Chapter 2

Using $F = ma$

The general goal of classical mechanics is to determine what happens to a given set of objects in a given physical situation. In order to figure this out, we need to know what makes the objects move the way they do. There are two main ways of going about this task. The first way, which you are undoubtedly familiar with, involves Newton's laws. This will be the subject of the present chapter. The second way, which is the more advanced one, is the *Lagrangian* method. This will be the subject of Chapter 5.

It should be noted that each of these methods is perfectly sufficient for solving any problem. They both produce the same information in the end, but they are based on vastly different principles. We'll talk more about this is Chapter 5.

2.1 Newton's Laws

Newton published his three laws in 1687 in his *Principia Mathematica*. The laws are fairly intuitive, although it seems a bit strange to attach the adjective "intuitive" to a set of statements that took millennia for humans to write down. The laws may be stated as follows.

- **First Law:** A body moves with constant velocity (which may be zero) unless acted on by a force.
- **Second Law:** The time rate of change of the momentum of a body equals the force acting on the body.
- **Third Law:** The forces two bodies apply to each other are equal in magnitude and opposite in direction.

We could discuss for days on end the degree to which these statements are physical laws, and the degree to which they are definitions. Sir Arthur Eddington once made the unflattering comment that the first law essentially says that "every particle continues in its state of rest or uniform motion in a straight line except insofar that it doesn't." Although Newton's laws might seem somewhat vacuous at first glance, there is actually a bit more content to them than Eddington's comment implies. Let's look at each in turn. The discussion will be brief, because we have to save time for other things in this book that we really *do* want to discuss for days on end.

First Law

One thing this law does is give a definition of zero force.

Another thing it does is give a definition of an *inertial frame*, which is defined simply as a reference frame in which the first law holds. Since the term "velocity" is used, we have to state what frame of reference we are measuring the velocity with respect to. The first law does *not* hold in an arbitrary frame. For example, it fails in the frame of a spinning turntable.[1] Intuitively, an inertial frame is one that moves at constant speed. But this is ambiguous, because we have to say what the frame is moving at constant speed *with respect to.* At any rate, an inertial frame is simply defined as the special type of frame where the first law holds.

So, what we now have are two intertwined definitions of "force" and "inertial frame." Not much physical content here. But, however sparse in content the law is, it still holds for *all* particles. So if we have a frame in which one free particle moves with constant velocity, than *all* free particles move with constant velocity. This is a statement with content.

Second Law

One thing this law does is give a definition of nonzero force. Momentum is defined[2] to be $m\mathbf{v}$. If m is constant,[3] then the second law says that

$$\mathbf{F} = m\mathbf{a}, \tag{2.1}$$

where $\mathbf{a} \equiv d\mathbf{v}/dt$. This law holds only in an inertial frame, which was defined by the first law.

> For things moving free or at rest,
> Observe what the first law does best.
> It defines a key frame,
> "Inertial" by name,
> Where the second law then is expressed.

So far, the second law merely gives a definition of $\mathbf{F}$. But the meaningful statement arises when we invoke the fact that the law holds for *all* particles. If the same force (for example, the same spring stretched by the same amount) acts on two

[1] It is, however, possible to modify things so that Newton's laws hold in such a frame, but we'll save this discussion for Chapter 9.

[2] We're doing everything nonrelativistically here, of course. Chapter 11 gives the relativistic modification of the $m\mathbf{v}$ expression.

[3] We'll assume in this chapter that m is constant. But don't worry, we'll get plenty of practice with changing mass (in rockets and such) in Chapter 4.

particles, with masses m_1 and m_2, then eq. (2.1) says that their accelerations must be related by

$$\frac{a_1}{a_2} = \frac{m_2}{m_1}\,. \tag{2.2}$$

This relation holds regardless of what the common force is. Therefore, once you've used one force to find the relative masses of two objects, then you know what the ratio of their a's will be when they are subjected to any other force.

Of course, we haven't really defined *mass* yet. But eq. (2.2) gives an experimental method for determining an object's mass in terms of a standard (say, 1 kg) mass. All you have to do is compare its acceleration with that of the standard mass, when acted on by the same force.

There is also another piece of substance in this law, in that it says $\mathbf{F} = m\mathbf{a}$, instead of, say, $\mathbf{F} = m\mathbf{v}$, or $\mathbf{F} = m\, d^3\mathbf{x}/dt^3$. This issue is related to the first law. $\mathbf{F} = m\mathbf{v}$ is not viable, because the first law says that it is possible to have a velocity without a force. And $\mathbf{F} = md^3\mathbf{x}/dt^3$ would make the first law incorrect, because it would then be true that a particle moves with constant acceleration (instead of constant velocity) unless acted on by a force.

Note that $\mathbf{F} = m\mathbf{a}$ is a vector equation, so it is really three equations in one. In Cartesian coordinates, it says that $F_x = ma_x$, $F_y = ma_y$, and $F_z = ma_z$.

Third Law

This law essentially postulates that momentum is conserved (that is, not dependent on time). To see this, note that

$$\begin{aligned} \frac{d\mathbf{p}}{dt} &= \frac{d(m_1\mathbf{v}_1 + m_2\mathbf{v}_2)}{dt} \\ &= m_1\mathbf{a}_1 + m_2\mathbf{a}_2 \\ &= \mathbf{F}_1 + \mathbf{F}_2, \end{aligned} \tag{2.3}$$

where $\mathbf{F}_1$ and $\mathbf{F}_2$ are the forces acting on m_1 and m_2, respectively. This demonstrates that momentum conservation (that is, $d\mathbf{p}/dt = 0$) is equivalent to Newton's third law (that is, $\mathbf{F}_1 = -\mathbf{F}_2$.)

There isn't much left to be defined via this law, so the third law is one of pure content. It says that if you have two isolated particles interacting through some force, then their accelerations are opposite in direction and inversely proportional to their masses.

This third law cannot be a definition, because it's actually not always valid. It only holds for forces of the "pushing" and "pulling" type. It fails for the magnetic force, for example. In that case, momentum is carried off in the electromagnetic field (so the total momentum of the particles *and* the field is conserved). But we won't deal with fields here. Just particles. So the third law will always hold in any situation we're concerned with.

2.2 Free-body diagrams

The law that allows us to be quantitative is the second law. Given a force, we can apply $\mathbf{F} = m\mathbf{a}$ to find the acceleration. And knowing the acceleration, we can determine the behavior of a given object (that is, where it is and what its velocity is), provided that we are given the initial position and velocity. This process sometimes takes a bit of work, but there are two basic types of situations that commonly arise.

- In many problems, all you are given is a physical situation (for example, a block resting on a plane, strings connecting masses, etc.), and it is up to you to find all the forces acting on all the objects. These forces generally point in various directions, so it is easy to lose track of them. It therefore proves useful to isolate the objects and draw all the forces acting on each of them. This is the subject of the present section.
- In other problems, you are *given* the force explicitly as a function of time, position, or velocity, and the task immediately becomes the mathematical one of solving the $F = ma \equiv m\ddot{x}$ equation (we'll just deal with one dimension here). These *differential equations* can be difficult (or impossible) to solve exactly. They are the subject of Section 2.3.

Let's now consider the first of these two types of scenarios, where we are presented with a physical situation, and where we must determine all the forces involved. The term *free-body diagram* is used to denote a diagram with all the forces drawn on a given object. After drawing such a diagram for each object in the setup, we simply write down all the $F = ma$ equations they imply. The result will be a system of linear equations in various unknown forces and accelerations, for which we must then solve. This procedure is best understood through an example.

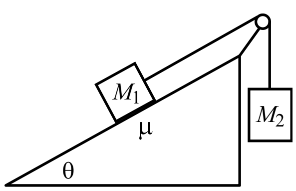


Figure 2.1

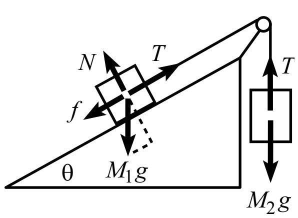


Figure 2.2

Example (A plane and masses): Mass M_1 is held on a plane with inclination angle θ, and mass M_2 hangs over the side. The two masses are connected by a massless string which runs over a massless pulley (see Fig. 2.1). The coefficient of kinetic friction between M_1 and the plane is μ. M_1 is released. Assuming that M_2 is sufficiently large so that M_1 gets pulled up the plane, what is the acceleration of the masses? What is the tension in the string?

Solution: The first thing to do is draw all the forces on the two masses. These are shown in Fig. 2.2. The forces on M_2 are gravity and the tension. The forces on M_1 are gravity, friction, the tension, and the normal force. Note that the friction force points down the plane, because we are assuming that M_1 moves up the plane.

Having drawn all the forces, we now simply have to write down all the $F = ma$ equations. When dealing with M_1, we could break things up into horizontal and vertical components, but it is much cleaner to use the components along and perpendicular to the plane.[4] These two components of $\mathbf{F} = m\mathbf{a}$, along with the vertical $F = ma$

[4]When dealing with inclined planes, one of these two coordinate systems will generally work much better than the other. Sometimes it's not clear which one, but if things get messy with one system, you can always try the other one.

equation for M_2, give

$$\begin{aligned} T - f - M_1 g\sin\theta &= M_1 a, \\ N - M_1 g\cos\theta &= 0, \\ M_2 g - T &= M_2 a, \end{aligned} \tag{2.4}$$

where we have used the fact that the two masses accelerate at the same rate (and we have defined the positive direction for M_2 to be downward). We have also used the fact that tension is the same at both ends of the string, because otherwise there would be a net force on some part of the string which would then have to undergo infinite acceleration, because it is massless.

There are four unknowns in eqs. (2.4) (namely T, a, N, and f), but only three equations. Fortunately, we have a fourth equation: $f = \mu N$. Using this in the second equation above gives $f = \mu M_1 g\cos\theta$. The first equation then becomes $T - \mu M_1 g\cos\theta - M_1 g\sin\theta = M_1 a$. Adding this to the third equation leave us with only a, so we find

$$a = \frac{g(M_2 - \mu M_1\cos\theta - M_1\sin\theta)}{M_1 + M_2}, \qquad \Longrightarrow \qquad T = \frac{M_1 M_2 g(1 + \mu\cos\theta + \sin\theta)}{M_1 + M_2}. \tag{2.5}$$

Note that in order for M_1 to move upward (that is, $a > 0$), we must have $M_2 > M_1(\mu\cos\theta + \sin\theta)$. This is clear from looking at the forces along the plane.

REMARK: If we had instead assumed that M_1 was sufficiently large so that it slides down the plane, then the friction force would point up the plane, and we would have found, as you can check,

$$a = \frac{g(M_2 + \mu M_1\cos\theta - M_1\sin\theta)}{M_1 + M_2}, \qquad \text{and} \qquad T = \frac{M_1 M_2 g}{M_1 + M_2}(1 - \mu\cos\theta + \sin\theta). \tag{2.6}$$

In order for M_1 to move downward (that is, $a < 0$), we must have $M_2 < M_1(\sin\theta - \mu\cos\theta)$. Therefore, the range of M_2 for which the system doesn't move is $M_1(\sin\theta - \mu\cos\theta) < M_2 < M_1(\sin\theta + \mu\cos\theta)$. ♣

In problems like the one above, it is clear what things you should pick as the objects on which you're going to draw forces. But in other problems, where there are various different subsystems you can choose, you must be careful to include all the relevant forces on a given subsystem. Which subsystems you want to pick depends on what quantities you're trying to find. Consider the following example.

Example (Platform and pulley): A person stands on a platform-and-pulley system, as shown in Fig. 2.3. The masses of the platform, person, and pulley[5] are M, m, and μ, respectively.[6] The rope is massless. Let the person pull up on the rope so that she has acceleration a upwards.[7]

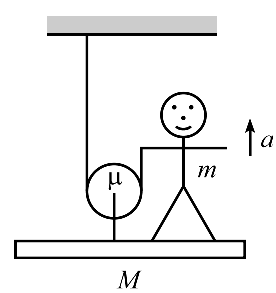


Figure 2.3

[5]Assume that the pulley's mass is concentrated at its center, so that we don't have to worry about any rotational dynamics (the subject of Chapter 7).

[6]My apologies for using μ as a mass here, since it usually denotes a coefficient of friction. Alas, there are only so many symbols for "m".

[7]Assume that the platform is somehow constrained to stay level, perhaps by having it run along some rails.

(a) What is the tension in the rope?

(b) What is the normal force between the person and the platform? What is the tension in the rod connecting the pulley to the platform?

Solution:

(a) To find the tension in the rope, we simply want to let our subsystem be the whole system (except the ceiling). If we imagine putting the system in a black box (to emphasize the fact that we don't care about any internal forces within the system), then the forces we see "protruding" from the box are the three weights (Mg, mg, and μg) downward, and the tension T upward. Applying $F = ma$ to the whole system gives

$$T-(M+m+\mu)g = (M+m+\mu)a \qquad \Longrightarrow \qquad T = (M+m+\mu)(g+a). \tag{2.7}$$

(b) To find the normal force, N, between the person and the platform, and also the tension, f, in the rod connecting the pulley to the platform, it is not sufficient to consider the system as a whole. We must consider subsystems.

- Let's apply $F = ma$ to the person. The forces acting on the person are gravity, the normal force from the platform, and the tension from the rope (pulling downward on her hand). Therefore, we have

$$N - T - mg = ma. \tag{2.8}$$

- Now apply $F = ma$ to the platform. The forces acting on the platform are gravity, the normal force from the person, and the force upward from the rod. Therefore, we have

$$f - N - Mg = Ma. \tag{2.9}$$

- Now apply $F = ma$ to the pulley. The forces acting on the pulley are gravity, the force downward from the rod, and *twice* the tension in the rope (because it pulls up on both sides). Therefore, we have

$$2T - f - \mu g = \mu a. \tag{2.10}$$

Note that if we add up the three previous equations, we obtain the $F = ma$ equation in eq. (2.7), as should be the case, because the whole system is the sum of the three above subsystems. Eqs. (2.8) – (2.10) are three equations in the three unknowns (T, N, and f). Their sum yields the T in (2.7), and then eqs. (2.8) and (2.10) give, respectively (as you can show),

$$N = (M + 2m + \mu)(g + a), \qquad \text{and} \qquad f = (2M + 2m + \mu)(g + a). \tag{2.11}$$

Remark: You can also obtain these results by considering subsystems different from the ones we chose above. For example, you can choose the pulley-plus-platform subsystem, etc. But no matter how you choose to break up the system, you will need to produce three independent $F = ma$ statements in order to solve for the three unknowns, T, N, and f.

In problems like this one, it is easy to forget to include one of the forces, such as the second T in eq. (2.10). The safest thing to do is to isolate each subsystem, draw a box around it, and then draw all the forces that "protrude" from the box. Fig. 2.4 shows the free-body diagram for the subsystem of the pulley. ♣

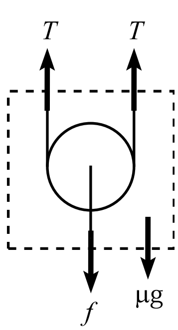


Figure 2.4

Another class of problems, similar to the previous example, goes by the name of *Atwood's machines*. An Atwood's machine is simply the name for any system that consists of a combination of masses, strings, and pulleys. In general, the pulleys and strings can have mass, but we'll just deal with massless ones in this chapter.

We'll do one example here, but additional (and stranger) setups are given in the exercises and problems for this chapter. As we'll see below, there are two basic steps in solving an Atwood's problem: (1) Write down all the $F = ma$ equations, and (2) Relate the accelerations of the various masses by noting that the length of the string doesn't change (a fact that we'll call "conservation of string").

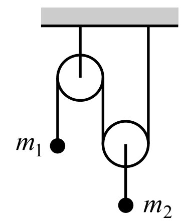


Figure 2.5

Example (An Atwood's machine): Consider the pulley system in Fig. 2.5, with masses m_1 and m_2. The strings and pulleys are massless. What are the accelerations of the masses? What is the tension in the string?

Solution: The first thing to note is that the tension, T, is the same everywhere throughout the massless string, because otherwise there would be infinite acceleration. It then follows that the tension in the short string connected to m_2 is $2T$. This is true because there must be zero net force on the massless right pulley, because otherwise it would have infinite acceleration. The $F = ma$ equations on the two masses are therefore

$$\begin{aligned} T - m_1 g &= m_1 a_1, \\ 2T - m_2 g &= m_2 a_2. \end{aligned} \tag{2.12}$$

We now have two equations in the three unknowns, a_1, a_2, and T. So we need one more equation. This is the "conservation of string" fact, which relates a_1 and a_2. If we imagine moving m_2 and the right pulley up a distance d, then a length $2d$ of string has disappeared from the two parts of the string touching the right pulley. This string has to go somewhere, so it ends up in the part of the string touching m_1. Therefore, m_1 goes down by a distance $2d$. In other words, $y_1 = -2y_2$ (where y_1 and y_2 are measured relative to the initial locations of the masses). Taking two time derivatives of this statement gives our desired relation between a_1 and a_2,

$$a_1 = -2a_2. \tag{2.13}$$

Combining this with eqs. (2.12), we can now solve for a_1, a_2, and T. The result is

$$a_1 = g\frac{2m_2 - 4m_1}{4m_1 + m_2}\,, \qquad a_2 = g\frac{2m_1 - m_2}{4m_1 + m_2}\,, \qquad T = \frac{3m_1 m_2 g}{4m_1 + m_2}\,. \tag{2.14}$$

REMARK: There are all sorts of limits and special cases that we can check here. A few are: (1) If $m_2 = 2m_1$, then eq. (2.14) gives $a_1 = a_2 = 0$, and $T = m_1 g$. Everything is at rest. (2) If $m_2 \gg m_1$, then eq. (2.14) gives $a_1 = 2g$, $a_2 = -g$, and $T = 3m_1 g$. In this case, m_2 is essentially in free fall, while m_1 gets yanked up with acceleration $2g$. The value of T is exactly what is needed to make the net force on m_1 equal to $m_1(2g)$, because $T - m_1 g = 3m_1 g - m_1 g = m_1(2g)$. We'll let you check the case where $m_1 \gg m_2$. ♣

In the problems for this chapter, you'll encounter some strange Atwood's setups. But no matter how complicated they get, there are only two things you need to do to solve them, as mentioned above: (1) Write down the $F = ma$ equations for all the masses (which may involve relating the tensions in various strings), and (2) relate the accelerations of the masses, using "conservation of string".

It may seem, with the angst it can bring,
That an Atwood's machine's a harsh thing.
But you just need to say
That F is ma,
And use conservation of string!

2.3 Solving differential equations

Let's now consider the type of problem where we are *given* the force as a function of time, position, or velocity, and where our task is to solve the $F = ma \equiv m\ddot{x}$ differential equation to find the position, $x(t)$, as a function of time. In what follows, we will develop a few techniques for solving differential equations. The ability to apply these techniques dramatically increases the number of problems we can solve.

In general, the force F can also be a function of higher derivatives of x, in addition to the quantities t, x, and $v \equiv \dot{x}$. But these cases don't arise much, so we won't worry about them. The $F = ma$ differential equation we want to solve is therefore (we'll just work in one dimension here)

$$m\ddot{x} = F(x, v, t). \tag{2.15}$$

In general, this equation cannot be solved exactly for $x(t)$.[8] But for most of the problems we will deal with, it can be solved. The problems we will encounter will often fall into one of three special cases, namely, where F is a function of t only, or x only, or v only. In all of these cases, we must invoke the given initial conditions, $x_0 \equiv x(t_0)$ and $v_0 \equiv v(t_0)$, to obtain our final solutions. These initial conditions will appear in the limits of the integrals in the following discussion.[9]

Note: You may want to just skim the following page and a half, and then refer back to it as needed. Don't try to memorize all the different steps. We present them only for completeness. The whole point here can basically be summarized by saying that sometimes you want to write $\ddot{x}$ as dv/dt, and sometimes you want to write it as $v\,dv/dx$ (see eq. (2.19)). Then you "simply" have to separate variables and integrate. We'll go through the three special cases, and then we'll do some examples.

[8] It can always be solved for $x(t)$ *numerically*, to any desired accuracy. This is discussed in Appendix D.

[9] It is no coincidence that we need *two* initial conditions to completely specify the solution to our *second*-order $F = m\ddot{x}$ differential equation. It is a general result (which we'll just accept here) that the solution to an nth-order differential equation has n free parameters, which must then be determined from the initial conditions.

- *F is a function of t only: $F = F(t)$.*

  Since $a = d^2x/dt^2$, we just need to integrate $F = ma$ twice to obtain $x(t)$. Let's do this in a very systematic way, to get used to the general procedure. First, write $F = ma$ as
$$m\frac{dv}{dt} = F(t). \tag{2.16}$$
  Then separate variables and integrate both sides to obtain[10]
$$m\int_{v_0}^{v(t)} dv' = \int_{t_0}^{t} F(t')\, dt'. \tag{2.17}$$
  We have put primes on the integration variables so that we don't confuse them with the limits of integration. Eq. (2.17) yields v as a function of t, $v(t)$. We then separate variables in $dx/dt = v(t)$ and integrate to obtain
$$\int_{x_0}^{x(t)} dx' = \int_{t_0}^{t} v(t')\, dt'. \tag{2.18}$$
  This yields x as a function of t, $x(t)$. This procedure might seem like a cumbersome way to simply integrate something twice. That's because it is. But the technique proves more useful in the following case.

- *F is a function of x only: $F = F(x)$.*

  We will use
$$a = \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx} \tag{2.19}$$
  to write $F = ma$ as
$$mv\frac{dv}{dx} = F(x). \tag{2.20}$$
  Now separate variables and integrate both sides to obtain
$$m\int_{v_0}^{v(x)} v'\, dv' = \int_{x_0}^{x} F(x')\, dx'. \tag{2.21}$$
  The left side will contain the square of $v(x)$. Taking a square root, this gives v as a function of x, $v(x)$. Separate variables in $dx/dt = v(x)$ to obtain
$$\int_{x_0}^{x(t)} \frac{dx'}{v(x')} = \int_{t_0}^{t} dt'. \tag{2.22}$$
  This gives t as a function of x, and hence (in principle) x as a function of t, $x(t)$. The unfortunate thing about this case is that the integral in eq. (2.22) might not be doable. And even if it is, it might not be possible to invert $t(x)$ to produce $x(t)$.

[10] If you haven't seen such a thing before, the act of multiplying both sides by the infinitesimal quantity dt' might make you feel a bit uneasy. But it is in fact quite legal. If you wish, you can imagine working with the small (but not infinitesimal) quantities Δv and Δt, for which it is certainly legal to multiply both sides by Δt. Then you can take a discrete sum over many Δt intervals, and then finally take the limit $\Delta t \to 0$, which results in eq. (2.17)

- *F is a function of v only: $F = F(v)$.*

  Write $F = ma$ as
  $$m\frac{dv}{dt} = F(v). \tag{2.23}$$
  Separate variables and integrate both sides to obtain
  $$m\int_{v_0}^{v(t)} \frac{dv'}{F(v')} = \int_{t_0}^{t} dt'. \tag{2.24}$$
  This yields t as a function of v, and hence (in principle) v as a function of t, $v(t)$. Integrate $dx/dt = v(t)$ to obtain $x(t)$ from
  $$\int_{x_0}^{x(t)} dx' = \int_{t_0}^{t} v(t')\, dt'. \tag{2.25}$$
  Note: In this $F = F(v)$ case, if you want to find v as a function of x, $v(x)$, then you should write a as $v(dv/dx)$ and integrate
  $$m\int_{v_0}^{v(x)} \frac{v'\, dv'}{F(v')} = \int_{x_0}^{x} dx'. \tag{2.26}$$
  You can then obtain $x(t)$ from eq. (2.22), if desired.

When dealing with the initial conditions, we have chosen to put them in the limits of integration above. If you wish, you can perform the integrals without any limits, and just tack on a constant of integration to your result. The constant is then determined from the initial conditions.

Again, as mentioned above, you do *not* have to memorize the above three procedures, because there are variations, depending on what you're given and what you want to solve for. All you have to remember is that $\ddot{x}$ can be written as either dv/dt or $v\, dv/dx$. One of these will get the job done (namely, the one that makes only two out of the three variables, t, x, and v, appear in your differential equation). And then be prepared to separate variables and integrate as many times as needed.

> a is dv by dt.
> Is this useful? There's no guarantee.
> If it leads to "Oh, heck!"'s,
> Take dv by dx,
> And then write down its product with v.

Example 1 (Gravitational force): A particle of mass m is subject to a constant force $F = -mg$. The particle starts at rest at height h. Because this constant force falls into all of the above three categories, we should be able to solve for $y(t)$ in two ways:

(a) Find $y(t)$ by writing a as dv/dt.

(b) Find $y(t)$ by writing a as $v\, dv/dy$.

Solution:

(a) $F = ma$ gives $dv/dt = -g$. Integrating this yields $v = -gt + C$, where C is a constant of integration.[11] The initial condition $v(0) = 0$ gives $C = 0$. Therefore, $dy/dt = -gt$. Integrating this and using $y(0) = h$ gives

$$y = h - \frac{1}{2}gt^2. \tag{2.27}$$

(b) $F = ma$ gives $v\,dv/dy = -g$. Separating variables and integrating yields $v^2/2 = -gy + C$. The initial condition $v(0) = 0$ gives $v^2/2 = -gy + gh$. Therefore, $v \equiv dy/dt = -\sqrt{2g(h-y)}$. We have chosen the negative square root, because the particle is falling. Separating variables gives

$$\int \frac{dy}{\sqrt{h-y}} = -\sqrt{2g}\int dt. \tag{2.28}$$

This yields $2\sqrt{h-y} = \sqrt{2g}\,t$, where we have used the initial condition $y(0) = h$. Hence, $y = h - gt^2/2$, in agreement with part (a). The solution in part (a) was clearly the simpler one.

Example 2 (Dropped ball): A beach-ball is dropped from rest at height h. Assume[12] that the drag force from the air takes the form, $F_d = -\beta v$. Find the velocity and height as a function of time.

Solution: For simplicity in future formulas, let's write the drag force as $F_d = -\beta v \equiv -m\alpha v$ (so we won't have a bunch of $1/m$'s floating around). Taking upward to be the positive y direction, the force on the ball is

$$F = -mg - m\alpha v. \tag{2.29}$$

Note that v is negative here, because the ball is falling, so the drag force points upward, as it should. Writing $F = m\,dv/dt$, and separating variables, gives

$$\int_0^{v(t)} \frac{dv'}{g + \alpha v'} = -\int_0^t dt'. \tag{2.30}$$

Integration yields $\ln(1 + \alpha v/g) = -\alpha t$. Exponentiation then gives

$$v(t) = -\frac{g}{\alpha}\left(1 - e^{-\alpha t}\right). \tag{2.31}$$

Writing $dy/dt \equiv v(t)$, and then separating variables and integrating to obtain $y(t)$, yields

$$\int_h^{y(t)} dy' = -\frac{g}{\alpha}\int_0^t \left(1 - e^{-\alpha t'}\right) dt'. \tag{2.32}$$

[11] We'll do this example by adding on constants of integration which are then determined from the initial conditions. We'll do the following example by putting the initial conditions in the limits of integration.

[12] The drag force is roughly proportional to v as long as the speed is fairly slow. For large speeds, the drag force is roughly proportional to v^2.

Therefore,

$$y(t) = h - \frac{g}{\alpha}\left(t - \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right)\right). \tag{2.33}$$

REMARKS:

(a) Let's look at some limiting cases. If t is very small (more precisely, if $\alpha t \ll 1$), then we can use $e^{-x} \approx 1 - x + x^2/2$ to make approximations to leading order in t. You can show that eq. (2.31) gives $v(t) \approx -gt$. This makes sense, because the drag force is negligible at the start, so the ball is essentially in free fall. And eq. (2.33) gives $y(t) \approx h - gt^2/2$, as expected.

We can also look at large t. In this case, $e^{-\alpha t}$ is essentially equal to zero, so eq. (2.31) gives $v(t) \approx -g/\alpha$. (This is the "terminal velocity." Its value makes sense, because it is the velocity for which the total force, $-mg - m\alpha v$, vanishes.) And eq. (2.33) gives $y(t) \approx h - (g/\alpha)t + g/\alpha^2$. Interestingly, we see that for large t, g/α^2 is the distance our ball lags behind another ball which started out already at the terminal velocity, g/α.

(b) The velocity of the ball obtained in eq. (2.31) depends on α, which was defined via $F_d = -m\alpha v$. We explicitly wrote the m here just to make all of our formulas look a little nicer, but it should *not* be inferred that the velocity of the ball is independent of m. The coefficient $\beta \equiv m\alpha$ depends (in some complicated way) on the cross-sectional area, A, of the ball. Therefore, $\alpha \propto A/m$. Two balls of the same size, one made of lead and one made of styrofoam, will have the same A but different m's. Hence, their α's will be different, and they will fall at different rates.

For heavy objects in a thin medium such as air, α is small, so the drag effects are not very noticeable over short distances. Heavy objects fall at roughly the same rate. If the air were a bit thicker, different objects would fall at noticeably different rates, and maybe it would have taken Galileo a bit longer to come to his conclusions.

What would you have thought, Galileo,
If instead you dropped cows and did say, "Oh!
To lessen the sound
Of the moos from the ground,
They should fall not through air, but through mayo!" ♣

2.4 Projectile motion

Consider a ball thrown through the air, not necessarily vertically. We will neglect air resistance in the following discussion.

Let x and y be the horizontal and vertical positions, respectively. The force in the x-direction is $F_x = 0$, and the force in the y-direction is $F_y = -mg$. So $\mathbf{F} = m\mathbf{a}$ gives

$$\ddot{x} = 0, \qquad \text{and} \qquad \ddot{y} = -g. \tag{2.34}$$

Note that these two equations are "decoupled." That is, there is no mention of y in the equation for $\ddot{x}$, and vice-versa. The motions in the x- and y-directions are therefore completely independent.

REMARK: The classic demonstration of the independence of the x- and y-motions is the following. Fire a bullet horizontally (or, preferably, just imagine firing a bullet horizontally), and at the same time drop a bullet from the height of the gun. Which bullet will hit the ground first? (Neglect air resistance, the curvature of the earth, etc.) The answer is that they will hit the ground at the same time, because the effect of gravity on the two y-motions is exactly the same, independent of what is going on in the x-direction. ♣

If the initial position and velocity are (X, Y) and (V_x, V_y), then we can easily integrate eqs. (2.34) to obtain

$$\begin{aligned} \dot{x}(t) &= V_x, \\ \dot{y}(t) &= V_y - gt. \end{aligned} \tag{2.35}$$

Integrating again gives

$$\begin{aligned} x(t) &= X + V_x t, \\ y(t) &= Y + V_y t - \frac{1}{2}gt^2. \end{aligned} \tag{2.36}$$

These equations for the speeds and positions are all you need to solve a projectile problem.

Example (Throwing a ball):

(a) For a given initial speed, at what inclination angle should a ball be thrown so that it travels the maximum horizontal distance by the time it returns to the ground? Assume that the ground is horizontal, and that the ball is released from ground level.

(b) What is the optimal angle if the ground is sloped upward at an angle β (or downward, if β is negative)?

Solution:

(a) Let the inclination angle be θ, and let the initial speed be V. Then the horizontal speed is always $V_x = V\cos\theta$, and the initial vertical speed is $V_y = V\sin\theta$.

The first thing we need to do is find the time t in the air. We know that the vertical speed is zero at time $t/2$, because the ball is moving horizontally at its highest point. So the second of eqs. (2.35) gives $V_y = g(t/2)$. Therefore, $t = 2V_y/g$.[13]

The first of eqs. (2.36) tells us that the horizontal distance traveled is $d = V_x t$. Using $t = 2V_y/g$ in this gives

$$d = \frac{2V_x V_y}{g} = \frac{V^2(2\sin\theta\cos\theta)}{g} = \frac{V^2\sin 2\theta}{g}\,. \tag{2.37}$$

The $\sin 2\theta$ factor has a maximum at

$$\theta = \frac{\pi}{4}\,. \tag{2.38}$$

[13] Alternatively, the time of flight can be found from the second of eqs. (2.36), which says that the ball returns to the ground when $V_y t = gt^2/2$. We will have to use this type of strategy in part (b), where the trajectory is not symmetric around the maximum.

The maximum horizontal distance traveled is then $d_{\max} = V^2/g$.

REMARKS: For $\theta = \pi/4$, you can show that the maximum height achieved is $V^2/4g$. This may be compared to the maximum height of $V^2/2g$ (as you can show) if the ball is thrown straight up. Note that any possible distance you might want to find in this problem must be proportional to V^2/g, by dimensional analysis. The only question is what the numerical factor is. ♣

(b) As in part (a), the first thing we need to do is find the time t in the air. If the ground is sloped at an angle β, then the equation for the line of the ground is

$$y = (\tan\beta)x. \tag{2.39}$$

The path of the ball is given in terms of t by

$$x = (V\cos\theta)t, \qquad \text{and} \qquad y = (V\sin\theta)t - \frac{1}{2}gt^2. \tag{2.40}$$

We must solve for the t that makes $y = (\tan\beta)x$, because this gives the place where the path of the ball intersects the line of the ground. Using eqs. (2.40), we find that $y = (\tan\beta)x$ when

$$t = \frac{2V}{g}(\sin\theta - \tan\beta\cos\theta). \tag{2.41}$$

(There is, of course, also the solution $t = 0$.) Plugging this into the expression for x in eq. (2.40) gives

$$x = \frac{2V^2}{g}(\sin\theta\cos\theta - \tan\beta\cos^2\theta). \tag{2.42}$$

We must now maximize this value for x, which is equivalent to maximizing the distance along the slope. Setting the derivative with respect to θ equal to zero, and using the double-angle formulas, $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$, we find $\tan\beta = -\cot 2\theta$. This can be rewritten as $\tan\beta = -\tan(\pi/2 - 2\theta)$. Therefore, $\beta = -(\pi/2 - 2\theta)$, so we have

$$\theta = \frac{1}{2}\left(\beta + \frac{\pi}{2}\right). \tag{2.43}$$

In other words, the throwing angle should bisect the angle between the ground and the vertical.

REMARKS: For $\beta \approx \pi/2$, we have $\theta \approx \pi/2$, as should be the case. For $\beta = 0$, we have $\theta = \pi/4$, as we found in part (a). And for $\beta \approx -\pi/2$, we have $\theta \approx 0$, which makes sense.

Substituting the value of θ from eq. (2.43) into eq. (2.42), you can show (after a bit of algebra) that the maximum distance traveled along the tilted ground is

$$d = \frac{x}{\cos\beta} = \frac{V^2/g}{1 + \sin\beta}. \tag{2.44}$$

This checks in the various limits for β. ♣

Along with the bullet example mentioned above, another classic example of the independence of the x- and y-motions is the "hunter and monkey" problem. In it, a hunter aims an arrow (made of styrofoam, of course) at a monkey hanging from a branch in a tree. The monkey, thinking he's being clever, tries to avoid the arrow by letting go of the branch right when he sees the arrow released. The unfortunate consequence of this action is that he *will* get hit, because gravity acts on both him and the arrow in the same way; they both fall the same distance relative to where they would have been if there were no gravity. And the monkey *would* get hit in such a case, because the arrow is initially aimed at him. You can work this out in Exercise 16, in a more peaceful setting involving fruit.

If a monkey lets go of a tree,
The arrow will hit him, you see,
Because both heights are pared
By a half gt^2
From what they would be with no g.

2.5 Motion in a plane, polar coordinates

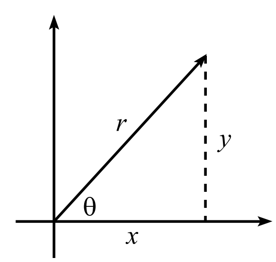


Figure 2.6

When dealing with problems where the motion lies in a plane, it is often convenient to work with polar coordinates, r and θ. These are related to the Cartesian coordinates by (see Fig. 2.6)

$$x = r\cos\theta, \qquad \text{and} \qquad y = r\sin\theta. \tag{2.45}$$

Depending on the problem, either Cartesian or polar coordinates will be easier to use. It is usually clear from the setup which is better. For example, if the problem involves circular motion, then polar coordinates are a good bet. But to use polar coordinates, we need to know what form Newton's second law takes in terms of them. Therefore, the goal of the present section is to determine what $\mathbf{F} = m\mathbf{a} \equiv m\ddot{\mathbf{r}}$ looks like when written in terms of polar coordinates.

At a given position $\mathbf{r}$ in the plane, the basis vectors in polar coordinates are $\hat{\mathbf{r}}$, which is a unit vector pointing in the radial direction; and $\hat{\boldsymbol{\theta}}$, which is a unit vector pointing in the counterclockwise tangential direction. In polar coords, a general vector may therefore be written as

$$\mathbf{r} = r\hat{\mathbf{r}}. \tag{2.46}$$

Note that the directions of the $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ basis vectors depend, of course, on $\mathbf{r}$.

Since the goal of this section is to find $\ddot{\mathbf{r}}$, we must, in view of eq. (2.46), get a handle on the time derivative of $\hat{\mathbf{r}}$. And we'll eventually need the derivative of $\hat{\boldsymbol{\theta}}$, too. In contrast with the fixed Cartesian basis vectors ($\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$), the polar basis vectors ($\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$) change as a point moves around in the plane.

We can find $\dot{\hat{\mathbf{r}}}$ and $\dot{\hat{\boldsymbol{\theta}}}$ in the following way. In terms of the Cartesian basis, Fig. 2.7 shows that

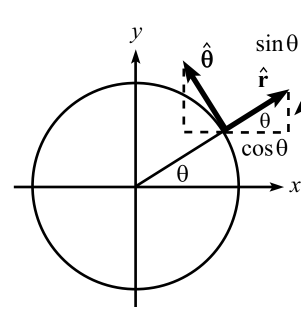


Figure 2.7

$$\begin{aligned}\hat{\mathbf{r}} &= \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}},\\ \hat{\boldsymbol{\theta}} &= -\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}.\end{aligned} \tag{2.47}$$

Taking the time derivative of these equations gives

$$\begin{aligned}\dot{\hat{\mathbf{r}}} &= -\sin\theta\,\dot\theta\hat{\mathbf{x}} + \cos\theta\,\dot\theta\hat{\mathbf{y}},\\ \dot{\hat{\boldsymbol{\theta}}} &= -\cos\theta\,\dot\theta\hat{\mathbf{x}} - \sin\theta\,\dot\theta\hat{\mathbf{y}}.\end{aligned} \tag{2.48}$$

Using eqs. (2.47), we arrive at the nice clean expressions,

$$\dot{\hat{\mathbf{r}}} = \dot\theta\hat{\boldsymbol{\theta}}, \qquad \text{and} \qquad \dot{\hat{\boldsymbol{\theta}}} = -\dot\theta\hat{\mathbf{r}}. \tag{2.49}$$

These relations are fairly evident if we look at what happens to the basis vectors as $\mathbf{r}$ moves a tiny distance in the tangential direction. Note that the basis vectors do not change as $\mathbf{r}$ moves in the radial direction.

We can now start differentiating eq. (2.46). One derivative gives (yes, the product rule works fine here)

$$\begin{aligned}\dot{\mathbf{r}} &= \dot r\hat{\mathbf{r}} + r\dot{\hat{\mathbf{r}}}\\ &= \dot r\hat{\mathbf{r}} + r\dot\theta\hat{\boldsymbol{\theta}}.\end{aligned} \tag{2.50}$$

This makes sense, because $\dot r$ is the speed in the radial direction, and $r\dot\theta$ is the speed in the tangential direction, which is often written as ωr (where $\omega \equiv \dot\theta$ is the angular speed, or "angular frequency").[14]

Differentiating eq. (2.50) then gives

$$\begin{aligned}\ddot{\mathbf{r}} &= \ddot r\hat{\mathbf{r}} + \dot r\dot{\hat{\mathbf{r}}} + \dot r\dot\theta\hat{\boldsymbol{\theta}} + r\ddot\theta\hat{\boldsymbol{\theta}} + r\dot\theta\dot{\hat{\boldsymbol{\theta}}}\\ &= \ddot r\hat{\mathbf{r}} + \dot r(\dot\theta\hat{\boldsymbol{\theta}}) + \dot r\dot\theta\hat{\boldsymbol{\theta}} + r\ddot\theta\hat{\boldsymbol{\theta}} + r\dot\theta(-\dot\theta\hat{\mathbf{r}})\\ &= (\ddot r - r\dot\theta^2)\hat{\mathbf{r}} + (r\ddot\theta + 2\dot r\dot\theta)\hat{\boldsymbol{\theta}}.\end{aligned} \tag{2.51}$$

Finally, equating $m\ddot{\mathbf{r}}$ with $\mathbf{F} \equiv F_r\hat{\mathbf{r}} + F_\theta\hat{\boldsymbol{\theta}}$ gives the radial and tangential forces as

$$\begin{aligned}F_r &= m(\ddot r - r\dot\theta^2),\\ F_\theta &= m(r\ddot\theta + 2\dot r\dot\theta).\end{aligned} \tag{2.52}$$

(See Exercise 32 for a slightly different derivation of these equations.) Let's look at each of the four terms on the right-hand sides of eqs. (2.52).

- The $m\ddot r$ term is quite intuitive. For radial motion, it simply states that $F = ma$ along the radial direction.
- The $mr\ddot\theta$ term is also quite intuitive. For circular motion, it states that $F = ma$ along the tangential direction, because $r\ddot\theta$ is the second derivative of the distance $r\theta$ along the circumference.

[14]For $r\dot\theta$ to be the tangential speed, we must measure θ in radians and not degrees. Then $r\theta$ is by definition the distance along the circumference, so $r\dot\theta$ is the speed along the circumference.

- The $-mr\dot{\theta}^2$ term is also fairly clear. For circular motion, it says that the radial force is $-m(r\dot{\theta})^2/r = -mv^2/r$, which is the familiar force that causes the centripetal acceleration, v^2/r. See Problem 19 for an alternate (and quicker) derivation of this v^2/r result.
- The $2m\dot{r}\dot{\theta}$ term isn't so obvious. It is called the *Coriolis* force. There are various ways to look at this term. One is that it exists in order to keep angular momentum conserved. We'll have a great deal to say about the Coriolis force in Chapter 9.

Example (Circular pendulum): A mass hangs from a massless string of length ℓ. Conditions have been set up so that the mass swings around in a horizontal circle, with the string making an angle β with the vertical (see Fig. 2.8). What is the angular frequency, ω, of this motion?

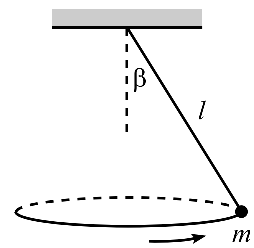


Figure 2.8

Solution: The mass travels in a circle, so the horizontal radial force must be $F_r = mr\dot{\theta}^2 \equiv mr\omega^2$ (with $r = \ell \sin\beta$), directed radially inward. The forces on the mass are the tension in the string, T, and gravity, mg (see Fig. 2.9). There is no acceleration in the vertical direction, so $F = ma$ in the vertical and radial directions gives, respectively,

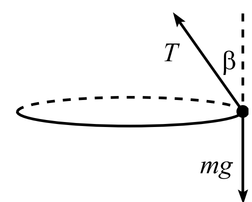


Figure 2.9

$$\begin{aligned} T\cos\beta &= mg, \\ T\sin\beta &= m(\ell\sin\beta)\omega^2. \end{aligned} \tag{2.53}$$

Solving for ω gives

$$\omega = \sqrt{\frac{g}{\ell\cos\beta}}\,. \tag{2.54}$$

Note that if $\beta \approx 0$, then $\omega \approx \sqrt{g/\ell}$, which equals the frequency of a plane pendulum of length ℓ. And if $\beta \approx 90^\circ$, then $\omega \to \infty$, which makes sense.

2.6 Exercises

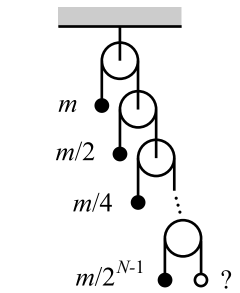


Figure 2.10

Section 2.2: Free-body diagrams

1. **A peculiar Atwood's machine**

   The Atwood's machine in Fig. 2.10 consists of N masses, m, $m/2$, $m/4$, $\ldots$, $m/2^{N-1}$. All the pulleys and strings are massless, as usual.

   (a) Put a mass $m/2^{N-1}$ at the free end of the bottom string. What are the accelerations of all the masses?

   (b) Remove the mass $m/2^{N-1}$ (which was arbitrarily small, for very large N) that was attached in part (a). What are the accelerations of all the masses, now that you've removed this infinitesimal piece?

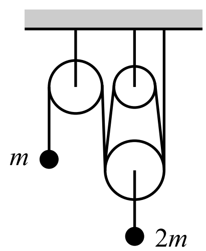


Figure 2.11

2. **Double-loop Atwood's** $*$

   Consider the Atwood's machine shown in Fig. 2.11. It consists of three pulleys, a short piece of string connecting one mass to the bottom pulley, and a continuous long piece of string that wraps twice around the bottom side of the bottom pulley, and once around the top side of the top two pulleys. The two masses are m and $2m$. Assume that the parts of the string connecting the pulleys are essentially vertical. Find the accelerations of the masses.

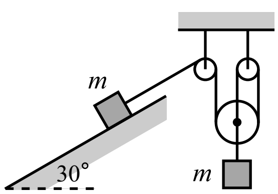


Figure 2.12

3. **Atwood's and a plane** $*$

   Consider the Atwoods machine shown in Fig. 2.12, with two masses m. The plane is frictionless, and it is inclined at a 30° angle. Find the accelerations of the masses.

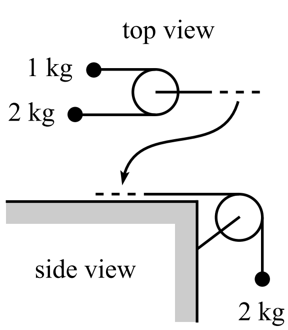


Figure 2.13

4. **Atwood's on a table** $*$

   Consider the Atwood's machine shown in Fig. 2.13, Masses of 1 kg and 2 kg lie on a frictionless table, connected by a string which passes around a pulley. The pulley is connected to another mass of 2 kg, which hangs down over another pulley, as shown. Find the accelerations of all three masses.

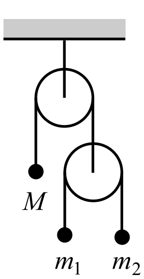


Figure 2.14

5. **Keeping the mass still** $*$

   In the Atwood's machine in Fig. 2.14, what should M be (in terms of m_1 and m_2) so that it doesn't move?

6. **Three-mass Atwood's** $**$

   Consider the Atwood's machine in Fig. 2.15, with masses m, $2m$, and $3m$. Find the accelerations of all three masses.

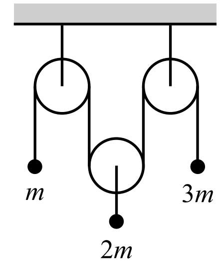


Figure 2.15

7. **Accelerating plane** $**$

   A block of mass m rests on a plane inclined at angle θ. The coefficient of static friction between the block and the plane is μ. The plane is accelerated to the right with acceleration a (which may be negative); see Fig. 2.16. For what range of a does the block remain at rest with respect to the plane?

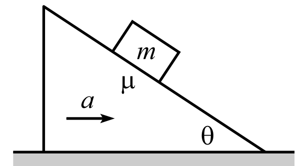


Figure 2.16

8. **Accelerating cylinders** $**$

   Three identical cylinders are arranged in a triangle as shown in Fig. 2.17, with the bottom two lying on the ground. The ground and the cylinders are frictionless. You apply a constant horizontal force (directed to the right) on the left cylinder. Let a be the acceleration you give to the system. For what range of a will all three cylinders remain in contact with each other?

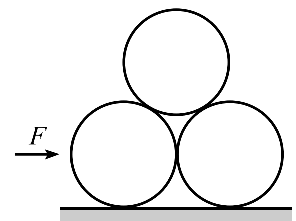


Figure 2.17

Section 2.3: Solving differential equations

9. $-bv^2$ **force** $*$

   A particle of mass m is subject to a force $F(v) = -bv^2$. The initial position is zero, and the initial speed is v_0. Find $x(t)$.

10. $-kx$ **force** $**$

    A particle of mass m is subject to a force $F(x) = -kx$. The initial position is zero, and the initial speed is v_0. Find $x(t)$.

11. kx **force** $**$

    A particle of mass m is subject to a force $F(x) = kx$. The initial position is zero, and the initial speed is v_0. Find $x(t)$.

12. **Motorcycle circle** $***$

    A motorcyclist wishes to travel in a circle of radius R on level ground. The coefficient of friction between the tires and the ground is μ. The motorcycle starts at rest. What is the minimum distance the motorcycle must travel in order to achieve its maximum allowable speed (that is, the speed above which it will skid out of the circular path)?

Section 2.4: Projectile motion

13. **Dropped balls**

    A ball is dropped from height $4h$. After it has fallen a distance d, a second ball is dropped from height h. What should d be (in terms of h) so that the balls hit the ground at the same time?

14. **Equal distances**

At what angle should a ball be thrown so that its maximum height equals the horizontal distance traveled?

15. **Redirected horizontal motion** $*$

A ball is dropped from rest at height h, and it bounces off a surface at height y, with no loss in speed. The surface is inclined at 45°, so that the ball bounces off horizontally. What should y be so that the ball travels the maximum horizontal distance?

16. **Newton's apple** $*$

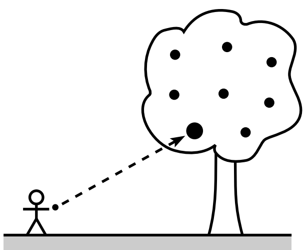


Figure 2.18

Newton is tired of apples falling on his head, so he decides to throw a rock at one of the larger and more formidable looking apples positioned directly above his favorite sitting spot. Forgetting all about his work on gravitation, he aims the rock directly at the apple (see Fig. 2.18). To his surprise, the apple falls from the tree just as he releases the rock. Show, by calculating the rock's height when it reaches the horizontal position of the apple, that the rock will hit the apple.[15]

17. **Throwing at a wall** $*$

You throw a ball with speed V_0 at a vertical wall, a distance ℓ away. At what angle should you throw the ball, so that it hits the wall at a maximum height? Assume $\ell < V_0^2/g$ (why?).

18. **Firing a cannon** $**$

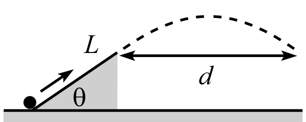


Figure 2.19

A cannon, when aimed vertically, is observed to fire a ball to a maximum height of L. Another ball is then fired with this same speed, but with the cannon now aimed up along a plane of length L, inclined at an angle θ, as shown in Fig. 2.19. What should θ be, so that the ball travels the largest horizontal distance, d, by the time it returns to the height of the top of the plane?

19. **Colliding projectiles** $*$

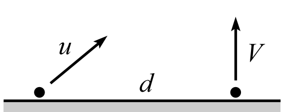


Figure 2.20

Two balls are fired from ground level, a distance d apart. The right one is fired vertically with speed V; see Fig. 2.20. You wish to simultaneously fire the left one at the appropriate velocity $\vec{u}$ so that it collides with the right ball when they reach their highest point. What should $\vec{u}$ be (give the horizontal and vertical components)? Given d, what should V be so that the speed u is minimum?

[15] This problem suggests a way in which William Tell and his son might survive their ordeal if they were plopped down on a planet with an unknown gravitational constant (provided that the son weren't too short or g weren't too big).

20. **Throwing in the wind** $*$

    A ball is thrown horizontally to the right, from the top of a vertical cliff of height h. A wind blows horizontally to the left, and assume (simplistically) that the effect of the wind is to provide a constant force to the left, equal in magnitude to the weight of the ball. How fast should the ball be thrown, so that it lands at the foot of the cliff?

21. **Throwing in the wind again** $*$

    A ball is thrown eastward across level ground. A wind blows horizontally to the east, and assume (simplistically) that the effect of the wind is to provide a constant force to the east, equal in magnitude to the weight of the ball. At what angle θ should the ball be thrown, so that it travels the maximum horizontal distance?

22. **Increasing gravity** $*$

    At $t = 0$ on the planet Gravitus Increasicus, a projectile is fired with speed V_0 at an angle θ above the horizontal. This planet is a strange one, in that the acceleration due to gravity increases linearly with time, starting with a value of zero when the projectile is fired. In other words, $g(t) = \beta t$, where β is a given constant. What horizontal distance does the projectile travel? What should θ be so that this horizonal distance is maximum?

23. **Cart, ball, and plane** $**$

    A cart rolls down an inclined plane. A ball is fired from the cart, perpendicularly to the plane. Will the ball eventually land in the cart? *Hint*: Choose your coordinate system wisely.

Section 2.5: Motion in a plane, polar coordinates

24. **Low-orbit satellite**

    What is the speed of a satellite whose orbit is just above the earth's surface? Give the numerical value.

25. **Weight at the equator** $*$

    A person stands on a scale at the equator. If the earth somehow stopped spinning but kept its same shape, would the reading on the scale increase or decrease? By what fraction?

26. **Banking an airplane** $*$

    An airplane flies at speed v in a horizontal circle of radius R. At what angle should the plane be banked so that you don't feel like you are getting flung to the side in your seat?

27. **Car on a banked track** $**$

    A car travels around a circular banked track with radius R. The coefficient of friction between the tires and the track is μ. What is the maximum allowable speed, above which the car slips?

28. **Driving on tilted ground** $**$

A driver encounters a large tilted parking lot, where the angle of the ground with respect to the horizontal is θ. The driver wishes to drive in a circle of radius R, at constant speed. The coefficient of friction between the tires and the ground is μ.

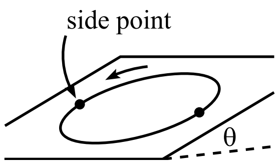


Figure 2.21

(a) What is the largest speed the driver can have if he wants to avoid slipping?

(b) What is the largest speed the driver can have, assuming he is concerned only with whether or not he slips at one of the "side" points on the circle (that is, halfway between the top and bottom points; see Fig. 2.21)?

29. **Rolling wheel** $*$

If you paint a dot on the rim of a rolling wheel, the coordinates of the dot may be written as[16]

$$(x, y) = (R\theta + R\sin\theta, R + R\cos\theta). \tag{2.55}$$

The path of the dot is called a *cycloid*. Assume that the wheel is rolling at constant speed, which implies $\theta = \omega t$.

(a) Find $\vec{v}(t)$ and $\vec{a}(t)$ of the dot.

(b) At the instant the dot is at the top of the wheel, it may be considered to be moving along the arc of a circle. What is the radius of this circle in terms of R? *Hint*: You know v and a.

30. **Bead on a hoop** $**$

A bead rests on top of a frictionless hoop of radius R which lies in a vertical plane. The bead is given a tiny push so that it slides down and around the hoop. At what points on the hoop (specify them by giving the angular position relative to the top) is the bead's acceleration vertical?[17] What is this vertical acceleration? *Note*: We haven't studied conservation of energy yet, but use the fact that the bead's speed after it has fallen a height h is given by $v = \sqrt{2gh}$.

31. **Another bead on a hoop** $**$

A bead rests on top of a frictionless hoop of radius R which lies in a vertical plane. The bead is given a tiny push so that it slides down and around the hoop. At what points on the hoop (specify them by giving the angular position relative to the horizontal) is the bead's acceleration horizontal? As in the previous exercise, use $v = \sqrt{2gh}$.

---

[16] This can be shown by writing (x, y) as $(R\theta, R) + (R\sin\theta, R\cos\theta)$. The first term here is the position of the center of the wheel, and the second term is the position of the dot relative to the center, where θ is measured clockwise from the top.

[17] One such point is the bottom of the hoop. Another point is technically the top, where $a = 0$. Find the other two more interesting points (one on each side).

32. **Derivation of F_r and F_θ** $**$

In Cartesian coordinates, a general vector takes the form,

$$\begin{aligned}\mathbf{r} &= x\hat{\mathbf{x}} + y\hat{\mathbf{y}} \\ &= r\cos\theta\,\hat{\mathbf{x}} + r\sin\theta\,\hat{\mathbf{y}}.\end{aligned} \tag{2.56}$$

Derive eqs. (2.52) by taking two derivatives of this expression for $\mathbf{r}$, and then using eqs. (2.47) to show that the result can be written in the form of eq. (2.51). Note that unlike $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$, the vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ do not change with time.

33. **A force $F_\theta = 2\dot{r}\dot{\theta}$** $**$

Consider a particle that feels an angular force only, of the form $F_\theta = 2m\dot{r}\dot{\theta}$. (As in Problem 21, there's nothing all that physical about this force; it simply makes the $F = ma$ equations solvable.) Show that the trajectory takes the form of an exponential spiral, that is, $r = Ae^{\theta}$.

34. **A force $F_\theta = 3\dot{r}\dot{\theta}$** $**$

Consider a particle that feels an angular force only, of the form $F_\theta = 3m\dot{r}\dot{\theta}$. (As in the previous exercise, we're solving this problem simply because we can.) Show that $\dot{r} = \sqrt{Ar^4 + B}$. Also, show that the particle reaches $r = \infty$ in a finite time.

2.7 Problems

Section 2.2: Free-body diagrams

1. **Sliding down a plane** $**$

   (a) A block starts at rest and slides down a frictionless plane inclined at angle θ. What should θ be so that the block travels a given horizontal distance in the minimum amount of time?

   (b) Same question, but now let there be a coefficient of kinetic friction, μ, between the block and the plane.

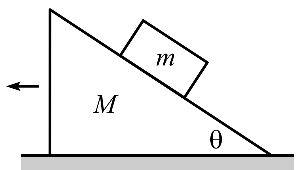


Figure 2.22

2. **Moving plane** $***$

   A block of mass m is held motionless on a frictionless plane of mass M and angle of inclination θ (see Fig. 2.22). The plane rests on a frictionless horizontal surface. The block is released. What is the horizontal acceleration of the plane?

3. **Sliding sideways on plane** $***$

   A block is placed on a plane inclined at angle θ. The coefficient of friction between the block and the plane is $\mu = \tan\theta$. The block is given a kick so that it initially moves with speed V horizontally along the plane (that is, in the direction perpendicular to the direction pointing straight down the plane). What is the speed of the block after a very long time?

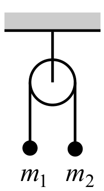


Figure 2.23

4. **Atwood's machine**

   A massless pulley hangs from a fixed support. A massless string connecting two masses, m_1 and m_2, hangs over the pulley (see Fig. 2.23). Find the acceleration of the masses and the tension in the string.

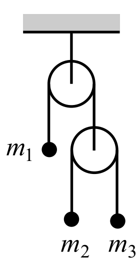


Figure 2.24

5. **Double Atwood's machine** $**$

   A double Atwood's machine is shown in Fig. 2.24, with masses m_1, m_2, and m_3. What are the accelerations of the masses?

6. **Infinite Atwood's machine** $***$

   Consider the infinite Atwood's machine shown in Fig. 2.25. A string passes over each pulley, with one end attached to a mass and the other end attached to another pulley. All the masses are equal to m, and all the pulleys and strings are massless. The masses are held fixed and then simultaneously released. What is the acceleration of the top mass?[18]

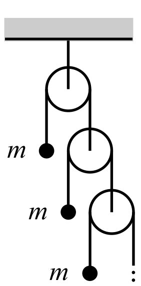


Figure 2.25

[18] You may define this infinite system as follows. Consider it to be made of N pulleys, with a non-zero mass replacing what would have been the $(N+1)$st pulley. Then take the limit as $N \to \infty$.

7. **Line of pulleys** $*$

   $N+2$ equal masses hang from a system of pulleys, as shown in Fig. 2.26. What are the accelerations of all the masses?

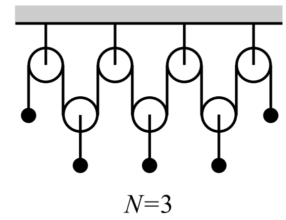


Figure 2.26

8. **Ring of pulleys** $**$

   Consider the system of pulleys shown in Fig. 2.27. The string (which is a loop with no ends) hangs over N fixed pulleys. N masses, $m_1, m_2, \ldots, m_N$, are attached to N pulleys that hang on the string. What are the accelerations of all the masses?

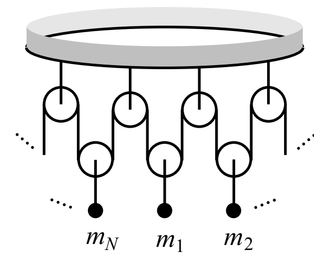


Figure 2.27

Section 2.3: Solving differential equations

9. **Exponential force**

   A particle of mass m is subject to a force $F(t) = me^{-bt}$. The initial position and speed are zero. Find $x(t)$.

10. **Falling chain** $**$

    A chain of length ℓ is held stretched out on a frictionless horizontal table, with a length y_0 hanging down through a hole in the table. The chain is released. As a function of time, find the length that hangs down through the hole (don't bother with t after the chain loses contact with the table). Also, find the speed of the chain right when it loses contact with the table.

11. **Circling around a pole** $**$

    A mass, which is free to move on a horizontal frictionless plane, is attached to one end of a massless string which wraps partially around a frictionless vertical pole of radius r (see the top view in Fig. 2.28). You hold onto the other end of the string. At $t = 0$, the mass has speed v_0 in the tangential direction along the dotted circle of radius R shown.

    Your task is to pull on the string so that the mass keeps moving along the dotted circle. You are required to do this in such a way that the string remains in contact with the pole at all times. (You will have to move your hand around the pole, of course.) What is the the speed of the mass as a function of time?

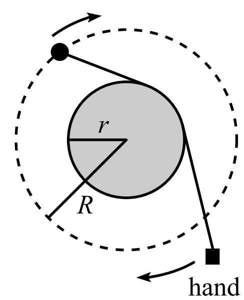


Figure 2.28

12. **Throwing a beach ball** $***$

    A beach ball is thrown upward with initial speed v_0. Assume that the drag force from the air is $F = -m\alpha v$. What is the speed of the ball, v_f, when it hits the ground? (An implicit equation is sufficient.) Does the ball spend more time or less time in the air than it would if it were thrown in vacuum?

13. **Balancing a pencil** $***$

    Consider a pencil that stands upright on its tip and then falls over. Let's idealize the pencil as a mass m sitting at the end of a massless rod of length ℓ.[19]

[19] It actually involves only a trivial modification to do the problem correctly using the moment of inertia and the torque. But the point-mass version will be quite sufficient for the present purposes.

(a) Assume that the pencil makes an initial (small) angle θ_0 with the vertical, and that its initial angular speed is ω_0. The angle will eventually become large, but while it is small (so that $\sin\theta \approx \theta$), what is θ as a function of time?

(b) You might think that it would be possible (theoretically, at least) to make the pencil balance for an arbitrarily long time, by making the initial θ_0 and ω_0 sufficiently small.

However, it turns out that due to Heisenberg's uncertainty principle (which puts a constraint on how well we can know the position and momentum of a particle), it is impossible to balance the pencil for more than a certain amount of time. The point is that you can't be sure that the pencil is initially both at the top *and* at rest. The goal of this problem is to be quantitative about this. The time limit is sure to surprise you.

Without getting into quantum mechanics, let's just say that the uncertainty principle says (up to factors of order 1) that $\Delta x \Delta p \geq \hbar$ (where $\hbar = 1.06 \cdot 10^{-34}$ Js is Planck's constant). The implications of this are somewhat vague, but we'll just take it to mean that the initial conditions satisfy $(\ell\theta_0)(m\ell\omega_0) \geq \hbar$.

With this condition, find the maximum time it can take your solution in part (a) to become of order 1. In other words, determine (roughly) the maximum time the pencil can balance. Assume $m = 0.01$ kg, and $\ell = 0.1$ m.

Section 2.4: Projectile motion

14. **Throwing a ball from a cliff** $**$

A ball is thrown with speed v from the edge of a cliff of height h. At what inclination angle should it be thrown so that it travels the maximum horizontal distance? What is this maximum distance? Assume that the ground below the cliff is horizontal.

15. **Redirected motion** $**$

A ball is dropped from rest at height h, and it bounces off a surface at height y (with no loss in speed). The surface is inclined so that the ball bounces off at an angle of θ with respect to the horizontal. What should y and θ be so that the ball travels the maximum horizontal distance?

16. **Maximum trajectory length** $***$

A ball is thrown at speed v from zero height on level ground. Let θ_0 be the angle at which the ball should be thrown so that the distance traveled *through the air* is maximum. Show that θ_0 satisfies

$$\sin\theta_0 \ln\left(\frac{1+\sin\theta_0}{\cos\theta_0}\right) = 1. \tag{2.57}$$

You can show numerically that $\theta_0 \approx 56.5°$.

17. **Maximum trajectory area** $*$

    A ball is thrown at speed v from zero height on level ground. At what angle should it be thrown so that the area under the trajectory is maximum?

18. **Bouncing ball** $*$

    A ball is thrown straight upward so that it reaches a height h. It falls down and bounces repeatedly. After each bounce, it returns to a certain fraction f of its previous height. Find the total distance traveled, and also the total time, before it comes to rest. What is its average speed?

Section 2.5: Motion in a plane, polar coordinates

19. **Centripetal acceleration** $*$

    Show that the acceleration of a particle moving in a circle is v^2/r. To do this, draw the position and velocity vectors at two nearby times, and then make use of some similar triangles.

20. **Free particle** $**$

    Consider a free particle in a plane. Using Cartesian coordinates, it is trivial to show that the particle moves in a straight line. The task of this problem is to demonstrate this result in a much more cumbersome way, using eqs. (2.52). More precisely, show that $\cos\theta = r_0/r$ for a free particle, where r_0 is the radius at closest approach to the origin, and θ is measured with respect to this radius.

21. **A force $F_\theta = \dot{r}\dot{\theta}$** $**$

    Consider a particle that feels an angular force only, of the form $F_\theta = m\dot{r}\dot{\theta}$. (There's nothing all that physical about this force. It simply makes the $F = ma$ equations solvable.) Show that $\dot{r} = \sqrt{A\ln r + B}$, where A and B are constants of integration, determined by the initial conditions.

2.8 Solutions

1. **Sliding down a plane**

(a) The component of gravity along the plane in $g\sin\theta$. The acceleration in the horizontal direction is therefore $a_x = (g\sin\theta)\cos\theta$. Our goal is to maximize a_x. By taking the derivative, or by noting that $\sin\theta\cos\theta = (\sin 2\theta)/2$, we obtain $\theta = \pi/4$.

(b) The normal force from the plane is $mg\cos\theta$, so the kinetic friction force is $\mu mg\cos\theta$. The acceleration along the plane is therefore $g(\sin\theta - \mu\cos\theta)$, and so the acceleration in the horizontal direction is $a_x = g(\sin\theta - \mu\cos\theta)\cos\theta$. We want to maximize this. Setting the derivative equal to zero gives

$$\begin{aligned}(\cos^2\theta - \sin^2\theta) + 2\mu\sin\theta\cos\theta = 0 \quad &\Longrightarrow \quad \cos 2\theta + \mu\sin 2\theta = 0\\ &\Longrightarrow \quad \tan 2\theta = -\frac{1}{\mu}\,. \end{aligned} \tag{2.58}$$

For $\mu \to 0$, this gives the $\pi/4$ result from part (a). For $\mu \to \infty$, we obtain $\theta \approx \pi/2$, which makes sense.

REMARK: The time to travel a horizontal distance d is obtained from $a_x t^2/2 = d$. In part (a), this gives a minimum time of $2\sqrt{d/g}$. In part (b), you can show that the maximum a_x is $(g/2)(\sqrt{1+\mu^2} - \mu)$, and that this leads to a minimum time of $2\sqrt{d/g}\sqrt{\sqrt{1+\mu^2}+\mu}$. This has the correct $\mu \to 0$ limit, and it behaves like $2\sqrt{2\mu d/g}$ for $\mu \to \infty$. ♣

2. **Moving plane**

Let N be the normal force between the block and the plane. Note that we *cannot* assume that $N = mg\cos\theta$, because the plane recoils. We can see that $N = mg\cos\theta$ is in fact incorrect, because in the limiting case where $M = 0$, we have no normal force at all.

The various $F = ma$ equations (vertical and horizontal for the block, and horizontal for the plane) are

$$\begin{aligned} mg - N\cos\theta &= ma_y,\\ N\sin\theta &= ma_x,\\ N\sin\theta &= MA_x, \end{aligned} \tag{2.59}$$

where we have chosen the positive directions for a_y, a_x, and A_x to be downward, rightward, and leftward, respectively. There are four unknowns here: a_x, a_y, A_x, and N. So we need one more equation. This fourth equation is the constraint that the block remains in contact with the plane. The horizontal distance between the block and its starting point on the plane is $(a_x + A_x)t^2/2$, and the vertical distance is $a_y t^2/2$. The ratio of these distances must equal $\tan\theta$ if the block is to remain on the plane. Therefore, we must have

$$\frac{a_y}{a_x + A_x} = \tan\theta. \tag{2.60}$$

Using eqs. (2.59), this becomes

$$\begin{aligned} \frac{g - \frac{N}{m}\cos\theta}{\frac{N}{m}\sin\theta + \frac{N}{M}\sin\theta} &= \tan\theta\\ \Longrightarrow \qquad N &= g\left(\sin\theta\tan\theta\left(\frac{1}{m}+\frac{1}{M}\right)+\frac{\cos\theta}{m}\right)^{-1}. \end{aligned} \tag{2.61}$$

(In the limit $M \to \infty$, this reduces to $N = mg\cos\theta$, as it should.) Having found N, the third of eqs. (2.59) gives A_x, which may be written as

$$A_x = \frac{N\sin\theta}{M} = \frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta}\,. \tag{2.62}$$

REMARKS: For given M and m, you can show that the angle θ_0 that maximizes A_x is

$$\tan\theta_0 = \sqrt{\frac{M}{M+m}}\,. \tag{2.63}$$

If $M \ll m$, then $\theta_0 \approx 0$. If $M \gg m$, then $\theta_0 \approx \pi/4$.

In the limit $M \ll m$, eq. (2.62) gives $A_x \approx g/\tan\theta$. This makes sense, because m falls essentially straight down, and the plane gets squeezed out to the left.

In the limit $M \gg m$, we have $A_x \approx g(m/M)\sin\theta\cos\theta$. This is more transparent if we instead look at $a_x = (M/m)A_x \approx g\sin\theta\cos\theta$. Since the plane is essentially at rest in this limit, this value of a_x implies that the acceleration of m *along* the plane is equal to $a_x/\cos\theta \approx g\sin\theta$, as expected. ♣

3. **Sliding sideways on plane**

The normal force from the plane is $N = mg\cos\theta$. Therefore, the friction force on the block is $\mu N = (\tan\theta)N = mg\sin\theta$. This force acts in the direction opposite to the motion. The block also feels the gravitational force of $mg\sin\theta$ pointing down the plane.

Because the magnitudes of the friction force and the gravitational force along the plane are equal, the acceleration along the direction of motion equals the negative of the acceleration in the direction down the plane. Therefore, in a small increment of time, the speed that the block loses along its direction of motion exactly equals the speed that it gains in the direction down the plane. Letting v be the speed of the block, and letting v_y be the component of the velocity in the direction down the plane, we therefore have

$$v + v_y = C, \tag{2.64}$$

where C is a constant. C is given by its initial value, which is $V + 0 = V$. The final value of C is $V_f + V_f = 2V_f$ (where V_f is the final speed of the block), because the block is essentially moving straight down the plane after a very long time. Therefore,

$$2V_f = V \qquad \Longrightarrow \qquad V_f = \frac{V}{2}\,. \tag{2.65}$$

4. **Atwood's machine**

Let T be the tension in the string, and let a be the acceleration of m_1 (with upward taken to be positive). Then $-a$ is the acceleration of m_2. So we have

$$\begin{aligned} T - m_1 g &= m_1 a, \\ T - m_2 g &= m_2(-a). \end{aligned} \tag{2.66}$$

Solving these two equations for a and T gives

$$a = \frac{(m_2 - m_1)g}{m_2 + m_1}\,, \qquad \text{and} \qquad T = \frac{2m_1 m_2 g}{m_2 + m_1}\,. \tag{2.67}$$

Remarks: As a double-check, a has the correct limits when $m_2 \gg m_1$, $m_1 \gg m_2$, and $m_2 = m_1$ (namely $a \approx g$, $a \approx -g$, and $a = 0$, respectively).

As far as T goes, if $m_1 = m_2 \equiv m$, then $T = mg$, as it should. And if $m_1 \ll m_2$, then $T \approx 2m_1 g$. This is correct, because it makes the net upward force on m_1 equal to $m_1 g$, which means that its acceleration is g upward, which is consistent with the fact that m_2 is essentially in free fall. ♣

5. **Double Atwood's machine**

Let the tension in the lower string be T. Then the tension in the upper string is $2T$ (by balancing the forces on the bottom pulley). The three $F = ma$ equations are therefore (with all the a's taken to be positive upward)

$$\begin{aligned} 2T - m_1 g &= m_1 a_1, \\ T - m_2 g &= m_2 a_2, \\ T - m_3 g &= m_3 a_3. \end{aligned} \tag{2.68}$$

And conservation of string says that the acceleration of m_1 is

$$a_1 = -\left(\frac{a_2 + a_3}{2}\right). \tag{2.69}$$

This follows from the fact that the average position of m_2 and m_3 moves the same distance as the bottom pulley, which in turn moves the same distance (but in the opposite direction) as m_1.

We now have four equations in the four unknowns, a_1, a_2, a_3, and T. With a little work, we can solve for the accelerations,

$$\begin{aligned} a_1 &= g\frac{4m_2m_3 - m_1(m_2 + m_3)}{4m_2m_3 + m_1(m_2 + m_3)}, \\ a_2 &= -g\frac{4m_2m_3 + m_1(m_2 - 3m_3)}{4m_2m_3 + m_1(m_2 + m_3)}, \\ a_3 &= -g\frac{4m_2m_3 + m_1(m_3 - 3m_2)}{4m_2m_3 + m_1(m_2 + m_3)}. \end{aligned} \tag{2.70}$$

Remarks: There are many limits we can check here. A couple are: (1) If $m_2 = m_3 = m_1/2$, then all the a's are zero, which is correct. (2) If m_3 is much less than both m_1 and m_2, then $a_1 = -g$, $a_2 = -g$, and $a_3 = 3g$. To understand this $3g$, convince yourself that if m_1 and m_2 go down by d, then m_3 goes up by $3d$.

Note that a_1 can be written as

$$a_1 = g\frac{\frac{4m_2m_3}{(m_2+m_3)} - m_1}{\frac{4m_2m_3}{(m_2+m_3)} + m_1}. \tag{2.71}$$

In view of the result of Problem 4 in eq. (2.67), we see that as far as m_1 is concerned, the m_2, m_3 pulley system acts just like a mass of $4m_2m_3/(m_2 + m_3)$. This has the expected properties of equaling zero when either m_2 or m_3 is zero, and equaling $2m$ if $m_2 = m_3 \equiv m$. ♣

6. **Infinite Atwood's machine**

First Solution: If the strength of gravity on the earth were multiplied by a factor η, then the tension in all of the strings in the Atwood's machine would likewise be multiplied by η. This is true because the only way to produce a quantity with the units of tension (that is, force) is to multiply a mass by g. Conversely, if we put the Atwood's machine on another planet and discover that all of the tensions are multiplied by η, then we know that the gravity there must be ηg.

Let the tension in the string above the first pulley be T. Then the tension in the string above the second pulley is $T/2$ (because the pulley is massless). Let the downward acceleration of the second pulley be a_2. Then the second pulley effectively lives in a world where gravity has strength $g - a_2$.

Consider the subsystem of all the pulleys except the top one. This infinite subsystem is identical to the original infinite system of all the pulleys. Therefore, by the arguments in the first paragraph above, we must have

$$\frac{T}{g} = \frac{T/2}{g - a_2}\,, \tag{2.72}$$

which gives $a_2 = g/2$. But a_2 is also the acceleration of the top mass, so our answer is $g/2$.

REMARKS: You can show that the relative acceleration of the second and third pulleys is $g/4$, and that of the third and fourth is $g/8$, etc. The acceleration of a mass far down in the system therefore equals $g(1/2 + 1/4 + 1/8 + \cdots) = g$, which makes intuitive sense.

Note that $T = 0$ also makes eq. (2.72) true. But this corresponds to putting a mass of zero at the end of a finite pulley system (see the following solution). ♣

Second Solution: Consider the following auxiliary problem.

Problem: Two setups are shown below in Fig. 2.29. The first contains a hanging mass m. The second contains a pulley, over which two masses, m_1 and m_2, hang. Let both supports have acceleration a_s downward. What should m be, in terms of m_1 and m_2, so that the tension in the top string is the same in both cases?

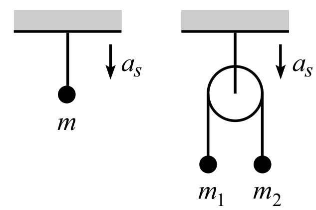


Figure 2.29

Answer: In the first case, we have

$$mg - T = ma_s. \tag{2.73}$$

In the second case, let a be the acceleration of m_2 relative to the support (with downward taken to be positive). Then we have

$$\begin{aligned} m_1 g - \frac{T}{2} &= m_1(a_s - a), \\ m_2 g - \frac{T}{2} &= m_2(a_s + a). \end{aligned} \tag{2.74}$$

Note that if we define $g' \equiv g - a_s$, then we may write the above three equations as

$$\begin{aligned} mg' &= T, \\ m_1 g' &= \frac{T}{2} - m_1 a, \\ m_2 g' &= \frac{T}{2} + m_2 a. \end{aligned} \tag{2.75}$$

Eliminating a from the last two of these equations gives $T = 4m_1 m_2 g'/(m_1 + m_2)$. Using this value of T in the first equation then gives

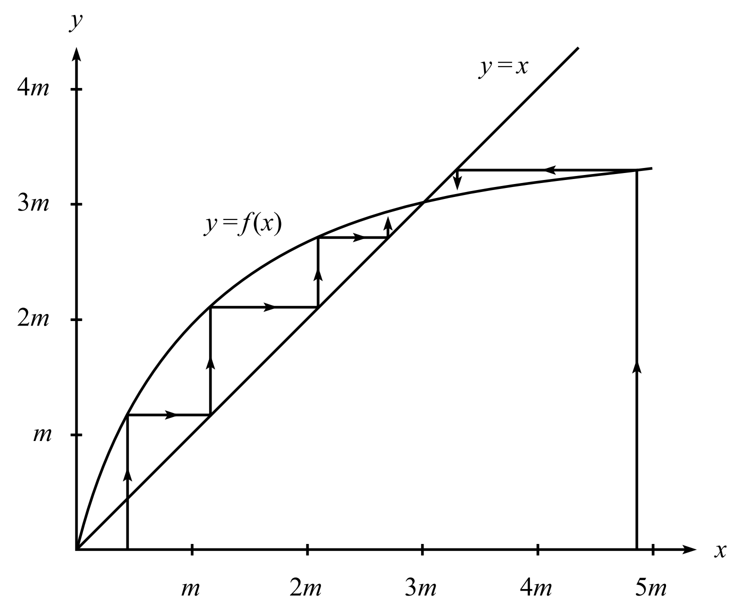
$$m = \frac{4m_1 m_2}{m_1 + m_2}\,. \tag{2.76}$$

Note that the value of a_s is irrelevant. We effectively have a fixed support in a world where the acceleration due to gravity is g' (see eqs. (2.75)), and the answer can't depend on g', by dimensional analysis. This auxiliary problem shows that the two-mass system in the second case may be equivalently treated as a mass m, given by eq. (2.76), as far as the upper string is concerned. ■

Now let's look at our infinite Atwood's machine. Assume that the system has N pulleys, where $N \to \infty$. Let the bottom mass be x. Then the auxiliary problem shows that the bottom two masses, m and x, may be treated as an effective mass $f(x)$, where

$$\begin{aligned} f(x) &= \frac{4mx}{m+x} \\ &= \frac{4x}{1+(x/m)} . \end{aligned} \tag{2.77}$$

We may then treat the combination of the mass $f(x)$ and the next m as an effective mass $f(f(x))$. These iterations may be repeated, until we finally have a mass m and a mass $f^{(N-1)}(x)$ hanging over the top pulley. So we must determine the behavior of $f^N(x)$, as $N \to \infty$. This behavior is clear if we look at the following plot of $f(x)$.



Note that $x = 3m$ is a fixed point of $f(x)$. That is, $f(3m) = 3m$. This plot shows that no matter what x we start with, the iterations approach $3m$ (unless we start at $x = 0$, in which case we remain there). These iterations are shown graphically by the directed lines in the plot. After reaching the value $f(x)$ on the curve, the line moves horizontally to the x value of $f(x)$, and then vertically to the value $f(f(x))$ on the curve, and so on.

Therefore, since $f^N(x) \to 3m$ as $N \to \infty$, our infinite Atwood's machine is equivalent to (as far as the top mass is concerned) just two masses, m and $3m$. You can then quickly show that that the acceleration of the top mass is $g/2$.

Note that as far as the support is concerned, the whole apparatus is equivalent to a mass $3m$. So $3mg$ is the upward force exerted by the support.

7. **Line of pulleys**

Let m be the common mass, and let T be the tension in the string. Let a be the acceleration of the end masses, and let a' be the acceleration of the other N masses, with upward taken to be positive. Note that these N accelerations are indeed all equal, because the same net force acts on all of the internal N masses, namely $2T$ upwards and mg downwards. The $F = ma$ equations for the end and internal masses are, respectively,

$$\begin{aligned} T - mg &= ma, \\ 2T - mg &= ma'. \end{aligned} \tag{2.78}$$

But the string has fixed length. Therefore,

$$N(2a') + a + a = 0. \tag{2.79}$$

Eliminating T from eqs. (2.78) gives $a' = 2a + g$. Combining this with eq. (2.79) then gives

$$a = -\frac{Ng}{2N+1}, \qquad \text{and} \qquad a' = \frac{g}{2N+1}. \tag{2.80}$$

REMARKS: For $N = 1$, we have $a = -g/3$ and $a' = g/3$. For larger N, a increases in magnitude and approaches $-g/2$ for $N \to \infty$, and a' decreases in magnitude and approaches zero for $N \to \infty$.

The signs of a and a' in eq. (2.80) may be surprising. You might think that if, say, $N = 100$, then these 100 masses will "“win” out over the two end masses, so that the N masses will fall. But this is not correct, because there are many ($2N$, in fact) tensions acting up on the N masses. They do *not* act like a mass Nm hanging below one pulley. In fact, two masses of $m/2$ on the ends will balance any number N of masses in the interior (with the help of the upward forces from the top row of pulleys). ♣

8. **Ring of pulleys**

Let T be the tension in the string. Then $F = ma$ for m_i gives

$$2T - m_i g = m_i a_i, \tag{2.81}$$

with upward taken to be positive. The a_i's are related by the fact that the string has fixed length, which implies that the sum of the displacements of all the masses is zero. In other words,

$$a_1 + a_2 + \cdots + a_N = 0. \tag{2.82}$$

If we divide eq. (2.81) by m_i, and then add the N such equations together, we obtain, using eq. (2.82),

$$2T\left(\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_N}\right) - Ng = 0. \tag{2.83}$$

Substituting this value for T into (2.81) gives

$$a_i = g\left(\frac{N}{m_i\left(\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_N}\right)} - 1\right). \tag{2.84}$$

A few special cases are: If all the masses are equal, then all $a_i = 0$. If $m_k = 0$ (and all the others are not zero), then $a_k = (N-1)g$, and all the other $a_i = -g$.

9. **Exponential force**

We are given $\ddot{x} = e^{-bt}$. Integrating this with respect to time gives $v(t) = -e^{-bt}/b + A$. Integrating again gives $x(t) = e^{-bt}/b^2 + At + B$. The initial condition, $v(0) = 0$, gives $-1/b + A = 0 \Longrightarrow A = 1/b$. And the initial condition, $x(0) = 0$, gives $1/b^2 + B = 0 \Longrightarrow B = -1/b^2$. Therefore,

$$x(t) = \frac{e^{-bt}}{b^2} + \frac{t}{b} - \frac{1}{b^2}\,. \tag{2.85}$$

Limits: For $t \to \infty$, v approaches $1/b$, and x approaches $t/b - 1/b^2$. We see that the particle eventually lags a distance $1/b^2$ behind another particle that started at the same position but with speed $v = 1/b$.

10. **Falling chain**

Let the density of the chain be ρ, and let $y(t)$ be the length hanging down through the hole at time t. Then the total mass is $\rho\ell$, and the mass hanging below the hole is ρy. The net downward force on the chain is $(\rho y)g$, so $F = ma$ gives

$$\rho g y = (\rho\ell)\ddot{y} \qquad \Longrightarrow \qquad \ddot{y} = \frac{g}{\ell}y. \tag{2.86}$$

At this point, there are two ways we can proceed:

First method: Since we have a function whose second derivative is proportional to itself, a good bet for the solution is an exponential function. And indeed, a quick check shows that the solution is

$$y(t) = Ae^{\alpha t} + Be^{-\alpha t}, \qquad \text{where} \quad \alpha \equiv \sqrt{\frac{g}{\ell}}\,. \tag{2.87}$$

Taking the derivative of this to obtain $\dot{y}(t)$, and using the given information that $\dot{y}(0) = 0$, we find $A = B$. Using $y(0) = y_0$, we then find $A = B = y_0/2$. So the length that hangs below the hole is

$$y(t) = \frac{y_0}{2}\left(e^{\alpha t} + e^{-\alpha t}\right) \equiv y_0 \cosh(\alpha t). \tag{2.88}$$

And the speed is

$$\dot{y}(t) = \frac{\alpha y_0}{2}\left(e^{\alpha t} - e^{-\alpha t}\right) \equiv \alpha y_0 \sinh(\alpha t). \tag{2.89}$$

The time T that satisfies $y(T) = \ell$ is given by $\ell = y_0 \cosh(\alpha T)$. Using $\sinh x = \sqrt{\cosh^2 x - 1}$, we find that the speed of the chain right when it loses contact with the table is

$$\dot{y}(T) = \alpha y_0 \sinh(\alpha T) = \alpha\sqrt{\ell^2 - y_0^2} \equiv \sqrt{g\ell}\sqrt{1 - \eta_0^2}, \tag{2.90}$$

where $\eta_0 \equiv y_0/\ell$ is the initial fraction hanging below the hole.

If $\eta_0 \approx 0$, then the speed at time T is $\sqrt{g\ell}$ (this quickly follows from conservation of energy, which is the subject of Chapter 4). Also, you can show that eq. (2.88) implies that T goes to infinity logarithmically as $\eta_0 \to 0$.

Second method: Write $\ddot{y}$ as $v\,dv/dy$ in eq. (2.86), and then separate variables and integrate to obtain

$$\int_0^v v\,dv = \alpha^2 \int_{y_0}^{y} y\,dy \qquad \Longrightarrow \qquad v^2 = \alpha^2(y^2 - y_0^2), \tag{2.91}$$

where $\alpha \equiv \sqrt{g/\ell}$. Now write v as dy/dt and separate variables again to obtain

$$\int_{y_0}^{y} \frac{dy}{\sqrt{y^2 - y_0^2}} = \alpha \int_0^t dt. \tag{2.92}$$

The integral on the left-hand side is $\cosh^{-1}(y/y_0)$, so we arrive at

$$y(t) = y_0 \cosh(\alpha t), \tag{2.93}$$

in agreement with eq. (2.88). The solution proceeds as above. However, an easier way to obtain the final speed with this method is to simply use the result for v in eq. (2.91). This tells us that the speed of the chain when it leaves the table (that is, when $y = \ell$) is $v = \alpha\sqrt{\ell^2 - y_0^2}$, in agreement with eq. (2.90).

11. **Circling around a pole**

Let F be the tension in the string. At the mass, the angle between the string and the radius of the dotted circle is $\theta = \sin^{-1}(r/R)$. In terms of θ, the radial and tangential $F = ma$ equations are

$$\begin{aligned} F\cos\theta &= \frac{mv^2}{R}, \qquad \text{and} \\ F\sin\theta &= m\dot{v}. \end{aligned} \tag{2.94}$$

Dividing these two equations gives $\tan\theta = (R\dot{v})/v^2$. Separating variables and integrating gives

$$\begin{aligned} \int_{v_0}^{v} \frac{dv}{v^2} &= \frac{\tan\theta}{R}\int_0^t dt \\ \Longrightarrow \quad \frac{1}{v_0} - \frac{1}{v} &= \frac{(\tan\theta)t}{R} \\ \Longrightarrow \quad v(t) &= \left(\frac{1}{v_0} - \frac{(\tan\theta)t}{R}\right)^{-1}. \end{aligned} \tag{2.95}$$

REMARK: Note that v becomes infinite when

$$t = T \equiv \frac{R}{v_0 \tan\theta}. \tag{2.96}$$

In other words, you can keep the mass moving in the desired circle only up to time T. After that, it is impossible. (Of course, it will become impossible, for all practical purposes, long before v becomes infinite.) The total distance, $d = \int v\,dt$, is infinite, because this integral diverges (barely, like a log) as t approaches T. ♣

12. **Throwing a beach ball**

On both the way up and the way down, the total force on the ball is

$$F = -mg - m\alpha v. \tag{2.97}$$

On the way up, v is positive, so the drag force points downward, as it should. And on the way down, v is negative, so the drag force points upward.

Our strategy for finding v_f will be to produce two different expressions for the maximum height, h, and then equate them. We'll find these two expressions by considering the upward and then the downward motion of the ball. In doing so, we will need to write the acceleration of the ball as $a = v\,dv/dy$.

For the upward motion, $F = ma$ gives

$$\begin{aligned} -mg - m\alpha v &= mv\frac{dv}{dy} \\ \Longrightarrow \qquad \int_0^h dy &= -\int_{v_0}^0 \frac{v\,dv}{g+\alpha v}\,. \end{aligned} \tag{2.98}$$

where we have taken advantage of the fact that we know that the speed of the ball at the top is zero. Writing $v/(g+\alpha v)$ as $[1 - g/(g+\alpha v)]/\alpha$, we may evaluate the integral to obtain

$$h = \frac{v_0}{\alpha} - \frac{g}{\alpha^2}\ln\left(1 + \frac{\alpha v_0}{g}\right). \tag{2.99}$$

Now let us consider the downward motion. Let v_f be the final speed, which is a positive quantity. The final velocity is then the negative quantity, $-v_f$. Using $F = ma$, we similarly obtain

$$\int_h^0 dy = -\int_0^{-v_f} \frac{v\,dv}{g+\alpha v}\,. \tag{2.100}$$

Performing the integration (or just replacing the v_0 in eq. (2.99) with $-v_f$) gives

$$h = -\frac{v_f}{\alpha} - \frac{g}{\alpha^2}\ln\left(1 - \frac{\alpha v_f}{g}\right). \tag{2.101}$$

Equating the expressions for h in eqs. (2.99) and (2.101) gives an implicit equation for v_f in terms of v_0,

$$v_0 + v_f = \frac{g}{\alpha}\ln\left(\frac{g+\alpha v_0}{g - \alpha v_f}\right). \tag{2.102}$$

Remarks: In the limit of small α (more precisely, in the limit $\alpha v_0/g \ll 1$), we can use $\ln(1+x) = x - x^2/2 + \cdots$ to obtain approximate values for h in eqs. (2.99) and (2.101). The results are, as expected,

$$h \approx \frac{v_0^2}{2g}\,, \qquad \text{and} \qquad h \approx \frac{v_f^2}{2g}\,. \tag{2.103}$$

We can also make approximations for large α (or large $\alpha v_0/g$). In this limit, the log term in eq. (2.99) is negligible, so we obtain $h \approx v_0/\alpha$. And eq. (2.101) gives $v_f \approx g/\alpha$, because the argument of the log must be very small in order to give a very large negative number, which is needed to produce a positive h on the left-hand side. There is no way to relate v_f and h is this limit, because the ball quickly reaches the terminal velocity of $-g/\alpha$ (which is the velocity that makes the net force equal to zero), independent of h. ♣

Let's now find the times it takes for the ball to go up and to go down. We'll present two methods for doing this.

First method: Let T_1 be the time for the upward path. If we write the acceleration of the ball as $a = dv/dt$, then $F = ma$ gives

$$\begin{aligned} -mg - m\alpha v &= m\frac{dv}{dt} \\ \Longrightarrow \qquad \int_0^{T_1} dt &= -\int_{v_0}^0 \frac{dv}{g+\alpha v}\,. \end{aligned} \tag{2.104}$$

$$T_1 = \frac{1}{\alpha}\ln\left(1 + \frac{\alpha v_0}{g}\right). \qquad (2.105)$$

In a similar manner, we find that the time T_2 for the downward path is

$$T_2 = -\frac{1}{\alpha}\ln\left(1 - \frac{\alpha v_f}{g}\right). \qquad (2.106)$$

Therefore,

$$T_1 + T_2 = \frac{1}{\alpha}\ln\left(\frac{g + \alpha v_0}{g - \alpha v_f}\right). \qquad (2.107)$$

Using eq. (2.102), we have

$$T_1 + T_2 = \frac{v_0 + v_f}{g}. \qquad (2.108)$$

This is shorter than the time in vacuum (namely $2v_0/g$) because $v_f < v_0$.

Second method: The very simple form of eq. (2.108) suggests that there is a cleaner way to calculate the total time of flight. And indeed, if we integrate $m\,dv/dt = -mg - m\alpha v$ with respect to time on the way up, we obtain $-v_0 = -gT_1 - \alpha h$ (because $\int v\,dt = h$). Likewise, if we integrate $m\,dv/dt = -mg - m\alpha v$ with respect to time on the way down, we obtain $-v_f = -gT_2 + \alpha h$ (because $\int v\,dt = -h$). Adding these two results gives eq. (2.108). This procedure only works, of course, because the drag force is proportional to v.

REMARKS: The fact that the time here is shorter than the time in vacuum isn't obvious. On one hand, the ball doesn't travel as high in air as it would in vacuum (so you might think that $T_1 + T_2 < 2v_0/g$). But on the other hand, the ball moves slower in air (so you might think that $T_1 + T_2 > 2v_0/g$). It isn't obvious which effect wins, without doing a calculation.

For any α, you can use eq. (2.105) to show that $T_1 < v_0/g$. But T_2 is harder to get a handle on, because it is given in terms of v_f. But in the limit of large α, the ball quickly reaches terminal velocity, so we have $T_2 \approx h/v_f \approx (v_0/\alpha)/(g/\alpha) = v_0/g$. Interestingly, this is the same as the downward (and upward) time for a ball thrown in vacuum. ♣

13. **Balancing a pencil**

(a) The component of gravity in the tangential direction is $mg\sin\theta \approx mg\theta$. Therefore, the tangential $F = ma$ equation is $mg\theta = m\ell\ddot{\theta}$, which may be written as $\ddot{\theta} = (g/\ell)\theta$. The general solution to this equation is[20]

$$\theta(t) = Ae^{t/\tau} + Be^{-t/\tau}, \qquad \text{where } \tau \equiv \sqrt{\ell/g}. \qquad (2.109)$$

The constants A and B are found from the initial conditions,

$$\begin{aligned} \theta(0) = \theta_0 \quad &\Longrightarrow \quad A + B = \theta_0, \\ \dot{\theta}(0) = \omega_0 \quad &\Longrightarrow \quad (A - B)/\tau = \omega_0. \end{aligned} \qquad (2.110)$$

Solving for A and B, and then plugging them into eq. (2.109) gives

$$\theta(t) = \frac{1}{2}(\theta_0 + \omega_0\tau)\,e^{t/\tau} + \frac{1}{2}(\theta_0 - \omega_0\tau)\,e^{-t/\tau}. \qquad (2.111)$$

[20] If you want, you can derive this by separating variables and integrating. The solution is essentially the same as in the second method presented in the solution to Problem 10.

(b) The constants A and B will turn out to be small (they will each be of order $\sqrt{\hbar}$). Therefore, by the time the positive exponential has increased enough to make θ of order 1, the negative exponential will have become negligible. We will therefore ignore the latter term from here on. In other words,

$$\theta(t) \approx \frac{1}{2}\left(\theta_0 + \omega_0\tau\right) e^{t/\tau}. \tag{2.112}$$

The goal is to keep θ small for as long as possible. Hence, we want to minimize the coefficient of the exponential, subject to the uncertainty-principle constraint, $(\ell\theta_0)(m\ell\omega_0) \geq \hbar$. This constraint gives $\omega_0 \geq \hbar/(m\ell^2\theta_0)$. Therefore,

$$\theta(t) \geq \frac{1}{2}\left(\theta_0 + \frac{\hbar\tau}{m\ell^2\theta_0}\right) e^{t/\tau}. \tag{2.113}$$

Taking the derivative with respect to θ_0 to minimize the coefficient, we find that the minimum value occurs at

$$\theta_0 = \sqrt{\frac{\hbar\tau}{m\ell^2}}\,. \tag{2.114}$$

Substituting this back into eq. (2.113) gives

$$\theta(t) \geq \sqrt{\frac{\hbar\tau}{m\ell^2}}\; e^{t/\tau}. \tag{2.115}$$

Setting $\theta \approx 1$, and then solving for t gives (using $\tau \equiv \sqrt{\ell/g}$)

$$t \leq \frac{1}{4}\sqrt{\frac{\ell}{g}}\ln\left(\frac{m^2\ell^3 g}{\hbar^2}\right). \tag{2.116}$$

With the given values, $m = 0.01\,\text{kg}$ and $\ell = 0.1\,\text{m}$, along with $g = 10\,\text{m/s}^2$ and $\hbar = 1.06\cdot 10^{-34}\,\text{Js}$, we obtain

$$t \leq \frac{1}{4}(0.1\,\text{s})\ln(9\cdot 10^{61}) \approx 3.5\,\text{s}. \tag{2.117}$$

No matter how clever you are, and no matter how much money you spend on the newest, cutting-edge pencil-balancing equipment, you can never get a pencil to balance for more than about four seconds.

REMARKS: This smallness of this answer is quite amazing. It is remarkable that a quantum effect on a macroscopic object can produce an everyday value for a time scale. Basically, the point here is that the fast exponential growth of θ (which gives rise to the log in the final result for t) wins out over the smallness of $\hbar$, and produces a result for t of order 1. When push comes to shove, exponential effects always win.

The above value for t depends strongly on ℓ and g, through the $\sqrt{\ell/g}$ term. But the dependence on m, ℓ, and g in the log term is very weak. If m were increased by a factor of 1000, for example, the result for t would increase by only about 10%. Note that this implies that any factors of order 1 that we neglected throughout this problem are completely irrelevant. They will appear in the argument of the log term, and will thus have negligible effect.

Note that dimensional analysis, which is generally a very powerful tool, won't get you too far in this problem. The quantity $\sqrt{\ell/g}$ has dimensions of time, and the quantity

$\eta \equiv m^2\ell^3 g/\hbar^2$ is dimensionless (it is the only such quantity), so the balancing time must take the form,

$$t \approx \sqrt{\frac{\ell}{g}}\, f(\eta), \tag{2.118}$$

where f is some function. If the leading term in f were a power (even, for example, a square root), then t would essentially be infinite ($t \approx 10^{30}$ s for the square root). But f in fact turns out to be a log (which you can't determine without solving the problem), which completely cancels out the smallness of $\hbar$, reducing an essentially infinite time down to a few seconds. ♣

14. **Throwing a ball from a cliff**

Let the inclination angle be θ. Then the horizontal speed is $v_x = v\cos\theta$, and the initial vertical speed is $v_y = v\sin\theta$. The time it takes for the ball to hit the ground is given by $h + (v\sin\theta)t - gt^2/2 = 0$. Therefore,

$$t = \frac{v}{g}\left(\sin\theta + \sqrt{\sin^2\theta + \beta}\right), \qquad \text{where} \quad \beta \equiv \frac{2gh}{v^2}\,. \tag{2.119}$$

(The "−" solution for t from the quadratic formula corresponds to the ball being thrown backwards down through the cliff.) The horizontal distance traveled is $d = (v\cos\theta)t$, which gives

$$d = \frac{v^2}{g}\cos\theta\left(\sin\theta + \sqrt{\sin^2\theta + \beta}\right). \tag{2.120}$$

We want to maximize this function of θ. Taking the derivative, multiplying through by $\sqrt{\sin^2\theta + \beta}$, and setting the result equal to zero, gives

$$(\cos^2\theta - \sin^2\theta)\sqrt{\sin^2\theta + \beta} = \sin\theta\big(\beta - (\cos^2\theta - \sin^2\theta)\big). \tag{2.121}$$

Using $\cos^2\theta = 1 - \sin^2\theta$, and then squaring and simplifying this equation, gives an optimal angle of

$$\sin\theta_{\max} = \frac{1}{\sqrt{2+\beta}} \equiv \frac{1}{\sqrt{2 + 2gh/v^2}}\,. \tag{2.122}$$

Plugging this into eq. (2.120), and simplifying, gives a maximum distance of

$$d_{\max} = \frac{v^2}{g}\sqrt{1+\beta} \equiv \frac{v^2}{g}\sqrt{1 + \frac{2gh}{v^2}}\,. \tag{2.123}$$

Remarks: If $h = 0$, then we obtain $\theta_{\max} = \pi/4$ and $d_{\max} = v^2/g$, in agreement with the example in Section 2.4. If $h \to \infty$ or $v \to 0$, then $\theta \approx 0$, which makes sense.

If we make use of conservation of energy (discussed in Chapter 4), it turns out that the final speed of the ball when it hits the ground is $v_f = \sqrt{v^2 + 2gh}$. The maximum distance in eq. (2.123) may therefore be written as (with $v_i \equiv v$ being the initial speed)

$$d_{\max} = \frac{v_i v_f}{g}\,. \tag{2.124}$$

Note that this is symmetric in v_i and v_f, as it must be, because we could imagine the trajectory running backwards. Also, it equals zero if v_i is zero, as it should. We can also write the angle θ in eq. (2.122) in terms of v_f (instead of h). You can show that the result is $\tan\theta = v_i/v_f$. You can further show that this implies that the initial and final velocities are perpendicular to each other. The simplicity of all these results suggests that there is an easier way to derive them, but I have no clue what it is. ♣

15. **Redirected motion**

First Solution: We will use the results of Problem 14, namely eqs. (2.123) and (2.122), which say that an object projected from height y at speed v travels a maximum horizontal distance of

$$d_{\max} = \frac{v^2}{g}\sqrt{1 + \frac{2gy}{v^2}}\,, \tag{2.125}$$

and the optimal angle yielding this distance is

$$\sin\theta = \frac{1}{\sqrt{2 + 2gy/v^2}}\,. \tag{2.126}$$

In the problem at hand, the object is dropped from a height h, so conservation of energy (or integration of $mv\,dv/dy = -mg$) says that the speed at height y is

$$v = \sqrt{2g(h-y)}. \tag{2.127}$$

Plugging this into eq. (2.125) shows that the maximum horizontal distance, as a function of y, is

$$d_{\max}(y) = 2\sqrt{h(h-y)}. \tag{2.128}$$

This is maximum when $y = 0$, in which case the distance is $d_{\max} = 2h$. Eq. (2.126) then gives the associated optimal angle as $\theta = 45^\circ$.

Second Solution: Assume that the greatest distance, d_0, is obtained when $y = y_0$ and $\theta = \theta_0$. And let the speed at y_0 be v_0. We will show that y_0 must be 0. We will do this by assuming that $y_0 \neq 0$ and explicitly constructing a situation that yields a greater distance.

Consider the situation where the ball falls all the way down to $y = 0$ and then bounces up at an angle such that when it reaches the height y_0, it is traveling at an angle θ_0 with respect to the horizontal. When it reaches the height y_0, the ball will have speed v_0 (by conservation of energy), so it will travel a horizontal distance d_0 from this point. But the ball already traveled a nonzero horizontal distance on its way up to the height y_0. We have therefore constructed a situation that yields a distance greater than d_0. Hence, the optimal setup must have $y_0 = 0$. Therefore, the maximum distance is obtained when $y = 0$, in which case the example in Section 2.4 says that the optimal angle is $\theta = 45^\circ$.

If we want the ball to go even further, we can simply dig a (wide enough) hole in the ground and have the ball bounce from the bottom of the hole.

16. **Maximum trajectory length**

Let θ be the angle at which the ball is thrown. Then the coordinates are given by $x = (v\cos\theta)t$ and $y = (v\sin\theta)t - gt^2/2$. The ball reaches its maximum height at $t = v\sin\theta/g$, so the length of the trajectory is

$$\begin{aligned} L &= 2\int_0^{v\sin\theta/g} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt \\ &= 2\int_0^{v\sin\theta/g} \sqrt{(v\cos\theta)^2 + (v\sin\theta - gt)^2}\, dt \\ &= 2v\cos\theta \int_0^{v\sin\theta/g} \sqrt{1 + \left(\tan\theta - \frac{gt}{v\cos\theta}\right)^2}\, dt. \end{aligned} \tag{2.129}$$

Letting $z \equiv \tan\theta - gt/v\cos\theta$, we obtain

$$L = -\frac{2v^2\cos^2\theta}{g}\int_{\tan\theta}^{0}\sqrt{1+z^2}\,dz. \tag{2.130}$$

We can either look up this integral, or we can derive it by making a $z \equiv \sinh\alpha$ substitution. The result is

$$\begin{aligned} L &= \frac{2v^2\cos^2\theta}{g}\cdot\frac{1}{2}\left(z\sqrt{1+z^2}+\ln\left(z+\sqrt{1+z^2}\right)\right)\Bigg|_0^{\tan\theta} \\ &= \frac{v^2}{g}\left(\sin\theta+\cos^2\theta\,\ln\left(\frac{\sin\theta+1}{\cos\theta}\right)\right). \end{aligned} \tag{2.131}$$

As a double-check, you can verify that $L=0$ when $\theta=0$, and $L=v^2/g$ when $\theta=90^\circ$. Taking the derivative of eq. (2.131) to find the maximum, we obtain

$$0 = \cos\theta - 2\cos\theta\sin\theta\ln\left(\frac{1+\sin\theta}{\cos\theta}\right)+\cos^2\theta\left(\frac{\cos\theta}{1+\sin\theta}\right)\frac{\cos^2\theta+(1+\sin\theta)\sin\theta}{\cos^2\theta}. \tag{2.132}$$

This reduces to

$$1 = \sin\theta\ln\left(\frac{1+\sin\theta}{\cos\theta}\right). \tag{2.133}$$

Finally, you can show numerically that the solution for θ is $\theta_0 \approx 56.5^\circ$.

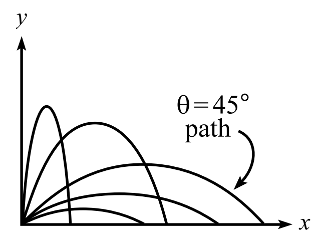


Figure 2.30

REMARK: A few possible trajectories are shown Fig. 2.30. Since it is well known that $\theta = 45^\circ$ provides the maximum *horizontal* distance, it follows from the figure that the θ_0 yielding the arc of maximum *length* must satisfy $\theta_0 \geq 45^\circ$. The exact angle, however, requires the above detailed calculation. ♣

17. **Maximum trajectory area**

Let θ be the angle at which the ball is thrown. Then the coordinates are given by $x=(v\cos\theta)t$ and $y=(v\sin\theta)t-gt^2/2$. The total time in the air is $2(v\sin\theta)/g$, so the area under the trajectory is

$$\begin{aligned} A &= \int_0^{x_{\max}} y\,dx \\ &= \int_0^{2v\sin\theta/g}\left((v\sin\theta)t-\frac{gt^2}{2}\right)v\cos\theta\,dt \\ &= \frac{2v^4}{3g^2}\sin^3\theta\cos\theta. \end{aligned} \tag{2.134}$$

Taking the derivative, we find that the maximum occurs when $\tan\theta=\sqrt{3}$, that is, when

$$\theta = 60^\circ. \tag{2.135}$$

The maximum area is then $A_{\max}=\sqrt{3}v^4/8g^2$. Note that by dimensional analysis, we know that the area, which has dimensions of distance squared, must be proportional to v^4/g^2.

18. **Bouncing ball**

The ball travels $2h$ during the first up-and-down journey. It travels $2hf$ during the second, then $2hf^2$ during the third, and so on. Therefore, the total distance traveled is

$$\begin{aligned} D &= 2h(1 + f + f^2 + f^3 + \cdots) \\ &= \frac{2h}{1-f} \, . \end{aligned} \tag{2.136}$$

The time it takes to fall down during the first up-and-down is obtained from $h = gt^2/2$. Therefore, the time for the first up-and-down equals $2t = 2\sqrt{2h/g}$. Likewise, the time for the second up-and-down equals $2\sqrt{2(hf)/g}$. Each successive up-and-down time decreases by a factor of $\sqrt{f}$, so the total time is

$$\begin{aligned} T &= 2\sqrt{\frac{2h}{g}}\left(1 + f^{1/2} + f^1 + f^{3/2} + \cdots\right) \\ &= 2\sqrt{\frac{2h}{g}} \cdot \frac{1}{1-\sqrt{f}} \, . \end{aligned} \tag{2.137}$$

The average speed equals

$$\frac{D}{T} = \frac{\sqrt{gh/2}}{1+\sqrt{f}} \, . \tag{2.138}$$

Remark: The average speed for $f \approx 1$ is roughly half of the average speed for $f \approx 0$. This may seem somewhat counterintuitive, because in the $f \approx 0$ case the ball slows down far more quickly than in the $f \approx 1$ case. But the $f \approx 0$ case consists of essentially only one bounce, and the average speed for that one bounce is the largest of any bounce. Both D and T are smaller for $f \approx 0$ than for $f \approx 1$, but T is smaller by a larger factor. ♣

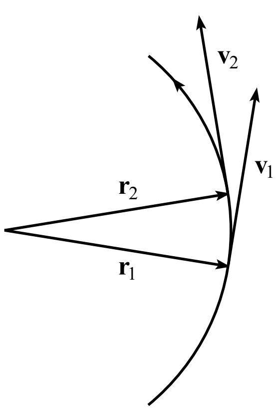


Figure 2.31

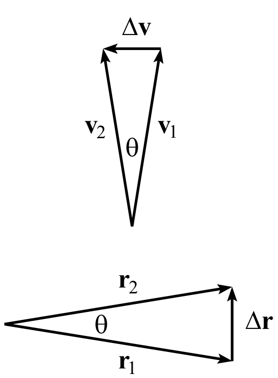


Figure 2.32

19. **Centripetal acceleration**

The position and velocity vectors at two nearby times are shown in Fig. 2.31. Their differences, $\Delta\mathbf{r} \equiv \mathbf{r}_2 - \mathbf{r}_1$ and $\Delta\mathbf{v} \equiv \mathbf{v}_2 - \mathbf{v}_1$, are shown in Fig. 2.32. The angle between the $\mathbf{v}$'s is the same as the angle between the $\mathbf{r}$'s, because each $\mathbf{v}$ makes a right angle with the corresponding $\mathbf{r}$. The triangles in Fig. 2.32 are therefore similar, so we have

$$\frac{|\Delta\mathbf{v}|}{v} = \frac{|\Delta\mathbf{r}|}{r} \, , \tag{2.139}$$

where $r \equiv |\mathbf{r}|$ and $v \equiv |\mathbf{v}|$. Dividing eq. (2.139) through by Δt gives

$$\frac{1}{v}\left|\frac{\Delta\mathbf{v}}{\Delta t}\right| = \frac{1}{r}\left|\frac{\Delta\mathbf{r}}{\Delta t}\right| \qquad \Longrightarrow \qquad \frac{|\mathbf{a}|}{v} = \frac{|\mathbf{v}|}{r} \qquad \Longrightarrow \qquad a = \frac{v^2}{r} \, . \tag{2.140}$$

We have assumed that Δt is infinitesimal here, which allows us to get rid of the Δ's in favor of instantaneous quantities.

20. **Free particle**

For zero force, eqs. (2.52) give

$$\begin{aligned} \ddot{r} &= r\dot{\theta}^2, \\ r\ddot{\theta} &= -2\dot{r}\dot{\theta}. \end{aligned} \tag{2.141}$$

Separating variables in the second equation and integrating yields

$$\int \frac{\ddot{\theta}}{\dot{\theta}} = -\int \frac{2\dot{r}}{r} \quad \Longrightarrow \quad \ln\dot{\theta} = -2\ln r + C \quad \Longrightarrow \quad \dot{\theta} = \frac{D}{r^2}\,, \tag{2.142}$$

where $D = e^C$ is a constant of integration, determined by the initial conditions.[21] Substituting this value of $\dot{\theta}$ into the first of eqs. (2.141), and then multiplying both sides by $\dot{r}$ and integrating, gives

$$\ddot{r} = r\left(\frac{D}{r^2}\right)^2 \quad \Longrightarrow \quad \int \ddot{r}\dot{r} = D^2 \int \frac{\dot{r}}{r^3} \quad \Longrightarrow \quad \frac{\dot{r}^2}{2} = -\frac{D^2}{2r^2} + E. \tag{2.143}$$

We want $\dot{r} = 0$ when $r = r_0$, which implies that $E = D^2/2r_0^2$. Therefore,

$$\dot{r} = V\sqrt{1 - \frac{r_0^2}{r^2}}\,, \tag{2.144}$$

where $V \equiv D/r_0$. Separating variables and integrating gives

$$\int \frac{r\dot{r}}{\sqrt{r^2 - r_0^2}} = V \quad \Longrightarrow \quad \sqrt{r^2 - r_0^2} = Vt \quad \Longrightarrow \quad r = \sqrt{r_0^2 + (Vt)^2}\,, \tag{2.145}$$

where the constant of integration is zero, because we have chosen $t = 0$ to correspond with $r = r_0$. Plugging this value for r into the $\dot{\theta} = D/r^2 \equiv Vr_0/r^2$ result in eq. (2.142) gives

$$\int d\theta = \int \frac{Vr_0\,dt}{r_0^2 + (Vt)^2} \quad \Longrightarrow \quad \theta = \tan^{-1}\left(\frac{Vt}{r_0}\right) \quad \Longrightarrow \quad \cos\theta = \frac{r_0}{\sqrt{r_0^2 + (Vt)^2}}\,. \tag{2.146}$$

Finally, combining this with the result for r in eq. (2.145) gives $\cos\theta = r_0/r$, as desired.

21. **A force $F_\theta = \dot{r}\dot{\theta}$**

With the given force, eqs. (2.52) become

$$\begin{aligned} 0 &= m(\ddot{r} - r\dot{\theta}^2), \\ m\dot{r}\dot{\theta} &= m(r\ddot{\theta} + 2\dot{r}\dot{\theta}). \end{aligned} \tag{2.147}$$

The second of these equations gives $-\dot{r}\dot{\theta} = r\ddot{\theta}$. Therefore,

$$\int \frac{\ddot{\theta}}{\dot{\theta}} = -\int \frac{\dot{r}}{r} \quad \Longrightarrow \quad \ln\dot{\theta} = -\ln r + C \quad \Longrightarrow \quad \dot{\theta} = \frac{D}{r}\,, \tag{2.148}$$

where $D = e^C$ is a constant of integration, determined by the initial conditions. Substituting this value of $\dot{\theta}$ into the first of eqs. (2.147), and then multiplying both sides by $\dot{r}$ and integrating, gives

$$\ddot{r} = r\left(\frac{D}{r}\right)^2 \quad \Longrightarrow \quad \int \ddot{r}\dot{r} = D^2 \int \frac{\dot{r}}{r} \quad \Longrightarrow \quad \frac{\dot{r}^2}{2} = D^2 \ln r + E. \tag{2.149}$$

Therefore,

$$\dot{r} = \sqrt{A\ln r + B}\,, \tag{2.150}$$

where $A \equiv 2D^2$ and $B \equiv 2E$.

[21] The statement that $r^2\dot{\theta}$ is constant is simply the statement of conservation of angular momentum, because $r^2\dot{\theta} = r(r\dot{\theta}) = rv_\theta$. More on this in Chapters 6 and 7.