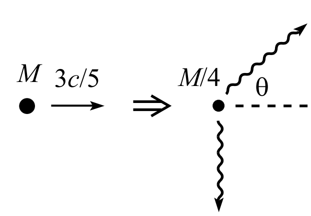

Figure 11.18

12. **Three masses colliding** $*$

    Three masses m, all traveling at speed $v = 4c/5$, collide at the origin and produce a particle of mass M. The three original velocities are in the northeast, north, and northwest directions. Find M and its velocity.

13. **Maximum mass** $*$

    A photon and a mass m move in opposite directions. They collide head-on and create a new particle. If the total energy of the system is E, how should it be divided between the photon and the mass m, so that the mass of the resulting particle is as large as possible?

Section 11.4: Particle-physics units

14. **Pion-muon race** $*$

A pion and a muon each have energy 10 GeV. They have a 100 m race. By how much distance does the muon win?

Section 11.5: Force

15. **Force and a collision** $*$

Two identical masses m are at rest, a distance x apart. A constant force F accelerates one of them towards the other until they collide and stick together. How much time does this take? What is the mass of the resulting particle?

16. **Pushing on a mass** $**$

A mass m starts at rest. You push on it with a constant force F.

(a) How much time, t, does it take to move the mass a distance x? (Both t and x here are measured in the lab frame.)

(b) After a very long time, the speed of m will approach the speed of light. Therefore, after a very long time, m will remain (approximately) a constant distance (as measured in the lab frame) behind a photon that was emitted at $t = 0$ from the starting position of m. Show that this distance equals mc^2/F.

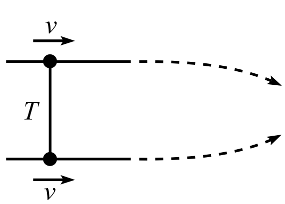

Figure 11.19

17. **Momentum paradox** $***$

Two equal masses are connected by a massless string with tension T. The masses are constrained to move with speed v along parallel lines, as shown in Fig. 11.19. The constraints are then removed, and the masses are drawn together. They collide and make one blob which continues to move to the right. Is the following reasoning correct? If your answer is "no", state what is invalid about whichever of the four sentences is/are invalid.

"The forces on the masses point in the y-direction. Therefore, there is no change in momentum in the x-direction. But the mass of the resulting blob is greater than the sum of the initial masses (because they collided with some relative speed). Therefore, the speed of the resulting blob must be less than v (to keep p_x constant), so the whole apparatus slows down in the x-direction."

Section 11.6: Rocket motion

18. **Rocket energy** $**$

As mentioned at the end of the first solution to the rocket problem in Section 11.6, the energy of the rocket in the ground frame equals $M/(1 + v)$. Derive this result again, by integrating up the amount of energy that the photons have in the ground frame.

Section 11.7: Relativistic strings

19. **Two masses** $*$

    A mass m is placed right in front of an identical one. They are connected by a relativistic string with tension T. The front one suddenly acquires a speed of $3c/5$. How far from the starting point will the masses collide with each other?

11.10 Problems

Section 11.1: Energy and momentum

1. **Deriving E and p** $**$

   Accepting the facts that the energy and momentum of a photon are $E = h\nu$ and $p = h\nu/c$ (where ν is the frequency of the light wave, and h is Planck's constant), derive the relativistic formulas for the energy and momentum of a massive particle, $E = \gamma mc^2$ and $p = \gamma mv$. *Hint:* Consider a mass m that decays into two photons. Look at this decay both in the rest frame of the mass, and in a frame where the mass has speed v. You'll need to use the Doppler effect.

Section 11.3: Collisions and decays

2. **Colliding photons**

   Two photons each have energy E. They collide at an angle θ and create a particle of mass M. What is M?

3. **Increase in mass**

   A large mass M, moving with speed V, collides and sticks to a small mass m, initially at rest. What is the mass of the resulting object? Work in the approximation where $M \gg m$.

4. **Compton scattering** $**$

   A photon collides with a stationary electron. If the photon scatters at an angle θ (see Fig. 11.20), show that the resulting wavelength, λ', is given in terms of the original wavelength, λ, by

   $$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta), \tag{11.81}$$

   where m is the mass of the electron. Note: The energy of a photon is $E = h\nu = hc/\lambda$.

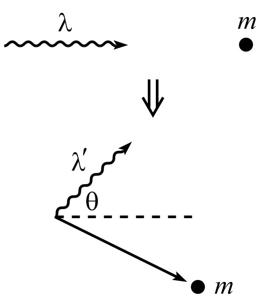

Figure 11.20

5. **Bouncing backwards** $**$

   A ball of mass M and energy E collides head-on elastically with a stationary ball of mass m. Show that the final energy of mass M is

   $$E' = \frac{2mM^2 + E(m^2 + M^2)}{m^2 + M^2 + 2Em}\,. \tag{11.82}$$

   Hint: This problem is a little messy, but you can save yourself a lot of trouble by noting that $E' = E$ must be a root of an equation you get for E'. (Why?)

6. **Two-body decay** $*$

   A mass M_A decays into masses M_B and M_C. What are the energies of M_B and M_C? What are their momenta?

7. **Threshold energy** $*$

   A particle of mass m and energy E collides with an identical stationary particle. What is the threshold energy for a final state containing N particles of mass m? ("Threshold energy" is the minimum energy for which the process occurs.)

Section 11.5: Force

8. **Relativistic harmonic oscillator** $**$

   A particle of mass m moves along the x-axis under a force $F = -m\omega^2 x$. The amplitude is b. Show that the period is given by

$$T = \frac{4}{c}\int_0^b \frac{\gamma}{\sqrt{\gamma^2 - 1}}\, dx, \tag{11.83}$$

   where

$$\gamma = 1 + \frac{\omega^2}{2c^2}(b^2 - x^2). \tag{11.84}$$

9. **System of masses** $**$

   Consider a dumbbell made of two equal masses, m. The dumbbell spins around, with its center pivoted at the end of a stick (see Fig. 11.21). If the speed of the masses is v, then the energy of the system is $2\gamma m$. Treated as a whole, the system is at rest. Therefore, the mass of the system must be $2\gamma m$. (Imagine enclosing it in a box, so that you can't see what is going on inside.)

   Convince yourself that the system does indeed behave like a mass of $M = 2\gamma m$, by pushing on the stick (when the dumbbell is in the "transverse" position shown in the figure) and showing that $F \equiv dp/dt = Ma$.

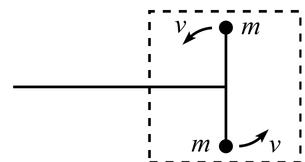

Figure 11.21

Section 11.6: Rocket motion

10. **Relativistic rocket** $**$

    Consider the relativistic rocket from Section 11.6. Let mass be converted to photons at a rate σ in the rest frame of the rocket. Find the time, t, in the ground frame as a function of v.[16] (Alas, it is not possible to invert this, to get v as a function of t.)

11. **Relativistic dustpan I** $*$

    A dustpan of mass M is given an initial relativistic speed. It gathers up dust with mass density λ per unit length on the floor (as measured in the lab frame). At the instant the speed is v, find the rate (as measured in the lab frame) at which the mass of the dustpan-plus-dust-inside system is increasing.

[16] This involves a slightly tricky integral. Pick your favorite method – pencil, book, or computer.

12. **Relativistic dustpan II** $**$

Consider the setup in Problem 11. If the initial speed of the dustpan is V, what are $v(x)$, $v(t)$, and $x(t)$? All quantities here are measured with respect to the lab frame.

13. **Relativistic dustpan III** $**$

Consider the setup in Problem 11. Calculate, in both the dustpan frame and lab frame, the force on the dustpan-plus-dust-inside system (due to the newly acquired dust particles smashing into it) as a function of v, and show that the results are equal.

14. **Relativistic cart I** $****$

A long cart moves at relativistic speed v. Sand is dropped into the cart at a rate $dm/dt = \sigma$ in the ground frame. Assume that you stand on the ground next to where the sand falls in, and you push on the cart to keep it moving at constant speed v. What is the force between your feet and the ground? Calculate this force in both the ground frame (your frame) and the cart frame, and show that the results are equal.

15. **Relativistic cart II** $****$

A long cart moves at relativistic speed v. Sand is dropped into the cart at a rate $dm/dt = \sigma$ in the ground frame. Assume that you grab the front of the cart and pull on it to keep it moving at constant speed v (while running with it). What force does your hand apply to the cart? (Assume that the cart is made of the most rigid material possible.) Calculate this force in both the ground frame and the cart frame (your frame), and show that the results are equal.

Section 11.6: Relativistic strings

16. **Different frames** $**$

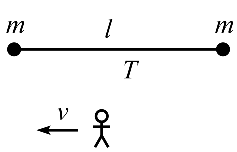

Figure 11.22

(a) Two masses m are connected by a string of length ℓ and constant tension T. The masses are released simultaneously. They collide and stick together. What is the mass, M, of the resulting blob?

(b) Consider this scenario from the point of view of a frame moving to the left with speed v (see Fig. 11.22). The energy of the resulting blob must be γMc^2, from part (a). Show that you obtain this same result by computing the work done on the two masses.

17. **Splitting mass** $**$

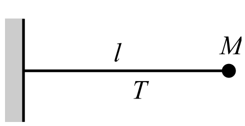

Figure 11.23

A massless string with constant tension T has one end attached to a wall and the other end attached to a mass M. The initial length of the string is ℓ (see Fig. 11.23). The mass is released. Halfway to the wall, the back half of the mass breaks away from the front half (with zero initial relative speed). What is the total time it takes the front half to reach the wall?

18. **Relativistic leaky bucket** ***

The mass M in Problem 17 is replaced by a massless bucket containing an initial mass M of sand (see Fig. 11.24). On the way to the wall, the bucket leaks sand at a rate $dm/dx = M/\ell$, where m denotes the mass at later positions. (Note that the rate is constant with respect to distance, not time.)

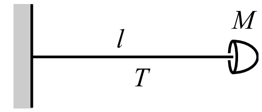

Figure 11.24

(a) What is the energy of the bucket, as a function of distance from the wall? What is its maximum value? What is the maximum value of the kinetic energy?

(b) What is the momentum of the bucket, as a function of distance from the wall? Where is it maximum?

19. **Relativistic bucket** ***

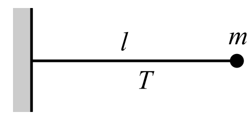

Figure 11.25

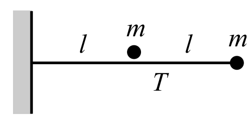

Figure 11.26

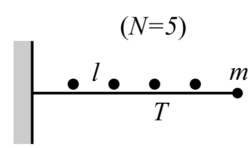

Figure 11.27

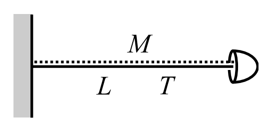

Figure 11.28

(a) A massless string with constant tension T has one end attached to a wall and the other end attached to a mass m. The initial length of the string is ℓ (see Fig. 11.25). The mass is released. How long does it take to reach the wall?

(b) Let the string now have length 2ℓ, with a mass m on the end. Let another mass m be positioned next to the ℓ mark on the string, but not touching the string (see Fig. 11.26). The right mass is released. It heads toward the wall (while the other mass is still motionless), and then sticks to the other mass to make one large blob, which then heads toward the wall.[17] How much time does this whole process take?[18]

(c) Let there now be N masses and a string of length $N\ell$ (see Fig. 11.27). How much time does this whole process take?

(d) Consider now a massless bucket at the end of the string (of length L) which gathers up a continuous stream of sand (of total mass M), as it gets pulled to the wall (see Fig. 11.28). How much time does this whole process take? What is the mass of the contents of the bucket right before it hits the wall?

[17] The left mass could actually be attached to the string, and we would still have the same situation. The mass wouldn't move during the first part of the process, because there would be equal tensions T on both sides of it.

[18] You can do this in various ways, but one method that generalizes nicely for the next part is to show that $\Delta(p^2) = (E_2^2 - E_1^2) + (E_4^2 - E_3^2)$, where the energies of the moving object (that is, the initial m or the resulting blob) are: E_1 right at the start, E_2 just before the collision, E_3 just after the collision, and E_4 right before the wall. Note that this method does not require knowledge of the mass of the blob (which is *not* $2m$).

11.11 Solutions

1. **Deriving E and p**

   We'll derive the energy formula, $E = \gamma mc^2$, first. Let the given mass decay into two photons, and let E_0 be the energy of the mass in its rest frame. Then each of the resulting photons has energy $E_0/2$ in this frame.

   Now look at the decay in a frame where the mass moves at speed v. From eq. (10.48), the frequencies of the photons are Doppler-shifted by the factors $\sqrt{(1+v)/(1-v)}$ and $\sqrt{(1-v)/(1+v)}$. Since the photons' energies are given by $E = h\nu$, their energies are shifted by these same factors, relative to the $E_0/2$ value in the original frame. Conservation of energy then says that in the moving frame, the mass (which is moving at speed v) has energy

$$E = \frac{E_0}{2}\sqrt{\frac{1+v}{1-v}} + \frac{E_0}{2}\sqrt{\frac{1-v}{1+v}} = \gamma E_0. \tag{11.85}$$

   We therefore see that a moving mass has an energy that is γ times its rest energy.

   We will now use the correspondence principle (which says that relativistic formulas must reduce to the familiar nonrelativistic ones, in the nonrelativistic limit) to find E_0 in terms of m. We just found that the difference between the energies of a moving mass and a stationary mass is $\gamma E_0 - E_0$. This must reduce to the familiar kinetic energy, $mv^2/2$, in the limit $v \ll c$. In other words,

$$\begin{aligned}
\frac{mv^2}{2} &\approx \frac{E_0}{\sqrt{1-v^2/c^2}} - E_0 \\
&\approx E_0\left(1 + \frac{v^2}{2c^2}\right) - E_0 \\
&= \left(\frac{E_0}{c^2}\right)\frac{v^2}{2},
\end{aligned} \tag{11.86}$$

   where we have used the Taylor series, $1/\sqrt{1-\epsilon} \approx 1 + \epsilon/2$. Therefore $E_0 = mc^2$, and hence $E = \gamma mc^2$.

   We can derive the momentum formula, $p = \gamma mv$, in a similar way. Let the magnitude of the photons' (equal and opposite) momenta in the particle's rest frame be $p_0/2$.[19] Using the Doppler-shifted frequencies as above, we see that the total momentum of the photons in the frame where the mass moves at speed v is

$$p = \frac{p_0}{2}\sqrt{\frac{1+v}{1-v}} - \frac{p_0}{2}\sqrt{\frac{1-v}{1+v}} = \gamma p_0 v. \tag{11.87}$$

   Putting the c's back in, we have $p = \gamma p_0 v/c$. By conservation of momentum, this is the momentum of the mass m moving at speed v.

   We can now use the correspondence principle to find p_0 in terms of m. If $p = \gamma(p_0/c)v$ is to reduce to the familiar $p = mv$ result in the limit $v \ll c$, then we must have $p_0 = mc$. Therefore, $p = \gamma mv$.

2. **Colliding photons**

   The 4-momenta of the photons are (see Fig. 11.29)

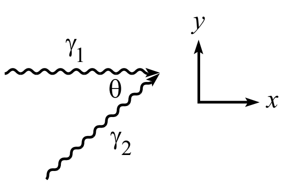

Figure 11.29

[19]With the given information that a photon has $E = h\nu$ and $p = h\nu/c$, we can use the preceding $E_0 = mc^2$ result to quickly conclude that $p_0 = mc$. But let's pretend that we haven't found E_0 yet. This will give us an excuse to use the correspondence principle again.

$$P_{\gamma_1} = (E, E, 0, 0), \qquad \text{and} \qquad P_{\gamma_2} = (E, E\cos\theta, E\sin\theta, 0). \tag{11.88}$$

Energy and momentum are conserved, so the 4-momentum of the final particle is $P_M = (2E, E + E\cos\theta, E\sin\theta, 0)$. Hence,

$$M^2 = P_M \cdot P_M = (2E)^2 - (E + E\cos\theta)^2 - (E\sin\theta)^2. \tag{11.89}$$

Therefore, the desired mass is

$$M = E\sqrt{2(1-\cos\theta)}\,. \tag{11.90}$$

If $\theta = 180°$ then $M = 2E$, as it should (none of the final energy is kinetic). And if $\theta = 0°$ then $M = 0$, as it should (all of the final energy is kinetic; we simply have a photon with twice the energy).

3. **Increase in mass**

In the lab frame, the energy of the resulting object is $\gamma M + m$, and the momentum is still γMV. The mass of the object is therefore

$$M' = \sqrt{(\gamma M + m)^2 - (\gamma MV)^2} = \sqrt{M^2 + 2\gamma Mm + m^2}\,. \tag{11.91}$$

The m^2 term is negligible compared to the other two terms, so we may approximate M' as

$$M' \approx M\sqrt{1 + \frac{2\gamma m}{M}} \approx M\left(1 + \frac{\gamma m}{M}\right) = M + \gamma m, \tag{11.92}$$

where we have used the Taylor series, $\sqrt{1+\epsilon} \approx 1 + \epsilon/2$. Therefore, the increase in mass is γ times the mass of the stationary object. (This increase must be greater than the nonrelativistic answer of "m", because heat is generated during the collision, and this heat shows up as mass in the final object.)

Remark: The γm result is quite clear if we work in the frame where M is initially at rest. In this frame, the mass m comes flying in with energy γm, and essentially all of this energy shows up as mass in the final object. That is, essentially none of it shows up as overall kinetic energy of the object.

This is a general result. Stationary large objects pick up negligible kinetic energy when hit by small objects. This is true because the speed of the large object is proportional to m/M, by momentum conservation (there's a factor of γ if things are relativistic), so the kinetic energy goes like $Mv^2 \propto M(m/M)^2 \approx 0$, if $M \gg m$. In other words, the smallness of v wins out over the largeness of M. When a snowball hits a tree, all of the initial energy goes into heat to melt the snowball; (essentially) none of it goes into changing the kinetic energy of the earth. ♣

4. **Compton scattering**

The 4-momenta before the collision are (see Fig. 11.30)

$$P_\gamma = \left(\frac{hc}{\lambda}, \frac{hc}{\lambda}, 0, 0\right), \qquad P_m = (mc^2, 0, 0, 0). \tag{11.93}$$

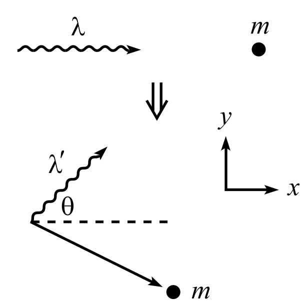

Figure 11.30

The 4-momenta after the collision are

$$P'_\gamma = \left(\frac{hc}{\lambda'}, \frac{hc}{\lambda'}\cos\theta, \frac{hc}{\lambda'}\sin\theta, 0\right), \qquad P'_m = (\text{we won't need this}). \tag{11.94}$$

If we wanted to, we could write P'_m in terms of its momentum and scattering angle. But the nice thing about the following method is that we don't need to introduce these quantities which we're not interested in.

Conservation of energy and momentum give $P_\gamma + P_m = P'_\gamma + P'_m$. Therefore,

$$\begin{aligned} (P_\gamma + P_m - P'_\gamma)^2 &= P'^2_m \\ \Longrightarrow P^2_\gamma + P^2_m + P'^2_\gamma + 2P_m(P_\gamma - P'_\gamma) - 2P_\gamma P'_\gamma &= P'^2_m \\ \Longrightarrow 0 + m^2c^4 + 0 + 2mc^2\left(\frac{hc}{\lambda} - \frac{hc}{\lambda'}\right) - 2\frac{hc}{\lambda}\frac{hc}{\lambda'}(1-\cos\theta) &= m^2c^4. \end{aligned} \quad (11.95)$$

Multiplying through by $\lambda\lambda'/(hmc^3)$ gives the desired result,

$$\lambda' = \lambda + \frac{h}{mc}(1-\cos\theta). \quad (11.96)$$

The nice thing about this solution is that all the unknown garbage in P'_m disappeared when we squared it.

If $\theta \approx 0$ (that is, not much scattering), then $\lambda' \approx \lambda$, as expected.

If $\theta = \pi$ (that is, backward scattering) and additionally $\lambda \ll h/mc$ (that is, $mc^2 \ll hc/\lambda = E_\gamma$, so the photon's energy is much larger than the electron's rest energy), then $\lambda' = 2h/mc$, so

$$E'_\gamma = \frac{hc}{\lambda'} \approx \frac{hc}{\frac{2h}{mc}} = \frac{1}{2}mc^2. \quad (11.97)$$

Therefore, the photon bounces back with an essentially fixed E'_γ, independent of the initial E_γ (as long as E_γ is large enough). This isn't all that obvious.

5. **Bouncing backwards**

The 4-momenta before the collision are

$$P_M = (E, p, 0, 0), \qquad P_m = (m, 0, 0, 0), \quad (11.98)$$

where $p = \sqrt{E^2 - M^2}$. The 4-momenta after the collision are

$$P'_M = (E', p', 0, 0), \qquad P'_m = (\text{we won't need this}), \quad (11.99)$$

where $p' = \sqrt{E'^2 - M^2}$. If we wanted to, we could write P'_m in terms of its momentum. But we don't need to introduce it. Conservation of energy and momentum give $P_M + P_m = P'_M + P'_m$. Therefore,

$$\begin{aligned} & (P_M + P_m - P'_M)^2 = P'^2_m \\ \Longrightarrow\quad & P^2_M + P^2_m + P'^2_M + 2P_m(P_M - P'_M) - 2P_M P'_M = P'^2_m \\ \Longrightarrow\quad & M^2 + m^2 + M^2 + 2m(E - E') - 2(EE' - pp') = m^2 \\ \Longrightarrow\quad & M^2 - EE' + m(E - E') = pp' \\ \Longrightarrow\quad & \Big((M^2 - EE') + m(E - E')\Big)^2 = \Big(\sqrt{E^2 - M^2}\sqrt{E'^2 - M^2}\Big)^2 \\ \Longrightarrow\quad & M^2(E^2 - 2EE' + E'^2) + 2(M^2 - EE')m(E - E') + m^2(E - E')^2 = 0. \end{aligned} \quad (11.100)$$

As claimed, $E' = E$ is a root of this equation. This is true because $E' = E$ and $p' = p$ certainly satisfy conservation of energy and momentum with the initial conditions, by definition. Dividing through by $(E - E')$ gives

$$M^2(E - E') + 2m(M^2 - EE') + m^2(E - E') = 0. \quad (11.101)$$

Solving for E' gives the desired result,

$$E' = \frac{2mM^2 + E(m^2 + M^2)}{m^2 + M^2 + 2Em}\,. \tag{11.102}$$

We can double-check a few limits:

(a) $E \approx M$ (barely moving): then $E' \approx M$, because M is still barely moving.

(b) $m \gg E$ (brick wall): then $E' \approx E$, because the heavy mass m picks up essentially no energy.

(c) $M \gg m$: then $E' \approx E$, because it's essentially like m is not there. Actually, this only holds if E isn't too big; more precisely, we need $Em \ll M^2$.

(d) $M = m$: then $E' = M$, because M stops and m picks up all the energy that M had.

(e) $E \gg m \gg M$: then $E' \approx m/2$. This isn't obvious, but it's similar to an analogous limit in the Compton scattering in Problem 4.

6. **Two-body decay**

B and C have equal and opposite momenta. Therefore,

$$E_B^2 - M_B^2 = p^2 = E_C^2 - M_C^2. \tag{11.103}$$

Also, conservation of energy gives

$$E_B + E_C = M_A. \tag{11.104}$$

Solving the two previous equations for E_B and E_C gives (using the shorthand $a \equiv M_A$, etc.)

$$E_B = \frac{a^2 + b^2 - c^2}{2a}, \qquad \text{and} \qquad E_C = \frac{a^2 + c^2 - b^2}{2a}. \tag{11.105}$$

Eq. (11.103) then gives the momentum of the particles as

$$p = \frac{\sqrt{a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2}}{2a}\,. \tag{11.106}$$

Remark: It turns out that the quantity under the radical may be factored into

$$(a + b + c)(a + b - c)(a - b + c)(a - b - c). \tag{11.107}$$

This makes it clear that if $a = b + c$, then $p = 0$, because there is no leftover energy for the particles to be able to move. ♣

7. **Threshold energy**

The initial 4-momenta are

$$(E, p, 0, 0), \qquad \text{and} \qquad (m, 0, 0, 0), \tag{11.108}$$

where $p = \sqrt{E^2 - m^2}$. Therefore, the final 4-momentum is $(E + m, p, 0, 0)$. The quantity $(E + m)^2 - p^2$ is an invariant, and it equals the square of the energy in the CM frame. At threshold, there is no relative motion among the final N particles (because there is no leftover energy for such motion; see the remark below). So the energy in the CM frame is simply the sum of the rest energies, or Nm. We therefore have

$$(E + m)^2 - (E^2 - m^2) = (Nm)^2 \qquad \Longrightarrow \qquad E = \left(\frac{N^2}{2} - 1\right) m. \tag{11.109}$$

Note that $E \propto N^2$, for large N.

REMARK: Let's justify rigorously that the final particles should travel as a blob (that is, with no relative motion). Using the invariance of $E^2 - p^2$, and the fact that $p^{\rm CM} = 0$, we have

$$\begin{aligned} \left(E_{\rm f}^{\rm lab}\right)^2 - \left(p_{\rm f}^{\rm lab}\right)^2 &= \left(E_{\rm f}^{\rm CM}\right)^2 - \left(p_{\rm f}^{\rm CM}\right)^2 \\ \Longrightarrow \quad (E+m)^2 - \left(\sqrt{E^2 - m^2}\right)^2 &= \left(E_{\rm f}^{\rm CM}\right)^2 - 0 \\ \Longrightarrow \quad 2Em + 2m^2 &= \left(E_{\rm f}^{\rm CM}\right)^2. \end{aligned} \tag{11.110}$$

Therefore, minimizing E is equivalent to minimizing $E_{\rm f}^{\rm CM}$. But $E_{\rm f}^{\rm CM}$ is clearly minimized when all the final particles are at rest in the CM frame (so there is no kinetic energy added to the rest energy). The minimum E is therefore achieved when there is no relative motion among the final particles in the CM frame, and hence in any other frame. ♣

8. **Relativistic harmonic oscillator**

$F = dp/dt$ gives $-m\omega^2 x = d(m\gamma v)/dt$. Using eq. (11.44), we have

$$-\omega^2 x = \gamma^3 \frac{dv}{dt}\,. \tag{11.111}$$

We must somehow solve this differential equation. A helpful thing to do is to multiply both sides by v to obtain $-\omega^2 x\dot{x} = \gamma^3 v\dot{v}$. The right-hand side of this is simply $d\gamma/dt$, as you can check. Integration then gives $-\omega^2 x^2/2 + C = \gamma$, where C is a constant of integration. We know that $\gamma = 1$ when $x = b$, so we find

$$\gamma = 1 + \frac{\omega^2}{2c^2}(b^2 - x^2), \tag{11.112}$$

where we have put the c's back in to make the units right.

The period is given by

$$T = 4\int_0^b \frac{dx}{v}\,. \tag{11.113}$$

But $\gamma \equiv 1/\sqrt{1 - v^2/c^2}$, and so $v = c\sqrt{\gamma^2 - 1}/\gamma$. Therefore,

$$T = \frac{4}{c}\int_0^b \frac{\gamma}{\sqrt{\gamma^2 - 1}}\,dx. \tag{11.114}$$

REMARK: In the limit $\omega b \ll c$ (so that $\gamma \approx 1$, from eq. (11.112); that is, the speed is always small), we must recover the Newtonian limit. Indeed, to lowest nontrivial order, $\gamma^2 \approx 1 + (\omega^2/c^2)(b^2 - x^2)$, and so

$$T \approx \frac{4}{c}\int_0^b \frac{dx}{(\omega/c)\sqrt{b^2 - x^2}}\,. \tag{11.115}$$

This is the correct result, because conservation of energy gives $v^2 = \omega^2(b^2 - x^2)$ for a nonrelativistic spring. ♣

9. **System of masses**

Let the speed of the stick go from 0 to ϵ, where $\epsilon \ll v$. Then the final speeds of the two masses are obtained by relativistically adding and subtracting ϵ from v. (Assume that the time involved is small, so that the masses are still essentially moving horizontally.) Repeating the derivation leading to eq. (11.17), we see that the final momenta of the two masses have magnitudes $\gamma_v\gamma_\epsilon(v \pm \epsilon)m$. But since ϵ is small, we may set $\gamma_\epsilon \approx 1$, to first order.

Therefore, the forward-moving mass now has momentum $\gamma_v(v+\epsilon)m$, and the backward-moving mass now has momentum $-\gamma_v(v-\epsilon)m$. The net increase in momentum is thus (with $\gamma_v \equiv \gamma$) $\Delta p = 2\gamma m\epsilon$. Hence,

$$F \equiv \frac{\Delta p}{\Delta t} = 2\gamma m \frac{\epsilon}{\Delta t} \equiv 2\gamma m a = Ma. \tag{11.116}$$

10. **Relativistic rocket**

The relation between m and v obtained in eq. (11.60) is independent of the rate at which mass is converted to photons. We now assume a certain rate, in order to obtain a relation between v and t.

In the frame of the rocket, we have $dm/d\tau = -\sigma$. From the usual time dilation effect, we then have $dm/dt = -\sigma/\gamma$ in the ground frame, because the ground frame sees the rocket's clocks run slow (that is, $dt = \gamma\, d\tau$).

Differentiating eq. (11.60), we have

$$dm = \frac{-M\, dv}{(1+v)\sqrt{1-v^2}}\,. \tag{11.117}$$

Using $dm = -(\sigma/\gamma)dt$, this becomes

$$\int_0^t \frac{\sigma}{M}\, dt = \int_0^v \frac{dv}{(1+v)(1-v^2)}\,. \tag{11.118}$$

We could simply use a computer to do this dv integral, but let's do it from scratch. Using a few partial-fraction tricks, we have

$$\begin{aligned}
\int \frac{dv}{(1+v)(1-v^2)} &= \int \frac{dv}{(1+v)(1-v)(1+v)} \\
&= \frac{1}{2}\int \left(\frac{1}{1+v} + \frac{1}{1-v}\right)\frac{dv}{1+v} \\
&= \frac{1}{2}\int \frac{dv}{(1+v)^2} + \frac{1}{4}\int \left(\frac{1}{1+v} + \frac{1}{1-v}\right) dv \\
&= -\frac{1}{2(1+v)} + \frac{1}{4}\ln\left(\frac{1+v}{1-v}\right)\,.
\end{aligned} \tag{11.119}$$

Equation (11.118) therefore gives

$$\frac{\sigma t}{M} = \frac{1}{2} - \frac{1}{2(1+v)} + \frac{1}{4}\ln\left(\frac{1+v}{1-v}\right)\,. \tag{11.120}$$

Remarks: If $v \ll 1$ (or rather, if $v \ll c$), then we may Taylor-expand the quantities in eq. (11.120) to obtain $\sigma t/M \approx v$. This may be rewritten as $\sigma \approx M(v/t) \equiv Ma$. But σ is simply the force acting on the rocket (or rather σc, to make the units correct), because this is the change in momentum of the photons. We therefore obtain the expected nonrelativistic $F = ma$ equation.

If $v = 1 - \epsilon$, where ϵ is very small (that is, if v is very close to c), then we can make approximations in eq. (11.120) to obtain $\epsilon \approx 2e/e^{4\sigma t/M}$. We see that the difference between v and 1 decreases exponentially with t. ♣

11. **Relativistic dustpan I**

This problem is essentially the same as Problem 3.

Let M be the mass of the dustpan-plus-dust-inside system (which we will label "S") when its speed is v. After a small time dt in the lab frame, S has moved a distance $v\,dt$, so it has basically collided with an infinitesimal mass $\lambda v\,dt$. Its energy therefore increases to $\gamma M + \lambda v\,dt$. Its momentum is still $\gamma M v$, so its mass is now

$$M' = \sqrt{(\gamma M + \lambda v\,dt)^2 - (\gamma M v)^2} \approx \sqrt{M^2 + 2\gamma M \lambda v\,dt}\,, \tag{11.121}$$

where we have dropped the second-order dt^2 terms. Using the Taylor series $\sqrt{1+\epsilon} \approx 1 + \epsilon/2$, we may approximate M' as

$$M' \approx M\sqrt{1 + \frac{2\gamma\lambda v\,dt}{M}} \approx M\left(1 + \frac{\gamma\lambda v\,dt}{M}\right) = M + \gamma\lambda v\,dt. \tag{11.122}$$

The rate of increase in S's mass is therefore $\gamma\lambda v$. (This increase must certainly be greater than the nonrelativistic answer of "λv", because heat is generated during the collision, and this heat shows up as mass in the final object.)

REMARKS: This result is quite clear if we work in the frame where S is at rest. In this frame, a mass $\lambda v\,dt$ comes flying in with energy $\gamma\lambda v\,dt$, and essentially all of this energy shows up as mass (heat) in the final object. That is, essentially none of it shows up as overall kinetic energy of the object, which is a general result when a small object hits a stationary large object.

Note that the rate at which the mass increases, as measured in S's frame, is $\gamma^2\lambda v$, due to time dilation. (The dust-entering-dustpan events happen at the same location in the dustpan frame, so we have indeed put the extra γ factor in the correct place.) Alternatively, you can view things in terms of length contraction. S sees the dust contracted, so its density is increased to $\gamma\lambda$. ♣

12. **Relativistic dustpan II**

The initial momentum is $\gamma_V MV \equiv P$. There are no external forces, so the momentum of the dustpan-plus-dust-inside system (denoted by "S") always equals P. That is, $\gamma m v = P$, where m and v are the mass and speed of S at later times.

Let's find $v(x)$ first. The energy of S, namely γm, increases due to the acquisition of new dust. Therefore, $d(\gamma m) = \lambda\,dx$, which we can write as

$$d\left(\frac{P}{v}\right) = \lambda\,dx. \tag{11.123}$$

Integrating this, and using the fact that the initial speed is V, gives $P/v - P/V = \lambda x$. Therefore,

$$v(x) = \frac{V}{1 + \frac{V\lambda x}{P}}\,. \tag{11.124}$$

Note that for large x, this approaches $P/(\lambda x)$. This makes sense, because the mass of S is essentially equal to λx, and it is moving at a slow, nonrelativistic speed.

To find $v(t)$, write the dx in eq. (11.123) as $v\,dt$ to obtain $(-P/v^2)\,dv = \lambda v\,dt$. Hence,

$$-\int_V^v \frac{P\,dv}{v^3} = \int_0^t \lambda\,dt \qquad \Longrightarrow \qquad \frac{P}{v^2} - \frac{P}{V^2} = 2\lambda t$$

$$\Longrightarrow \qquad v(t) = \frac{V}{\sqrt{1 + \frac{2\lambda V^2 t}{P}}}\,. \tag{11.125}$$

At this point, there are various ways to find $x(t)$. The simplest one is to just integrate eq. (11.125). The result is

$$x(t) = \frac{P}{V\lambda}\left(\sqrt{1+\frac{2V^2\lambda t}{P}}-1\right). \tag{11.126}$$

You can show that this reduces to $x = Vt$ for small t, as it should. For large t, x has the interesting property of being proportional to $\sqrt{t}$.

13. **Relativistic dustpan III**

Let S denote the dustpan-plus-dust-inside system at a given time, and consider a small bit of dust (call this subsystem s) that enters the dustpan. In S's frame, the density of the dust is $\gamma\lambda$, due to length contraction. Therefore, in a time $d\tau$ (where τ is the time in the dustpan frame), a little s system of dust with mass $\gamma\lambda v\, d\tau$ crashes into S and loses its (negative) momentum of $-(\gamma\lambda v\, d\tau)(\gamma v) = -\gamma^2 v^2\lambda\, d\tau$. The force on s is therefore $+\gamma^2 v^2\lambda$. The desired force on S is equal and opposite to this, so

$$F = -\gamma^2 v^2\lambda. \tag{11.127}$$

Now consider the lab frame. In a time dt (where t is the time in the lab frame), a little s system of dust with mass $\lambda v\, dt$ gets picked up by the dustpan. What is the change in momentum of s? It is tempting to say that it is $(\lambda v\, dt)(\gamma v)$, but this would lead to a force of $-\gamma v^2\lambda$ on the dustpan, which doesn't agree with the result we found above in the dustpan frame. This would be a problem, because longitudinal forces should be the same in different frames.

The key point to realize is that the mass of whatever is moving increases at a rate $\gamma\lambda v$, and not λv (see Problem 11). We therefore see that the change in momentum of the additional moving mass is $(\gamma\lambda v\, dt)(\gamma v) = \gamma^2 v^2\lambda\, dt$. The original moving system S therefore loses this much momentum, and so the force on it is $F = -\gamma^2 v^2\lambda$, in agreement with the result in the dustpan frame.

14. **Relativistic cart I**

Ground frame (your frame): Using reasoning similar to that in Problem 3 or Problem 11, we see that the mass of the cart-plus-sand-inside system increases at a rate $\gamma\sigma$. Therefore, its momentum increases at a rate

$$\frac{dP}{dt} = \gamma(\gamma\sigma)v = \gamma^2\sigma v. \tag{11.128}$$

This is the force you exert on the cart, so it is also the force the ground exerts on your feet (because the net force on you is zero).

Cart frame: The sand-entering-cart events happen at the same location in the ground frame, so time dilation says that the sand enters the cart at a slower rate in the cart frame; that is, at a rate σ/γ. The sand flies in at speed v, and then eventually comes at rest on the cart, so its momentum decreases at a rate $\gamma(\sigma/\gamma)v = \sigma v$. This is the force your hand applies to the cart.

If this were the only change in momentum in the problem, then we would be have a problem, because the force on your feet would be σv in the cart frame, whereas we found above that it is $\gamma^2\sigma v$ in the ground frame. This would contradict the fact that longitudinal forces are the same in different frames. What is the resolution to this apparent paradox?

The resolution is that while you are pushing on the cart, *your mass is decreasing*. You are moving with speed v in the cart frame, and mass is continually being transferred from you (who are moving) to the cart (which is at rest). This is the missing change in momentum we need. Let's be quantitative about this.

Go back to the ground frame for a moment. We found above that the mass the cart-plus-sand-inside system (call this "C") increases at rate $\gamma\sigma$ in the ground frame. Therefore, the energy of C increases at a rate $\gamma(\gamma\sigma)$ in the ground frame. The sand provides σ of this energy, so you must provide the remaining $(\gamma^2-1)\sigma$ part. Therefore, since you are losing energy at this rate, you must also be losing mass at this rate in the ground frame (because you are at rest there).

Now go back to the cart frame. Due to time dilation, you lose mass at a rate of only $(\gamma^2-1)\sigma/\gamma$. This mass goes from moving at speed v (that is, along with you), to speed zero (that is, at rest on the cart). Therefore, the rate of decrease in momentum of this mass is $\gamma\big((\gamma^2-1)\sigma/\gamma\big)v = (\gamma^2-1)\sigma v$.

Adding this result to the σv result due to the sand, we see that the total rate of decrease in momentum is $\gamma^2\sigma v$. This is therefore the force that the ground applies to your feet, in agreement with the calculation in the ground frame.

15. **Relativistic cart II**

Ground frame: Using reasoning similar to that in Problem 3 or Problem 11, we see that the mass of the cart-plus-sand-inside system increases at a rate $\gamma\sigma$. Therefore, its momentum increases at a rate $\gamma(\gamma\sigma)v = \gamma^2\sigma v$.

However, this is *not* the force that your hand exerts on the cart. The reason is that the sand enters the cart at locations that are receding from your hand, so your hand cannot immediately be aware of the additional need for momentum. No matter how rigid the cart is, it can't transmit information faster than c. In a sense, there is a sort of Doppler effect going on, and your hand only needs to be responsible for a certain fraction of the momentum increase. Let's be quantitative about this.

Consider two grains of sand that enter the cart a time t apart. What is the difference between the two times that your hand becomes aware that the grains have entered the cart? Assuming maximal rigidity (that is, assuming that signals propagate along the cart at speed c), then the relative speed (as measured by someone on the ground) of the signals and your hand is $c-v$. The distance between the two signals is ct. Therefore, they arrive at your hand separated by a time of $ct/(c-v)$. In other words, the rate at which you feel sand entering the cart is $(c-v)/c$ times the given σ rate. This is the factor by which we must multiply the naive $\gamma^2\sigma v$ result for the force we found above. The force you must apply is therefore

$$F = \left(1-\frac{v}{c}\right)\gamma^2\sigma v = \frac{\sigma v}{1+v}\,. \tag{11.129}$$

Cart frame (your frame): The sand-entering-cart events happen at the same location in the ground frame, so time dilation says that the sand enters the cart at a slower rate in the cart frame; that is, at a rate σ/γ. The sand flies in at speed v, and then eventually comes to rest on the cart, so its momentum decreases at a rate $\gamma(\sigma/\gamma)v = \sigma v$.

But again, this is *not* the force that your hand exerts on the cart. As before, the sand enters the cart at a location far from your hand, so your hand cannot immediately be aware of the additional need for momentum. Let's be quantitative about this.

Consider two grains of sand that enter the cart a time t apart. What is the difference between the two times that your hand becomes aware that the grains have entered the cart? Assuming maximal rigidity (that is, assuming that signals propagate along the cart at speed c), then the relative speed (as measured by someone on the cart) of the signals and your hand is c (because you are at rest). The distance between the two signals is $ct + vt$, because the sand source is moving away from you at speed v. Therefore, the signals arrive at your hand separated by a time of $(c+v)t/c$. In other words, the rate at which you feel sand entering the cart is $c/(c+v)$ times the σ/γ rate found above. This is the factor by which we must multiply the naive σv result for the force we found above. The force you must apply is therefore

$$F = \left(\frac{1}{1+v/c}\right)\sigma v = \frac{\sigma v}{1+v}\,, \tag{11.130}$$

in agreement with eq. (11.129).

In a nutshell, the two naive results in the two frames, $\gamma^2\sigma v$ and σv, differ by two factors of γ. The ratio of the two "Doppler-effect" factors (which arose from the impossibility of absolute rigidity) precisely remedies this discrepancy.

16. **Different frames**

(a) The energy of the resulting blob is $2m + T\ell$. Since the blob is at rest, we have

$$M = 2m + T\ell. \tag{11.131}$$

(b) Let the new frame be S. Let the original frame be S'. The critical point to realize is that in frame S the left mass begins to accelerate before the right mass does. This is due to the loss of simultaneity between the frames. Note that the longitudinal force is the same in the two frames, so the masses still feel a tension T in frame S.

Consider the two events when the two masses start to move. Let the left mass and right mass start moving at positions x_l and x_r in S. The Lorentz transformation $\Delta x = \gamma(\Delta x' + v\Delta t')$ tells us that $x_r - x_l = \gamma\ell$, because $\Delta x' = \ell$ and $\Delta t' = 0$ for these events.

Let the masses collide at position x_c in S. Then the gain in energy of the left mass is $T(x_c - x_l)$, and the gain in energy of the right mass is $(-T)(x_c - x_r)$ (which is negative if $x_c > x_r$). The gain in the sum of the energies is therefore

$$\Delta E = T(x_c - x_l) + (-T)(x_c - x_r) = T(x_r - x_l) = T\gamma\ell. \tag{11.132}$$

The initial sum of energies was $2\gamma m$, so the final energy is

$$E = 2\gamma m + \gamma T\ell = \gamma M, \tag{11.133}$$

as desired.

17. **Splitting mass**

We'll calculate the times for the two parts of the process to occur.

The energy of the mass right before it splits is $E_b = M + T(\ell/2)$, so the momentum is $p_b = \sqrt{E_b^2 - M^2} = \sqrt{MT\ell + T^2\ell^2/4}$. Using $F = dp/dt$, the time for the first part of the process is

$$t_1 = \frac{1}{T}\sqrt{MT\ell + T^2\ell^2/4}\,. \tag{11.134}$$

The momentum of the front half of the mass immediately after it splits is $p_a = p_b/2 = (1/2)\sqrt{MT\ell + T^2\ell^2/4}$. The energy at the wall is $E_w = E_b/2 + T(\ell/2) = M/2 + 3T\ell/4$, so the momentum at the wall is $p_w = \sqrt{E_w^2 - (M/2)^2} = (1/2)\sqrt{3MT\ell + 9T^2\ell^2/4}$. The change in momentum during the second part of the process is therefore $\Delta p = p_w - p_a = (1/2)\sqrt{3MT\ell + 9T^2\ell^2/4} - (1/2)\sqrt{MT\ell + T^2\ell^2/4}$. The time for the second part is thus

$$t_2 = \frac{1}{2T}\left(\sqrt{3MT\ell + 9T^2\ell^2/4} - \sqrt{MT\ell + T^2\ell^2/4}\right). \tag{11.135}$$

The total time is $t_1 + t_2$, which simply changes the minus sign in the above expression to a plus sign.

18. **Relativistic leaky bucket**

(a) Let the wall be at $x = 0$, and let the initial position be at $x = \ell$. Consider a small interval during which the bucket moves from x to $x + dx$ (where dx is negative). The bucket's energy changes by $(-T)\,dx$ due to the string, and it also changes by a fraction dx/x, due to the leaking. Therefore, $dE = (-T)\,dx + E\,dx/x$, or

$$\frac{dE}{dx} = -T + \frac{E}{x}\,. \tag{11.136}$$

In solving this differential equation, it is convenient to introduce the variable $y = E/x$. Then $E' = xy' + y$, where a prime denotes differentiation with respect to x. Eq. (11.136) then becomes $xy' = -T$, or

$$dy = \frac{-T\,dx}{x}\,. \tag{11.137}$$

Integration gives $y = -T\ln x + C$, which we may write as $y = -T\ln(x/\ell) + B$, in order to have a dimensionless argument in the log. Since $E = xy$, we therefore have

$$E = Bx - Tx\ln(x/\ell). \tag{11.138}$$

The reasoning up to this point is valid for both the total energy and the kinetic energy. Let's now look at each of these cases.

- *Total energy:* Eq. (11.138) gives

$$E = M(x/\ell) - Tx\ln(x/\ell), \tag{11.139}$$

where the constant of integration, B, has been chosen so that $E = M$ when $x = \ell$. To find the maximum of E, it is more convenient to work with the fraction $z \equiv x/\ell$, in terms of which $E = Mz - T\ell z\ln z$. Setting dE/dz equal to zero gives

$$\ln z = \frac{M}{T\ell} - 1 \qquad \Longrightarrow \qquad E_{\max} = \frac{T\ell}{e}\, e^{M/T\ell}. \tag{11.140}$$

The fraction z must satisfy $z \le 1$, so we must have $\ln z \le 0$. Therefore, a solution for z exists only for $M \le T\ell$. If $M \ge T\ell$, then the energy decreases all the way to the wall.

If M is slightly less then $T\ell$, then z is slightly less than 1, so E quickly achieves a maximum of slightly more than M, then decreases for the rest of the way to the wall.
If $M \ll T\ell$, then E achieves its maximum at $z \approx 1/e$, where it has the value $T\ell/e$.

- *Kinetic energy:* Eq. (11.138) gives

$$KE = -Tx\ln(x/\ell), \tag{11.141}$$

where the constant of integration, B, has been chosen so that $KE = 0$ when $x = \ell$. Equivalently, $E - KE$ must equal the mass $M(x/\ell)$. In terms of the fraction $z \equiv x/\ell$, we have $KE = -T\ell z\ln z$. Setting $d(KE)/dz$ equal to zero gives

$$z = \frac{1}{e} \qquad \Longrightarrow \qquad KE_{\text{max}} = \frac{T\ell}{e}\,, \tag{11.142}$$

which is independent of M. Since this result is independent of M, it must hold in the nonrelativistic limit. And indeed, the analogous "Leaky-bucket" problem in Chapter 4 (Problem 4.16) gave the same result.

(b) Eq. (11.139) gives, with $z \equiv x/\ell$,

$$\begin{aligned} p = \sqrt{E^2 - (Mz)^2} &= \sqrt{(Mz - T\ell z\ln z)^2 - (Mz)^2} \\ &= \sqrt{-2MT\ell z^2\ln z + T^2\ell^2 z^2\ln^2 z}\,. \end{aligned} \tag{11.143}$$

Setting the derivative equal to zero gives $T\ell\ln^2 z + (T\ell - 2M)\ln z - M = 0$. The maximum momentum therefore occurs at

$$\ln z = \frac{2M - T\ell - \sqrt{T^2\ell^2 + 4M^2}}{2T\ell}\,. \tag{11.144}$$

We have ignored the other root, because it gives $\ln z > 0$.

If $M \ll T\ell$, then the maximum p occurs at $z \approx 1/e$. In this case, the bucket immediately becomes relativistic, so we have $E \approx pc$. Therefore, both E and p should achieve their maxima at the same place. This agrees with the result for E above.

If $M \gg T\ell$, then the maximum p occurs at $z \approx 1/\sqrt{e}$. In this case, the bucket is nonrelativistic, so the result should agree with the analogous "Leaky-bucket" problem in Chapter 4 (Problem 4.16), which it does.

19. **Relativistic bucket**

(a) The mass's energy just before it hits the wall is $E = m + T\ell$. Therefore, the momentum just before it hits the wall is $p = \sqrt{E^2 - m^2} = \sqrt{2mT\ell + T^2\ell^2}$. $F = \Delta p/\Delta t$ then gives (using the fact that the tension is constant)

$$\Delta t = \frac{\Delta p}{F} = \frac{\sqrt{2mT\ell + T^2\ell^2}}{T}\,. \tag{11.145}$$

If $m \ll T\ell$, then $\Delta t \approx \ell$ (or ℓ/c in normal units), which is correct, because the mass essentially travels at speed c.

If $m \gg T\ell$, then $\Delta t \approx \sqrt{2m\ell/T}$. This is the nonrelativistic limit, and it agrees with the result obtained from the familiar $\ell = at^2/2$, where $a = T/m$ is the acceleration.

(b) *Straightforward method:* The energy of the blob right before it hits the wall is $E_w = 2m + 2T\ell$. If we can find the mass, M, of the blob, then we can use $p = \sqrt{E^2 - M^2}$ to get the momentum, and then use $\Delta t = \Delta p/F$ to get the time.[20]

[20] Note that although the tension T acts on two different things (the mass m initially, and then the blob), it is valid to use the total Δp to obtain the total time Δt via $\Delta t = \Delta p/F$, simply because we could break up the Δp into its two parts, and then find the two partial times, and then add them back together to get the total Δt.

The momentum right before the collision is $p_b = \sqrt{2mT\ell + T^2\ell^2}$, and this is also the momentum of the blob right after the collision, p_a.

The energy of the blob right after the collision is $E_a = 2m + T\ell$. So the mass of the blob after the collision is $M = \sqrt{E_a^2 - p_a^2} = \sqrt{4m^2 + 2mT\ell}$.

Therefore, the momentum at the wall is $p_w = \sqrt{E_w^2 - M^2} = \sqrt{6mT\ell + 4T^2\ell^2}$, and hence

$$\Delta t = \frac{\Delta p}{F} = \frac{\sqrt{6mT\ell + 4T^2\ell^2}}{T}. \tag{11.146}$$

Note that if $m = 0$ then $\Delta t = 2\ell$, as it should.

Better method: In the notation in the footnote in the statement of the problem, the change in p^2 from the start to just before the collision is $\Delta(p^2) = E_2^2 - E_1^2$. This is true because

$$E_1^2 - m^2 = p_1^2, \qquad \text{and} \qquad E_2^2 - m^2 = p_2^2, \tag{11.147}$$

and since m is the same throughout the first half of the process, we have $\Delta(E^2) = \Delta(p^2)$.

Likewise, the change in p^2 during the second half of the process is $\Delta(p^2) = E_4^2 - E_3^2$, because

$$E_3^2 - M^2 = p_3^2, \qquad \text{and} \qquad E_4^2 - M^2 = p_4^2, \tag{11.148}$$

and since M is the same throughout the second half of the process,[21] we have $\Delta(E^2) = \Delta(p^2)$.

The total change in p^2 is the sum of the above two changes, so the final p^2 is

$$\begin{aligned} p^2 &= (E_2^2 - E_1^2) + (E_4^2 - E_3^2) \\ &= \Big((m + T\ell)^2 - m^2\Big) + \Big((2m + 2T\ell)^2 - (2m + T\ell)^2\Big) \\ &= 6mT\ell + 4T^2\ell^2, \end{aligned} \tag{11.149}$$

as in eq. (11.146). The first solution above basically performs the same calculation, but in a more obscure manner.

(c) The reasoning in part (b) tells us that the final p^2 equals the sum of the $\Delta(E^2)$ terms over the N parts of the process. So we have, using an indexing notation analogous to that in part (b),

$$\begin{aligned} p^2 &= \sum_{k=1}^{N} \Big(E_{2k}^2 - E_{2k-1}^2\Big) \\ &= \sum \Big((km + kT\ell)^2 - \big(km + (k-1)T\ell\big)^2\Big) \\ &= \sum \Big(2kmT\ell + \big(k^2 - (k-1)^2\big)T^2\ell^2\Big) \\ &= N(N+1)mT\ell + N^2T^2\ell^2. \end{aligned} \tag{11.150}$$

Therefore,

$$\Delta t = \frac{\Delta p}{F} = \frac{\sqrt{N(N+1)mT\ell + N^2T^2\ell^2}}{T}. \tag{11.151}$$

This checks with the results from parts (a) and (b).

[21] M happens to be $\sqrt{4m^2 + 2mT\ell}$, but the nice thing about this solution is that we don't need to know this. All we need to know is that it is constant.

(d) We want to take the limit $N \to \infty$, $\ell \to 0$, $m \to 0$, with the restrictions that $N\ell = L$ and $Nm = M$. Written in terms of M and L, the result in part (c) is

$$\Delta t = \frac{\sqrt{(1+1/N)MTL + T^2L^2}}{T} \quad \longrightarrow \quad \frac{\sqrt{MTL + T^2L^2}}{T}\,, \tag{11.152}$$

as $N \to \infty$. Note that this time equals the time is takes for one particle of mass $m = M/2$ to reach the wall, from part (a).

The mass, M_f, of the final blob at the wall is

$$\begin{aligned} M_f = \sqrt{E_w^2 - p_w^2} &= \sqrt{(M+TL)^2 - (MTL + T^2L^2)} \\ &= \sqrt{M^2 + MTL}\,. \end{aligned} \tag{11.153}$$

If $TL \ll M$, then $M_f \approx M$, which makes sense. If $M \ll TL$, then $M_f \approx \sqrt{MTL}$, so M_f is the geometric mean between the given mass and the energy stored in the string, which isn't entirely obvious.