Chapter 8

Angular Momentum, Part II (General $\hat{L}$)

In the Chapter 7, we discussed situations where the direction of the vector **L** remains constant, and only its magnitude changes. In this chapter, we will look at the more complicated situations where the direction of **L** is allowed to change. The vector nature of **L** will prove to be vital, and we will arrive at all sorts of strange results for spinning tops and such things.

This chapter is rather long, alas. The first three sections consist of general theory, and then in Section 8.4 we start solving some actual problems.

8.1 Preliminaries concerning rotations

8.1.1 The form of general motion

Before getting started, we should make sure we're all on the same page concerning a few important things about rotations. Because rotations generally involve three dimensions, they can often be hard to visualize. A rough drawing on a piece of paper might not do the trick. For this reason, this topic is one of the more difficult ones in this book.

The next few pages consist of some definitions and helpful theorems. This first theorem describes the form of general motion. You might consider it obvious, but let's prove it anyway.

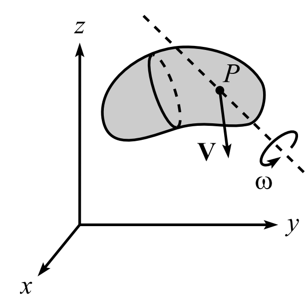

Figure 8.1

Theorem 8.1 *Consider a rigid body undergoing arbitrary motion. Pick any point P in the body. Then at any instant (see Fig. 8.1), the motion of the body may be written as the sum of the translational motion of P, plus a rotation around some axis, $\boldsymbol{\omega}$, through P (the axis $\boldsymbol{\omega}$ may change with time).*[1]

Proof: The motion of the body may be written as the sum of the translational motion of P, plus some other motion relative to P (this is true because relative

[1] In other words, what we mean here is that a person at rest with respect to a frame whose origin is P, and whose axes are parallel to the fixed-frame axes, will see the body undergoing a rotation around some axis through P.

coordinates are additive quantities). We must show that this latter motion is simply a rotation. This seems quite plausible, and it holds because the body is rigid; that is, all points keep the same relative distances. (If the body weren't rigid, then this theorem wouldn't be true.)

To be rigorous, consider a sphere fixed in the body, centered at P. The motion of the body is completely determined by the motion of the points on this sphere, so we need only examine what happens to the sphere. And because we are looking at motion relative to P, we have reduced the problem to the following: In what manner can a rigid sphere transform into itself? We claim that *any such transformation requires that two points end up where they started.*[2]

If this claim is true, then we are done, because for an infinitesimal transformation, a given point moves in only one direction (since there is no time to do any bending). So a point that ends up where it started must have always been fixed. Therefore, the diameter joining the two fixed points remains stationary (because distances are preserved), and we are left with a rotation around this axis.

This claim is quite believable, but nevertheless tricky to prove. I can't resist making you think about it, so I've left it as a problem (Problem 1). Try to solve it on your own. ∎

We will invoke this theorem repeatedly in this chapter (often without bothering to say so). Note that it is required that P be a point in the body, since we used the fact that P keeps the same distances from other points in the body.

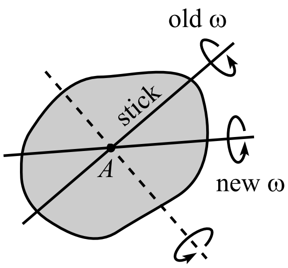

Figure 8.2

Remark: A situation where our theorem is not so obvious is the following. Consider an object rotating around a fixed axis, the stick shown in Fig. 8.2. In this case, $\boldsymbol{\omega}$ simply points along the stick. But now imagine grabbing the stick and rotating it around some other axis (the dotted line shown). It is not immediately obvious that the resulting motion is (instantaneously) a rotation around some new axis through A. But indeed it is. (We'll be quantitative about this in the "Rotating Sphere" example near the end of this section.) ♣

8.1.2 The angular velocity vector

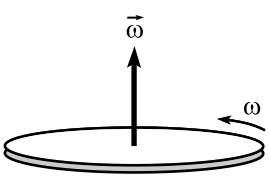

Figure 8.3

It is extremely useful to introduce the angular velocity vector, $\boldsymbol{\omega}$, which is defined to point along the axis of rotation, with a magnitude equal to the angular speed. The choice of the two possible directions is given by the right-hand rule. (Curl your right-hand fingers in the direction of the spin, and your thumb will point in the direction of $\boldsymbol{\omega}$.) For example, a spinning record has $\boldsymbol{\omega}$ perpendicular to the record, through its center (as shown in Fig. 8.3), with magnitude equal to the angular speed, ω.

Remark: You could, of course, break the mold and use the left-hand rule, as long as you use it consistently. The direction of $\vec{\omega}$ would be opposite, but that doesn't matter, because $\vec{\omega}$ isn't really physical. Any physical result (for example, the velocity of a particle,

[2]This claim is actually true for *any* transformation of a rigid sphere into itself, but for the present purposes we are concerned only with infinitesimal transformations (because we are only looking at what happens at a given instant in time).

or the force on it) will come out the same, independent of which hand you (consistently) use.

> When studying vectors in school,
> You'll use your right hand as a tool.
> But look in a mirror,
> And then you'll see clearer,
> You can just use the left-handed rule. ♣

The points on the axis of rotation are the ones that (instantaneously) do not move. Of, course, the direction of $\boldsymbol{\omega}$ may change over time, so the points that were formerly on $\boldsymbol{\omega}$ may now be moving.

Remark: The fact that we can specify a rotation by specifying a vector $\boldsymbol{\omega}$ is a peculiarity to three dimensions. If we lived in one dimension, then there would be no such thing as a rotation. If we lived in two dimensions, then all rotations would take place in that plane, so we could label a rotation by simply giving its speed, ω. In three dimensions, rotations take place in $\binom{3}{2} = 3$ independent planes. And we choose to label these, for convenience, by the directions orthogonal to these planes, and by the angular speed in each plane. If we lived in four dimensions, then rotations could take place in $\binom{4}{2} = 6$ planes, so we would have to label a rotation by giving 6 planes and 6 angular speeds. Note that a vector (which has four components in four dimensions) would not do the trick here. ♣

In addition to specifying the points that are instantaneously motionless, $\boldsymbol{\omega}$ also easily produces the velocity of any point in the rotating object. Consider the case where the axis of rotation passes through the origin (which we will generally assume to be the case in this chapter, unless otherwise stated). Then we have the following theorem.

Theorem 8.2 *Given an object rotating with angular velocity $\boldsymbol{\omega}$, the velocity of any point in the object is given by (with $\mathbf{r}$ being the position of the point)*

$$\boxed{\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}}. \tag{8.1}$$

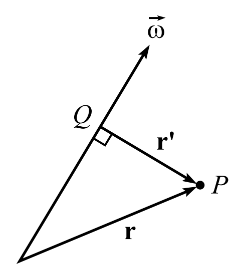

Figure 8.4

Proof: Drop a perpendicular from the point in question (call it P) to the axis $\boldsymbol{\omega}$ (call the point there Q). Let $\mathbf{r}'$ be the vector from Q to P (see Fig. 8.4). From the properties of the cross product, $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ is orthogonal to $\boldsymbol{\omega}$, $\mathbf{r}$, and also $\mathbf{r}'$ (since $\mathbf{r}'$ is a linear combination of $\boldsymbol{\omega}$ and $\mathbf{r}$). Therefore, the direction of $\mathbf{v}$ is correct (it lies in a plane perpendicular to $\boldsymbol{\omega}$, and is also perpendicular to $\mathbf{r}'$, so it describes circular motion around the axis $\boldsymbol{\omega}$; also, by the right-hand rule, it points in the proper orientation around $\boldsymbol{\omega}$). And since

$$|\mathbf{v}| = |\boldsymbol{\omega}||\mathbf{r}| \sin\theta = \omega r', \tag{8.2}$$

which is the speed of the circular motion around $\boldsymbol{\omega}$, we see that $\mathbf{v}$ has the correct magnitude. So $\mathbf{v}$ is indeed the correct velocity vector. ∎

Note that if we have the special case where P lies along ω, then $\mathbf{r}$ is parallel to $\boldsymbol{\omega}$, and so the cross product gives a zero result for $\mathbf{v}$, as it should.

Eq. (8.1) is extremely useful and will be applied repeatedly in this chapter. Even if it's hard to visualize what's going on with a given rotation, all you have to do to find the speed of any given point is calculate the cross product $\boldsymbol{\omega} \times \mathbf{r}$.

Conversely, if the speed of every point in a moving body is given by $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, then the body is undergoing a rotation with angular velocity $\boldsymbol{\omega}$ (because all points on the axis $\boldsymbol{\omega}$ are motionless, and all other points move with the proper speed for this rotation).

A very nice thing about angular velocities is that they simply add. Stated more precisely, we have the following theorem.

Theorem 8.3 *Let coordinate systems S_1, S_2, and S_3 have the same origin. Let S_1 rotate with angular velocity $\boldsymbol{\omega}_{1,2}$ with respect to S_2. Let S_2 rotate with angular velocity $\boldsymbol{\omega}_{2,3}$ with respect to S_3. Then S_1 rotates (instantaneously) with angular velocity*

$$\boldsymbol{\omega}_{1,3} = \boldsymbol{\omega}_{1,2} + \boldsymbol{\omega}_{2,3} \tag{8.3}$$

with respect to S_3.

Proof: If $\boldsymbol{\omega}_{1,2}$ and $\boldsymbol{\omega}_{2,3}$ point in the same direction, then the theorem is clear; the angular speeds just add. If, however, they don't point in the same direction, then things are a little harder to visualize. But we can prove the theorem by simply making abundant use of the definition of $\boldsymbol{\omega}$.

Pick a point P_1 at rest in S_1. Let $\mathbf{r}$ be the vector from the origin to P_1. The velocity of P_1 (relative to a very close point P_2 at rest in S_2) due to the rotation about $\boldsymbol{\omega}_{1,2}$ is $\mathbf{V}_{P_1P_2} = \boldsymbol{\omega}_{1,2} \times \mathbf{r}$. The velocity of P_2 (relative to a very close point P_3 at rest in S_3) due to the rotation about $\boldsymbol{\omega}_{2,3}$ is $\mathbf{V}_{P_2P_3} = \boldsymbol{\omega}_{2,3} \times \mathbf{r}$ (because P_2 is also located essentially at position $\mathbf{r}$). Therefore, the velocity of P_1 (relative to P_3) is $\mathbf{V}_{P_1P_2} + \mathbf{V}_{P_2P_3} = (\boldsymbol{\omega}_{1,2} + \boldsymbol{\omega}_{2,3}) \times \mathbf{r}$. This holds for any point P_1 at rest in S_1. So the frame S_1 rotates with angular velocity $(\boldsymbol{\omega}_{1,2} + \boldsymbol{\omega}_{2,3})$ with respect to S_3. ■

Note that if $\boldsymbol{\omega}_{1,2}$ is constant in S_2, then the vector $\boldsymbol{\omega}_{1,3} = \boldsymbol{\omega}_{1,2} + \boldsymbol{\omega}_{2,3}$ will change with respect to S_3 as time goes by (because $\boldsymbol{\omega}_{1,2}$, which is fixed in S_2, is changing with respect to S_3). But at any instant, $\boldsymbol{\omega}_{1,3}$ may be obtained by simply adding the present values of $\boldsymbol{\omega}_{1,2}$ and $\boldsymbol{\omega}_{2,3}$. Consider the following example.

Example (Rotating sphere): A sphere rotates with angular speed ω_3 around a stick that initially points in the $\hat{\mathbf{z}}$ direction. You grab the stick and rotate it around the $\hat{\mathbf{y}}$-axis with angular speed ω_2. What is the angular velocity of the sphere, with respect to the lab frame, as time goes by?

Solution: In the language of Theorem 8.3, the sphere defines the S_1 frame; the stick and the $\hat{\mathbf{y}}$-axis define the S_2 frame; and the lab frame is the S_3 frame. The instant after you grab the stick, we are given that $\boldsymbol{\omega}_{1,2} = \omega_3\hat{\mathbf{z}}$, and $\boldsymbol{\omega}_{2,3} = \omega_2\hat{\mathbf{y}}$. Therefore, the angular velocity of the sphere with respect to the lab frame is $\boldsymbol{\omega}_{1,3} = \boldsymbol{\omega}_{1,2} + \boldsymbol{\omega}_{2,3} = \omega_3\hat{\mathbf{z}} + \omega_2\hat{\mathbf{y}}$. This is shown is Fig. 8.5. As time goes by, the stick (and hence $\boldsymbol{\omega}_{1,2}$) rotates around the $\mathbf{y}$ axis, so $\boldsymbol{\omega}_{1,3} = \boldsymbol{\omega}_{1,2} + \boldsymbol{\omega}_{2,3}$ traces out a cone around the $\mathbf{y}$ axis, as shown.

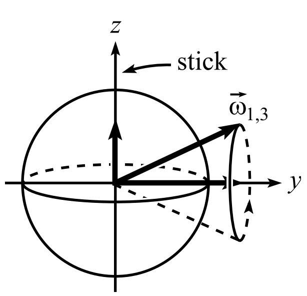

Figure 8.5

REMARK: Note the different behavior of $\vec{\omega}_{1,3}$ for a slightly different statement of the problem: Let the sphere initially rotate with angular velocity $\omega_2\hat{\mathbf{y}}$. Grab the axis (which points in the $\hat{\mathbf{y}}$ direction) and rotate it with angular velocity $\omega_3\hat{\mathbf{z}}$. For this situation, $\vec{\omega}_{1,3}$ initially points in the same direction as in the above statement of the problem (it is initially equal to $\omega_3\hat{\mathbf{z}} + \omega_2\hat{\mathbf{y}}$), but as time goes by, it is the $\omega_2\hat{\mathbf{y}}$ vector that will change, so $\vec{\omega}_{1,3} = \vec{\omega}_{1,2} + \vec{\omega}_{2,3}$ traces out a cone around the $\mathbf{z}$ axis, as shown in Fig. 8.6. ♣

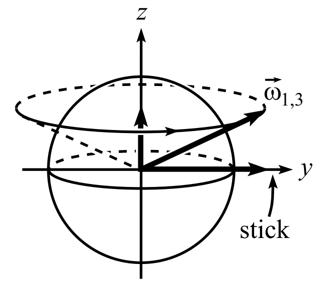

Figure 8.6

An important point concerning rotations in that they are defined with respect to a *coordinate system*. It makes no sense to ask how fast an object is rotating with respect to a certain point, or even a certain axis. Consider, for example, an object rotating with angular velocity $\boldsymbol{\omega} = \omega_3\hat{\mathbf{z}}$, with respect to the lab frame. Saying only, "The object has angular velocity $\boldsymbol{\omega} = \omega_3\hat{\mathbf{z}}$," is not sufficient, because someone standing in the frame of the object would measure $\boldsymbol{\omega} = 0$, and would therefore be very confused by your statement.

Throughout this chapter, we'll try to remember to state the coordinate system with respect to which $\boldsymbol{\omega}$ is measured. But if we forget, the default frame is the lab frame.

If you want to strain some brain cells thinking about $\boldsymbol{\omega}$ vectors, you are encouraged to solve Problem 3, and then also to look at the three given solutions.

This section was a bit abstract, so don't worry too much about it at the moment. The best strategy is probably to read on, and then come back for a second pass after digesting a few more sections. At any rate, we'll be discussing many other aspects of $\boldsymbol{\omega}$ in Section 8.7.2.

8.2 The inertia tensor

Given an object undergoing general motion, the *inertia tensor* is what relates the angular momentum, $\mathbf{L}$, to the angular velocity, $\boldsymbol{\omega}$. This tensor[3] depends on the geometry of the object, as we will see. In finding the $\mathbf{L}$ due to general motion, we will (in the same spirit as in Section 7.1) first look at the special case of rotation around an axis through the origin. Then we will look at the most general possible motion.

8.2.1 Rotation about an axis through the origin

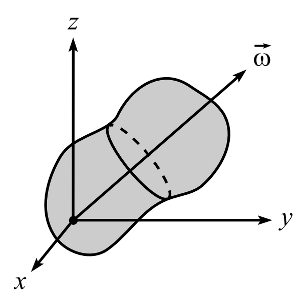

Figure 8.7

The three-dimensional object in Fig. 8.7 rotates with angular velocity $\boldsymbol{\omega}$. Consider a little piece of the body, with mass dm and position $\mathbf{r}$. The velocity of this piece is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. So the angular momentum (relative to the origin) of this piece is equal to $\mathbf{r} \times \mathbf{p} = (dm)\mathbf{r} \times \mathbf{v} = (dm)\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})$. The angular momentum of the entire body is therefore

$$\mathbf{L} = \int \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})\, dm, \qquad (8.4)$$

where the integration runs over the volume of the body.

[3] "Tensor" is just a fancy name for "matrix" here.

In the case where the rigid body is made up of a collection of point masses, m_i, the angular momentum is simply

$$\mathbf{L} = \sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i). \tag{8.5}$$

This double cross-product looks a bit intimidating, but it's actually not so bad. First, we have

$$\begin{aligned}\boldsymbol{\omega} \times \mathbf{r} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} \\ &= (\omega_2 z - \omega_3 y)\hat{\mathbf{x}} + (\omega_3 x - \omega_1 z)\hat{\mathbf{y}} + (\omega_1 y - \omega_2 x)\hat{\mathbf{z}}. \end{aligned} \tag{8.6}$$

Therefore,

$$\begin{aligned}\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & y & z \\ (\omega_2 z - \omega_3 y) & (\omega_3 x - \omega_1 z) & (\omega_1 y - \omega_2 x) \end{vmatrix} \\ &= \Big(\omega_1(y^2+z^2) - \omega_2 xy - \omega_3 zx\Big)\hat{\mathbf{x}} \\ &\quad + \Big(\omega_2(z^2+x^2) - \omega_3 yz - \omega_1 xy\Big)\hat{\mathbf{y}} \\ &\quad + \Big(\omega_3(x^2+y^2) - \omega_1 zx - \omega_2 yz\Big)\hat{\mathbf{z}}. \end{aligned} \tag{8.7}$$

The angular momentum in eq. (8.4) may therefore be written in the nice, concise, matrix form,

$$\begin{aligned}\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} &= \begin{pmatrix} \int(y^2+z^2) & -\int xy & -\int zx \\ -\int xy & \int(z^2+x^2) & -\int yz \\ -\int zx & -\int yz & \int(x^2+y^2) \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \\ &\equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \\ &\equiv \mathbf{I}\boldsymbol{\omega} \end{aligned} \tag{8.8}$$

For sake of clarity, we have not bothered to write the dm part of each integral. The matrix **I** is called the *inertia tensor*. If the word "tensor" scares you, just ignore it. **I** is simply a matrix. It acts on one vector (the angular velocity) to yield another vector (the angular momentum).

REMARKS:

1. **I** is a rather formidable-looking object. Therefore, you will undoubtedly be very pleased to hear that you will rarely have to use it. It's nice to know that it's there if you do need it, but the concept of *principal axes* in Section 8.3 provides a much better way of solving problems, which avoids the use of the inertia tensor.
2. **I** is a symmetric matrix. (This fact will be important in Section 8.3.) There are therefore only six independent entries, instead of nine.

3. In the case where the rigid body is made up of a collection of point masses, m_i, the entries in the matrix are just sums. For example, the upper left entry is $\sum m_i(y_i^2+z_i^2)$.
4. $\mathbf{I}$ depends only on the geometry of the object, and not on $\boldsymbol{\omega}$.
5. To construct an $\mathbf{I}$, you not only need to specify the origin, you also need to specify the x,y,z axes of your coordinate system. (These basis vectors must be orthogonal, because the cross-product calculation above is valid only for an orthonormal basis.) If someone else comes along and chooses a different orthonormal basis (but the same origin), then her $\mathbf{I}$ will have different *entries*, as will her $\boldsymbol{\omega}$, as will her $\mathbf{L}$. But her $\boldsymbol{\omega}$ and $\mathbf{L}$ will be exactly the same *vectors* as your $\boldsymbol{\omega}$ and $\mathbf{L}$. They will only appear different because they are written in a different coordinate system. (A vector is what it is, independent of how you choose to look at it. If you each point your arm in the direction of what you calculate $\mathbf{L}$ to be, then you will both be pointing in the same direction.) ♣

All this is fine and dandy. Given any rigid body, we can calculate $\mathbf{I}$ (relative to a given origin, using a given set of axes). And given $\boldsymbol{\omega}$, we can then apply $\mathbf{I}$ to it to find $\mathbf{L}$ (relative to the origin). But what do these entries in $\mathbf{I}$ really mean? How do we interpret them? Note, for example, that the L_3 in eq. (8.8) contains terms involving ω_1 and ω_2. But ω_1 and ω_2 have to do with rotations around the x and y axes, so what in the world are they doing in L_3? Consider the following examples.

Example 1 (Point-mass in x-y plane): Consider a point-mass m traveling in a circle (centered at the origin) in the x-y plane, with frequency ω_3. Let the radius of the circle be r (see Fig. 8.8).

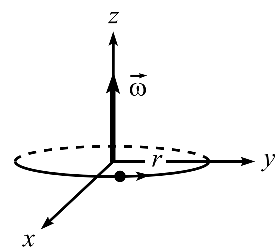

Figure 8.8

Using $\boldsymbol{\omega} = (0, 0, \omega_3)$, $x^2 + y^2 = r^2$, and $z = 0$ in eq. (8.8) (with a discrete sum of only one object, instead of the integrals), the angular momentum with respect to the origin is

$$\mathbf{L} = (0, 0, mr^2\omega_3). \tag{8.9}$$

The z-component is mrv, as it should be. And the x- and y-components are 0, as they should be. This case where $\omega_1 = \omega_2 = 0$ and $z = 0$ is simply the case we studied in the Chapter 7.

Example 2 (Point-mass in space): Consider a point-mass m traveling in a circle of radius r, with frequency ω_3. But now let the circle be centered at the point $(0, 0, z_0)$, with the plane of the circle parallel to the x-y plane (see Fig. 8.9).

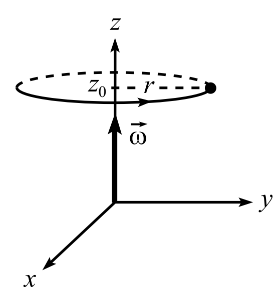

Figure 8.9

Using $\boldsymbol{\omega} = (0, 0, \omega_3)$, $x^2 + y^2 = r^2$, and $z = z_0$ in eq. (8.8), the angular momentum with respect to the origin is

$$\mathbf{L} = m\omega_3(-xz_0, -yz_0, r^2). \tag{8.10}$$

The z-component is mrv, as it should be. But, surprisingly, we have nonzero L_1 and L_2, even though our mass is simply rotating around the z-axis. What's going on?

Consider the instant when the mass is in the x-z plane. The velocity of the mass is then in the $\hat{\mathbf{y}}$ direction. Therefore, the particle most certainly has angular momentum around the x-axis, as well as the z-axis. (Someone looking at a split-second movie of the particle at this point could not tell whether the mass was rotating around the

x-axis, the z-axis, or undergoing some complicated motion. But the past and future motion is irrelevant; at any instant in time, as far as the angular momentum goes, we are concerned only with what is happening at that instant.)

At this instant, the angular momentum around the x-axis is $-mz_0v$ (since z_0 is the distance from the x-axis; and the minus sign comes from the right-hand rule). Using $v = \omega_3 x$, we have $L_1 = -mxz_0\omega_3$, in agreement with eq. (8.10).

At this instant, L_2 is zero, since the velocity is parallel to the y-axis. This agrees with eq. (8.10), since $y = 0$. And you can check that eq. (8.10) is indeed correct when the mass is at a general point (x, y, z_0).

For a point mass, $\mathbf{L}$ is much more easily obtained by simply calculating $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ (you should use this to check the results of this example). But for more complicated objects, the tensor $\mathbf{I}$ must be used.

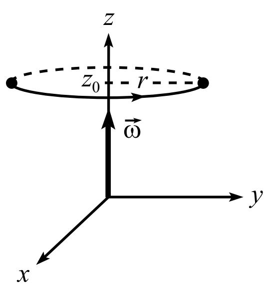

Figure 8.10

Example 3 (Two point-masses): Add another point-mass m to the previous example. Let it travel in the same circle, at the diametrically opposite point (see Fig. 8.10).

Using $\omega = (0, 0, \omega_3)$, $x^2 + y^2 = r^2$, and $z = z_0$ in eq. (8.8), you can show that the angular momentum with respect to the origin is

$$\mathbf{L} = 2m\omega_3(0, 0, r^2). \tag{8.11}$$

The z-component is $2mrv$, as it should be. And L_1 and L_2 are zero, unlike in the previous example, because these components of the $\mathbf{L}$'s of the two particles cancel. This occurs because of the symmetry of the masses around the z-axis, which causes the I_{zx} and I_{zy} entries in the inertia tensor to vanish (because they are each the sum of two terms, with opposite x values, or opposite y values).

Let's now look at the kinetic energy of our object (which is rotating about an axis passing through the origin). To find this, we need to add up the kinetic energies of all the little pieces. A little piece has energy $(dm)\,v^2/2 = dm\,|\boldsymbol{\omega} \times \mathbf{r}|^2/2$. So, using eq. (8.6), the total kinetic energy is

$$T = \frac{1}{2}\int \Big((\omega_2 z - \omega_3 y)^2 + (\omega_3 x - \omega_1 z)^2 + (\omega_1 y - \omega_2 x)^2 \Big)\, dm. \tag{8.12}$$

Multiplying this out, we see (after a little work) that we may write T as

$$\begin{aligned} T &= \frac{1}{2}(\omega_1, \omega_2, \omega_3) \cdot \begin{pmatrix} \int (y^2 + z^2) & -\int xy & -\int zx \\ -\int xy & \int (z^2 + x^2) & -\int yz \\ -\int zx & -\int yz & \int (x^2 + y^2) \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \\ &= \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L}. \end{aligned} \tag{8.13}$$

If $\boldsymbol{\omega} = \omega_3\hat{\mathbf{z}}$, then this reduces to the $T = I_{33}\omega_3^2/2$ result in eq. (7.8) in Chapter 7 (with a slight change in notation).

8.2.2 General motion

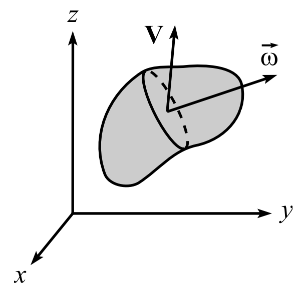

Figure 8.11

How do we deal with general motion in space? For the motion in Fig. 8.11, the various pieces of mass are not traveling in circles about the origin, so we cannot write $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, as we did prior to eq. (8.4).

To determine $\mathbf{L}$ (relative to the origin), and also the kinetic energy T, we will invoke Theorem 8.1. In applying this theorem, we may choose any point in the body to be the point P in the theorem. However, only in the case that P is the object's CM can we extract anything useful. The theorem then says that the motion of the body is the sum of the motion of the CM plus a rotation about the CM. So, let the CM move with velocity $\mathbf{V}$, and let the body instantaneously rotate with angular velocity $\boldsymbol{\omega}'$ around the CM. (That is, with respect to the frame whose origin is the CM, and whose axes are parallel to the fixed-frame axes.)

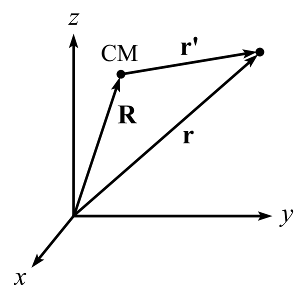

Figure 8.12

Let the CM coordinates be $\mathbf{R} = (X, Y, Z)$, and let the coordinates relative to the CM be $\mathbf{r}' = (x', y', z')$. Then $\mathbf{r} = \mathbf{R} + \mathbf{r}'$ (see Fig. 8.12). Let the velocity relative to the CM be $\mathbf{v}'$ (so $\mathbf{v}' = \boldsymbol{\omega}' \times \mathbf{r}'$). Then $\mathbf{v} = \mathbf{V} + \mathbf{v}'$.

Let's look at L first. The angular momentum is

$$\begin{aligned}
\mathbf{L} &= \int \mathbf{r} \times \mathbf{v}\, dm \\
&= \int (\mathbf{R} + \mathbf{r}') \times \Big(\mathbf{V} + (\boldsymbol{\omega}' \times \mathbf{r}')\Big)\, dm \\
&= \int (\mathbf{R} \times \mathbf{V})\, dm + \int \mathbf{r}' \times (\boldsymbol{\omega}' \times \mathbf{r}')\, dm \\
&= M(\mathbf{R} \times \mathbf{V}) + \mathbf{L}_{\text{CM}}.
\end{aligned} \tag{8.14}$$

The cross terms vanish because the integrands are linear in $\mathbf{r}'$ (and so the integrals, which involve $\int \mathbf{r}'\, dm$, are zero by definition of the CM). $\mathbf{L}_{\text{CM}}$ is the angular momentum relative to the CM.[4]

As in the pancake case Section 7.1.2, we see that the angular momentum (relative to the origin) of a body can be found by treating the body as a point mass located at the CM and finding the angular momentum of this point mass (relative to the origin), and by then adding on the angular momentum of the body, relative to the CM. Note that these two parts of the angular momentum need not point in the same direction (as they did in the pancake case).

Now let's look at T. The kinetic energy is

$$\begin{aligned}
T &= \int \frac{1}{2} v^2\, dm \\
&= \int \frac{1}{2} |\mathbf{V} + \mathbf{v}'|^2\, dm \\
&= \int \frac{1}{2} V^2\, dm + \int \frac{1}{2} v'^2\, dm \\
&= \frac{1}{2} M V^2 + \int \frac{1}{2} |\boldsymbol{\omega}' \times \mathbf{r}'|^2\, dm
\end{aligned}$$

[4]By this, we mean the angular momentum as measured in the coordinate system whose origin is the CM, and whose axes are parallel to the fixed-frame axes.

$$\equiv \frac{1}{2}MV^2 + \frac{1}{2}\boldsymbol{\omega}' \cdot \mathbf{L}_{\text{CM}}, \tag{8.15}$$

where the last line follows from the steps leading to eq. (8.13). The cross term $\int \mathbf{V}\cdot\mathbf{v}'\,dm = \int \mathbf{V}\cdot(\boldsymbol{\omega}'\times\mathbf{r}')\,dm$ vanishes because the integrand is linear in $\mathbf{r}'$ (and thus yields a zero integral, by definition of the CM).

As in the pancake case Section 7.1.2, we see that the kinetic energy of a body can be found by treating the body as a point mass located at the CM, and by then adding on the kinetic energy of the body due to motion relative to the CM.

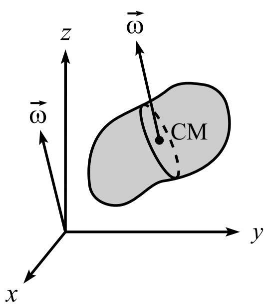

Figure 8.13

8.2.3 The parallel-axis theorem

Consider the special case where the CM rotates around the origin with the same angular velocity at which the body rotates around the CM (see Fig. 8.13). That is, $\mathbf{V} = \boldsymbol{\omega}'\times\mathbf{R}$, (This may be achieved, for example, by having a rod stick out of the body and pivoting one end of the rod at the origin.) This means that we have the nice situation where all points in the body travel in fixed circles around the axis of rotation (because $\mathbf{v} = \mathbf{V}+\mathbf{v}' = \boldsymbol{\omega}'\times\mathbf{R}+\boldsymbol{\omega}'\times\mathbf{r}' = \boldsymbol{\omega}'\times\mathbf{r}$). Dropping the prime on $\boldsymbol{\omega}$, eq. (8.14) becomes

$$\mathbf{L} = M\mathbf{R}\times(\boldsymbol{\omega}\times\mathbf{R}) + \int \mathbf{r}'\times(\boldsymbol{\omega}\times\mathbf{r}')\,dm \tag{8.16}$$

Expanding the double cross-products as in the steps leading to eq. (8.8), we may write this as

$$\begin{aligned}\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} &= M\begin{pmatrix} Y^2+Z^2 & -XY & -ZX \\ -XY & Z^2+X^2 & -YZ \\ -ZX & -YZ & X^2+Y^2 \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \\ &\quad + \begin{pmatrix} \int(y'^2+z'^2) & -\int x'y' & -\int z'x' \\ -\int x'y' & \int(z'^2+x'^2) & -\int y'z' \\ -\int z'x' & -\int y'z' & \int(x'^2+y'^2) \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \\ &\equiv (\mathbf{I}_{\text{R}}+\mathbf{I}_{\text{CM}})\boldsymbol{\omega}. \end{aligned} \tag{8.17}$$

This is the generalized parallel-axis theorem. It says that once you've calculated $\mathbf{I}_{\text{CM}}$ for an axis through the CM, then if you want to calculate $\mathbf{I}$ around any parallel axis, you simply have to add on the $\mathbf{I}_{\text{R}}$ matrix (obtained by treating the object like a point-mass at the CM). So you have to compute six numbers (there are only six, instead of nine, because the matrix is symmetric) instead of just the one MR^2 in the parallel-axis theorem in Chapter 7, given in eq. (7.12).

Likewise, if $\mathbf{V} = \boldsymbol{\omega}'\times\mathbf{R}$, then eq. (8.15) gives (dropping the prime on $\boldsymbol{\omega}$) a kinetic energy of

$$T = \frac{1}{2}\boldsymbol{\omega}\cdot(\mathbf{I}_{\text{R}}+\mathbf{I}_{\text{CM}})\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}. \tag{8.18}$$

8.3 Principal axes

The cumbersome expressions in the previous section may seem a bit unsettling, but it turns out that you will rarely have to invoke them. The strategy for avoiding all the previous mess is to use the *principal axes* of a body, which we will define below.

In general, the inertia tensor $\mathbf{I}$ in eq. (8.8) has nine nonzero entries (six independent ones). In addition to depending on the origin chosen, this inertia tensor depends on the set of orthonormal basis vectors chosen for the coordinate system. (The x,y,z variables in the integrals in $\mathbf{I}$ depend on the coordinate system with respect to which they are measured, of course.)

Given a blob of material, and given an arbitrary origin,[5] any orthonormal set of basis vectors is usable, but there is one special set that makes all our calculations very nice. These special basis vectors are called the *principal axes.* They can be defined in various equivalent ways.

- The principal axes are the orthonormal basis vectors for which $\mathbf{I}$ is diagonal, that is, for which[6]

$$\mathbf{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}. \tag{8.19}$$

  I_1, I_2, and I_3 are called the *principal moments.*

  For many objects, it is quite obvious what the principal axes are. For example, consider a uniform rectangle in the x-y plane, and let the CM be the origin (and let the sides be parallel to the coordinate axes). Then the principal axes are clearly the x, y, and z axes, because all the off-diagonal elements of the inertia tensor in eq. (8.8) vanish, by symmetry. For example $I_{xy} \equiv -\int xy\, dm$ equals zero, because for every point (x, y) in the rectangle, there is a corresponding point $(-x, y)$. So the contributions to $\int xy\, dm$ cancel in pairs. Also, the integrals involving z are identically zero, because $z = 0$.

- The principal axes are the orthonormal set $\hat{\boldsymbol{\omega}}_1$, $\hat{\boldsymbol{\omega}}_2$, $\hat{\boldsymbol{\omega}}_3$ with the property that

$$\mathbf{I}\hat{\boldsymbol{\omega}}_1 = I_1\hat{\boldsymbol{\omega}}_1, \qquad \mathbf{I}\hat{\boldsymbol{\omega}}_2 = I_2\hat{\boldsymbol{\omega}}_2, \qquad \mathbf{I}\hat{\boldsymbol{\omega}}_3 = I_3\hat{\boldsymbol{\omega}}_3. \tag{8.20}$$

  (That is, they are the $\boldsymbol{\omega}$'s for which $\mathbf{L}$ points in the same direction as $\boldsymbol{\omega}$.) These three statements are equivalent to eq. (8.19), because the vectors $\hat{\boldsymbol{\omega}}_1$, $\hat{\boldsymbol{\omega}}_2$, and $\hat{\boldsymbol{\omega}}_3$ are simply $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ in the frame in which they are the basis vectors.

- The principal axes are the axes around which the object can rotate with constant speed, without the need for any torque. (So in some sense, the object is

[5] The CM is often chosen to be the origin, but it need not be. There are principal axes associated with any origin.

[6] Technically, we should be writing I_{11} instead of I_1, etc., in this matrix, because we're talking about elements of a matrix. (The one-index object I_1 looks like a component of a vector.) But the two-index notation gets cumbersome, so we'll be sloppy and just use I_1, etc.

"happy" to spin around a principal axis.) This is equivalent to the previous definition for the following reason. Assume the object rotates around an axis $\hat{\boldsymbol{\omega}}_1$, for which $\mathbf{L} = \mathbf{I}\hat{\boldsymbol{\omega}}_1 = I_1\hat{\boldsymbol{\omega}}_1$, as in eq. (8.20). Then, since $\hat{\boldsymbol{\omega}}_1$ is assumed to be fixed, we see that $\mathbf{L}$ is also fixed. Therefore, $\boldsymbol{\tau} = d\mathbf{L}/dt = \mathbf{0}$.

The lack of need for any torque, for rotation around a principal axis $\hat{\boldsymbol{\omega}}$, means that if the object is pivoted at the origin, and if the origin is the only place where any force is applied, then the object can undergo rotation with constant angular velocity $\boldsymbol{\omega}$. If you try to set up this scenario with a non-principal axis, it won't work.

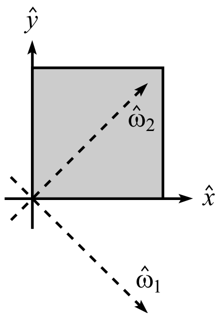

Figure 8.14

Example (Square with origin at corner): Consider the uniform square in Fig. 8.14. In Appendix G, we show that the principal axes are the dotted lines shown (and also the z-axis perpendicular to the page). But there is no need to use the techniques of the appendix to see this, because in this basis it is clear that the integral $\int x_1 x_2$ is zero, by symmetry. (And $x_3 \equiv z$ is identically zero, which makes the other off-diagonal terms in $\mathbf{I}$ also equal to zero.)

Furthermore, it is intuitively clear that the square will be happy to rotate around any one of these axes indefinitely. During such a rotation, the pivot will certainly be supplying a *force* (if the axis is $\hat{\boldsymbol{\omega}}_1$ or $\hat{\mathbf{z}}$), to provide the centripetal acceleration for the circular motion of the CM. But it will not be applying a *torque* relative to the origin (because the $\mathbf{r}$ in $\mathbf{r} \times \mathbf{F}$ is $\mathbf{0}$). This is good, because for a rotation around one of these principal axes, $d\mathbf{L}/dt = \mathbf{0}$, and there is no need for any torque.

It is fairly clear that it is impossible to make the square rotate around, say, the x-axis, assuming that its only contact with the world is through a free pivot at the origin. The square simply doesn't want to remain in that circular motion. There are various ways to demonstrate this rigorously. One is to show that $\mathbf{L}$ (relative to the origin) will not point along the x-axis, so it will therefore precess around the x-axis along with the square, tracing out the surface of a cone. This means that $\mathbf{L}$ is changing. But there is no torque available (relative to the origin) to provide for this change in $\mathbf{L}$. Hence, such a rotation cannot exist.

Note also that the integral $\int xy$ is not equal to zero (every point gives a positive contribution). So the inertia tensor is not diagonal in the x-y basis, which means that $\hat{x}$ and $\hat{y}$ are not principal axes.

At the moment, it is not at all obvious that an orthonormal set of principal axes exists for an arbitrary object. This is the task of Theorem 8.4 below. But assuming that principal axes do exist, the $\mathbf{L}$ and T in eqs. (8.8) and (8.13) take on the particularly nice forms,

$$\begin{aligned} \mathbf{L} &= (I_1\omega_1, I_2\omega_2, I_3\omega_3), \\ T &= \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2). \end{aligned} \tag{8.21}$$

in the basis of the principal axes. (The numbers ω_1, ω_2, and ω_3 are the components of a general vector $\boldsymbol{\omega}$ written in the principal-axis basis; that is, $\boldsymbol{\omega} = \omega_1\hat{\boldsymbol{\omega}}_1 + \omega_2\hat{\boldsymbol{\omega}}_2 +$

$\omega_3\hat{\boldsymbol{\omega}}_3$.) This is a vast simplification over the general formulas in eqs. (8.8) and (8.13). We will therefore invariably work with principal axes in the remainder of this chapter.

Remark: Note that the directions of the principal axes (relative to the body) depend only on the geometry of the body. They may therefore be considered to be painted onto the object. Hence, they will generally move around in space as the body rotates. (For example, in the special case where the object is rotating happily around a principal axis, then that axis will stay fixed, and the other two principal axes will rotate around it in space.) In equations like $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ and $\mathbf{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$, the components ω_i and $I_i\omega_i$ are measured along the *instantaneous* principal axes $\hat{\boldsymbol{\omega}}_i$. Since these axes change with time, the components ω_i and $I_i\omega_i$ will generally change with time (except in the case where we have a nice rotation around a principal axis). ♣

Let us now prove that a set of principal axes does indeed exist, for any object, and any origin. Actually, we'll just state the theorem here. The proof involves a rather slick and useful technique, but it's slightly off the main line of thought, so we'll relegate it to Appendix F. Take a look at the proof if you wish, but if you want to simply accept the fact that the principal axes exist, that's fine.

Theorem 8.4 *Given a real symmetric* 3×3 *matrix,* $\mathbf{I}$*, there exist three orthonormal real vectors,* $\hat{\boldsymbol{\omega}}_k$*, and three real numbers,* I_k*, with the property that*

$$\mathbf{I}\hat{\boldsymbol{\omega}}_k = I_k\hat{\boldsymbol{\omega}}_k. \tag{8.22}$$

Proof: See Appendix F. ■

Since the inertia tensor in eq. (8.8) is indeed symmetric, for any body and any origin, this theorem says that we can always find three orthogonal basis-vectors for which $\mathbf{I}$ is a diagonal matrix. That is, principal axes always exist. Invariably, it is best to work in a coordinate system that has this basis. (As mentioned above, the CM is generally chosen to be the origin, but this is not necessary. There are principal axes associated with any origin.)

Problem 5 gives another way to show the existence of principal axes in the special case of a pancake object.

For an object with a fair amount of symmetry, the principal axes are usually the obvious choices and can be written down by simply looking at the object (examples are given below). If, however, you are given an unsymmetrical body, then the only way to determine the principal axes is to pick an arbitrary basis, then find $\mathbf{I}$ in this basis, then go through a diagonalization procedure. This diagonalization procedure basically consists of the steps at the beginning of the proof of Theorem 8.4 (given in Appendix F), with the addition of one more step to get the actual vectors, so we'll relegate it to Appendix G. You need not worry much about this method. Virtually every problem we encounter will involve an object with sufficient symmetry to enable you to simply write down the principal axes.

Let's now prove two very useful (and very similar) theorems, and then we'll give some examples.

Theorem 8.5 *If two principal moments are equal ($I_1 = I_2 \equiv I$), then any axis (through the chosen origin) in the plane of the corresponding principal axes is a principal axis (and its moment is also I).*

Similarly, if all three principal moments are equal ($I_1 = I_2 = I_3 \equiv I$), then any axis (through the chosen origin) in space is a principal axis (and its moment is also I).

Proof: This first part was already proved at the end of the proof in Appendix F, but we'll do it again here. Let $I_1 = I_2 \equiv I$. Then $\mathbf{I}\mathbf{u}_1 = I\mathbf{u}_1$, and $\mathbf{I}\mathbf{u}_2 = I\mathbf{u}_2$. Hence, $\mathbf{I}(a\mathbf{u}_1 + b\mathbf{u}_2) = I(a\mathbf{u}_1 + b\mathbf{u}_2)$. Therefore, any linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ is a solution to $\mathbf{I}\mathbf{u} = I\mathbf{u}$ and is thus a principal axis, by definition.

Similarly, let $I_1 = I_2 = I_3 \equiv I$. Then $\mathbf{I}\mathbf{u}_1 = I\mathbf{u}_1$, $\mathbf{I}\mathbf{u}_2 = I\mathbf{u}_2$, and $\mathbf{I}\mathbf{u}_3 = I\mathbf{u}_3$. Hence, $\mathbf{I}(a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3) = I(a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3)$. Therefore, any linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ (that is, any vector in space) is a solution to $\mathbf{I}\mathbf{u} = I\mathbf{u}$ and is thus a principal axis, by definition.

Basically, if $I_1 = I_2 \equiv I$, then $\mathbf{I}$ is (up to a multiple) the identity matrix in the space spanned by $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. And if $I_1 = I_2 = I_3 \equiv I$, then $\mathbf{I}$ is (up to a multiple) the identity matrix in the entire space. ■

If two or three moments are equal, so that there is freedom in choosing the principal axes, then it is possible to pick a non-orthogonal group of them. We will, however, always choose ones that are orthogonal. So when we say "a set of principal axes", we mean an orthonormal set.

Theorem 8.6 *If a pancake object is symmetric under a rotation through an angle $\theta \neq 180°$ in the x-y plane (for example, a hexagon), then every axis in the x-y plane (with the origin chosen to be the center of the symmetry rotation) is a principal axis.*

Proof: Let $\hat{\boldsymbol{\omega}}_0$ be a principal axis in the plane, and let $\hat{\boldsymbol{\omega}}_\theta$ be the axis obtained by rotating $\hat{\boldsymbol{\omega}}_0$ through the angle θ. Then $\hat{\boldsymbol{\omega}}_\theta$ is also a principal axis with the same principal moment (due to the symmetry of the object). Therefore, $\mathbf{I}\hat{\boldsymbol{\omega}}_0 = I\hat{\boldsymbol{\omega}}_0$, and $\mathbf{I}\hat{\boldsymbol{\omega}}_\theta = I\hat{\boldsymbol{\omega}}_\theta$.

Now, any vector $\boldsymbol{\omega}$ in the x-y plane can be written as a linear combination of $\hat{\boldsymbol{\omega}}_0$ and $\hat{\boldsymbol{\omega}}_\theta$, provided that $\theta \neq 180°$ (this is where we use that assumption). That is, $\hat{\boldsymbol{\omega}}_0$ and $\hat{\boldsymbol{\omega}}_\theta$ span the x-y plane. Therefore, any vector $\boldsymbol{\omega}$ may be written as $\boldsymbol{\omega} = a\hat{\boldsymbol{\omega}}_0 + b\hat{\boldsymbol{\omega}}_\theta$, and so

$$\mathbf{I}\boldsymbol{\omega} = \mathbf{I}(a\hat{\boldsymbol{\omega}}_0 + b\hat{\boldsymbol{\omega}}_\theta) = aI\hat{\boldsymbol{\omega}}_0 + bI\hat{\boldsymbol{\omega}}_\theta = I\boldsymbol{\omega}. \tag{8.23}$$

Hence, $\boldsymbol{\omega}$ is also a principal axis. (Problem 6 gives another proof of this theorem.) ■

Let's now give some examples. We'll state the principal axes for the following objects (relative to the origin). Your exercise is to show that these are correct. Usually, a quick symmetry argument shows that

$$\mathbf{I} \equiv \begin{pmatrix} \int(y^2+z^2) & -\int xy & -\int zx \\ -\int xy & \int(z^2+x^2) & -\int yz \\ -\int zx & -\int yz & \int(x^2+y^2) \end{pmatrix} \tag{8.24}$$

is diagonal. In all of these examples (see Fig. 8.15), the origin for the principal axes is the origin of the given coordinate system (which is not necessarily the CM). In describing the axes, they thus all pass through the origin, in addition to having the other properties stated.

Example 1: Point mass at the origin.
principal axes: any axes.

Example 2: Point mass at the point (x_0, y_0, z_0).
principal axes: axis through point, any axes perpendicular to this.

Example 3: Rectangle centered at the origin, as shown.
principal axes: z-axis, axes parallel to sides.

Example 4: Cylinder with axis as z-axis.
principal axes: z-axis, any axes in x-y plane.

Example 5: Square with one corner at origin, as shown.
principal axes: z axis, axis through CM, axis perp to this.

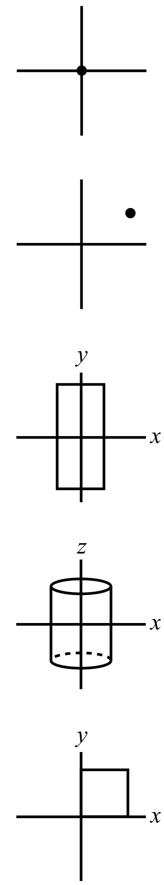

Figure 8.15

8.4 Two basic types of problems

The previous three sections introduced many new, and somewhat abstract, concepts. We will now (finally) get our hands dirty and solve some actual problems. The concept of principal axes, in particular, gives us the ability to solve many kinds of problems. Two types, however, come up again and again. There are variations on these, of course, but they may be generally stated as follows.

- Strike a rigid object with an impulsive (that is, quick) blow. What is the motion of the object immediately after the blow?
- An object rotates around a fixed axis. A given torque is applied. What is the frequency of the rotation? (Or conversely, given the frequency, what is the required torque?)

Let's work through an example for each of these problems. In both cases, the solution involves a few standard steps, so we'll write them out explicitly.

8.4.1 Motion after an impulsive blow

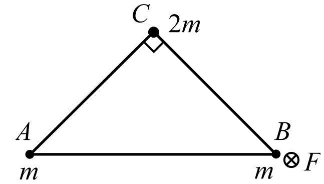

Figure 8.16

Problem: Consider the rigid object in Fig. 8.16. Three masses are connected by three massless rods, in the shape of an isosceles right triangle with hypotenuse length $4a$. The mass at the right angle is $2m$, and the other two masses are m. Label them A, B, C, as shown. Assume that the object is floating freely in space. (Alternatively, let the object hang from a long thread attached to mass C.)

Mass B is struck with a quick blow, directed into the page. Let the imparted impulse have magnitude $\int F\,dt = P$. (See Section 7.6 for a discussion of impulse and angular impulse.) What are the velocities of the three masses immediately after the blow?

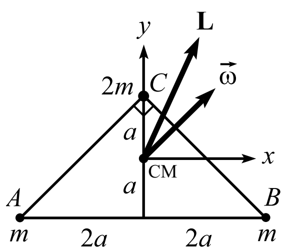

Figure 8.17

Solution: The strategy of the solution will be to find the angular momentum of the system (relative to the CM) using the angular impulse, then calculate the principal moments and find the angular velocity vector (which will give the velocities relative to the CM), and then add on the CM motion.

The altitude from the right angle to the hypotenuse has length $2a$, and the CM is easily seen to be located at its midpoint (see Fig. 8.17). Picking the CM as our origin, and letting the plane of the paper be the x-y plane, the positions of the three masses are $\mathbf{r}_A = (-2a, -a, 0)$, $\mathbf{r}_B = (2a, -a, 0)$, and $\mathbf{r}_C = (0, a, 0)$. There are now five standard steps that we must perform.

- **Find L:** The positive z-axis is directed out of the page, so the impulse vector is $\mathbf{P} \equiv \int \mathbf{F}\,dt = (0, 0, -P)$. Therefore, the angular momentum of the system (relative to the CM) is

$$\begin{aligned} \mathbf{L} = \int \boldsymbol{\tau}\,dt &= \int (\mathbf{r}_B \times \mathbf{F})\,dt = \mathbf{r}_B \times \int \mathbf{F}\,dt \\ &= (2a, -a, 0) \times (0, 0, -P) = aP(1, 2, 0), \end{aligned} \tag{8.25}$$

as shown in Fig. 8.17. We have used the fact that $\mathbf{r}_B$ is essentially constant during the blow (because the blow is assumed to happen very quickly) in taking $\mathbf{r}_B$ outside the integral in the above equation.

- **Calculate the principal moments:** The principal axes are clearly the x, y, and z axes. The moments (relative to the CM) are

$$\begin{aligned} I_x &= ma^2 + ma^2 + (2m)a^2 = 4ma^2, \\ I_y &= m(2a)^2 + m(2a)^2 + (2m)0^2 = 8ma^2, \\ I_z &= I_x + I_y = 12ma^2. \end{aligned} \tag{8.26}$$

We have used the perpendicular-axis theorem, eq. (7.17), to obtain I_z. But I_z will not be needed to solve the problem.

- **Find $\boldsymbol{\omega}$:** We now have two expressions for the angular momentum of the system. One expression is in terms of the given impulse, eq. (8.25). The other is in terms of the moments and the angular velocity components, eq. (8.21). Therefore,

$$\begin{aligned} (I_x\omega_x, I_y\omega_y, I_z\omega_z) &= aP(1, 2, 0) \\ \Longrightarrow \quad (4ma^2\omega_x, 8ma^2\omega_y, 12ma^2\omega_z) &= aP(1, 2, 0) \\ \Longrightarrow \quad (\omega_x, \omega_y, \omega_z) &= \frac{P}{4ma}(1, 1, 0), \end{aligned} \tag{8.27}$$

as shown in Fig. 8.17.

- **Calculate speeds relative to CM:** Right after the blow, the object rotates around the CM with the angular velocity found above. The speeds relative to the CM are therefore $\mathbf{u}_i = \boldsymbol{\omega} \times \mathbf{r}_i$. That is,

$$\begin{aligned}\mathbf{u}_A &= \boldsymbol{\omega} \times \mathbf{r}_A = \frac{P}{4ma}(1,1,0) \times (-2a,-a,0) = (0,0,P/4m),\\ \mathbf{u}_B &= \boldsymbol{\omega} \times \mathbf{r}_B = \frac{P}{4ma}(1,1,0) \times (2a,-a,0) = (0,0,-3P/4m),\\ \mathbf{u}_C &= \boldsymbol{\omega} \times \mathbf{r}_C = \frac{P}{4ma}(1,1,0) \times (0,a,0) = (0,0,P/4m).\end{aligned} \tag{8.28}$$

- **Add on speed of CM:** The impulse (that is, the change in linear momentum) supplied to the whole system is $\mathbf{P} = (0,0,-P)$. The total mass of the system is $M = 4m$. Therefore, the velocity of the CM is

$$V_{\rm CM} = \frac{\mathbf{P}}{M} = (0,0,-P/4m). \tag{8.29}$$

  The total velocities of the masses are therefore

$$\begin{aligned}\mathbf{v}_A &= \mathbf{u}_A + V_{\rm CM} = (0,0,0),\\ \mathbf{v}_B &= \mathbf{u}_B + V_{\rm CM} = (0,0,-P/m),\\ \mathbf{v}_C &= \mathbf{u}_C + V_{\rm CM} = (0,0,0).\end{aligned} \tag{8.30}$$

Remarks:

1. We see that masses A and C are instantaneously at rest immediately after the blow, and mass B acquires all of the imparted impulse. In retrospect, this is quite clear. Basically, it is possible for both A and C to remain at rest while B moves a tiny bit, so this is what happens. (If B moves into the page by a small distance ϵ, then A and C won't know that B has moved, since their distances to B will change only by a distance of order ϵ^2.) If we changed the problem and added a mass D at, say, the midpoint of the hypotenuse, then this would not be the case; it would not be possible for A, C, and D to remain at rest while B moved a tiny bit. So there would be some other motion, in addition to B's.
2. As time goes on, the system will undergo a rather complicated motion. What will happen is that the CM will move with constant velocity, and the masses will rotate around it in a messy (but understandable) manner. Since there are no torques acting on the system (after the initial blow), we know that $\mathbf{L}$ will forever remain constant. It turns out that $\boldsymbol{\omega}$ will move around $\mathbf{L}$, and the body will rotate around this changing $\boldsymbol{\omega}$. These matters are the subject of Section 8.6. (Although in that discussion, we restrict ourselves to symmetric tops; that is, ones with two equal moments.) But these issues aside, it's good to know that we can, without too much difficulty, determine what's going on immediately after the blow.
3. The body in the above problem was assumed to be floating freely in space. If we instead have an object that is pivoted at a given (fixed) point, then we simply want to use the pivot as our origin, and there is no need to perform the last step of adding on the velocity of the origin (which was the CM, above), since this velocity is now zero. Equivalently, just consider the pivot to be an infinite mass, which is therefore the location of the (motionless) CM. ♣

8.4.2 Frequency of motion due to a torque

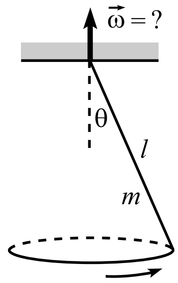

Figure 8.18

Problem: Consider a stick of length ℓ, mass m, and uniform mass density. The stick is pivoted at its top end and swings around the vertical axis. Assume conditions have been set up so that the stick always makes an angle θ with the vertical, as shown in Fig. 8.18. What is the frequency, ω, of this motion?

Solution: The strategy of the solution will be to find the principal moments and then the angular momentum of the system (in terms of ω), then find the rate of change of $\mathbf{L}$, and then calculate the torque and equate it with $d\mathbf{L}/dt$. We will choose the pivot to be the origin.[7] Again, there are five standard steps that we must perform.

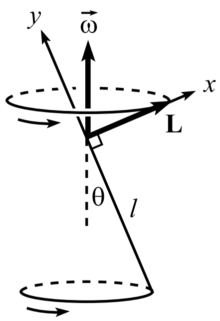

Figure 8.19

- **Calculate the principal moments:** The principal axes are clearly the axis along the stick, along with any two orthogonal axes perpendicular to the stick. So let the x- and y-axes be as shown in Fig. 8.19, and let the z-axis point out of the page. The moments (relative to the pivot) are $I_x = m\ell^2/3$, $I_y = 0$, and $I_z = m\ell^2/3$. (I_z won't be needed in this solution.)

- **Find L:** The angular velocity vector points vertically,[8] so in the basis of the principal axes, the angular velocity vector is $\boldsymbol{\omega} = (\omega\sin\theta, \omega\cos\theta, 0)$, where ω is yet to be determined. The angular momentum of the system (relative to the pivot) is thus

$$\mathbf{L} = (I_x\omega_x, I_y\omega_y, I_z\omega_z) = (m\ell^2\omega\sin\theta/3, 0, 0). \tag{8.31}$$

- **Find $d\mathbf{L}/dt$:** The vector $\mathbf{L}$ therefore points upwards to the right, along the x-axis (at the instant shown in Fig. 8.19), with magnitude $L = m\ell^2\omega\sin\theta/3$. As the stick rotates around the vertical axis, $\mathbf{L}$ traces out the surface of a cone. That is, the tip of $\mathbf{L}$ traces out a horizontal circle. The radius of this circle is the horizontal component of $\mathbf{L}$, which is $L\cos\theta$. The speed of the tip (that is, the magnitude of $d\mathbf{L}/dt$) is therefore $(L\cos\theta)\omega$, because $\mathbf{L}$ rotates around the vertical axis with the same frequency as the stick. So, $d\mathbf{L}/dt$ has magnitude

$$\left|\frac{d\mathbf{L}}{dt}\right| = (L\cos\theta)\omega = \frac{1}{3}m\ell^2\omega^2\sin\theta\cos\theta, \tag{8.32}$$

and it points into the page.

Remark: In more complicated problems (where $I_y \neq 0$), $\mathbf{L}$ will point in some messy direction (not along a principal axis), and the length of the horizontal component (that is, the radius of the circle $\mathbf{L}$ traces out) won't be immediately obvious. In this case, you can either explicitly calculate the horizontal component (see the Gyroscope example in Section 8.7.5), or you can simply do things the formal (and easier) way by

[7] This is a better choice than the CM, because this way we won't have to worry about any messy forces acting at the pivot, when computing the torque.

[8] However, see the third Remark, following this solution.

finding the rate of change of $\mathbf{L}$ via the expression $d\mathbf{L}/dt = \boldsymbol{\omega} \times \mathbf{L}$ (which holds for all the same reasons that $\mathbf{v} \equiv d\mathbf{r}/dt = \boldsymbol{\omega} \times \mathbf{r}$ holds). In the present problem, we obtain

$$d\mathbf{L}/dt = (\omega \sin\theta, \omega\cos\theta, 0) \times (m\ell^2\omega\sin\theta/3, 0, 0) = (0, 0, -m\ell^2\omega^2 \sin\theta\cos\theta/3), \tag{8.33}$$

which agrees with eq. (8.32). And the direction is correct, since the negative z-axis points into the page. Note that we calculated this cross-product in the principal-axis basis. Although these axes are changing in time, they present a perfectly good set of basis vectors at any instant. ♣

- **Calculate the torque:** The torque (relative to the pivot) is due to gravity, which effectively acts on the CM of the stick. So $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ has magnitude

$$\tau = rF\sin\theta = (\ell/2)(mg)\sin\theta, \tag{8.34}$$

and it points into the page.

- **Equate $\boldsymbol{\tau}$ with $d\mathbf{L}/dt$:** The vectors $d\mathbf{L}/dt$ and $\boldsymbol{\tau}$ both point into the page (they had better point in the same direction). Equating their magnitudes gives

$$\begin{aligned} \frac{m\ell^2\omega^2\sin\theta\cos\theta}{3} &= \frac{mg\ell\sin\theta}{2} \\ \Longrightarrow \qquad \omega &= \sqrt{\frac{3g}{2\ell\cos\theta}}\,. \end{aligned} \tag{8.35}$$

REMARKS:

1. This frequency is slightly larger than the frequency obtained if we instead have a mass at the end of a massless stick of length ℓ. From Problem 12, the frequency in that case is $\sqrt{g/\ell\cos\theta}$. So, in some sense, a uniform stick of length ℓ behaves like a mass at the end of a massless stick of length $2\ell/3$, as far as these rotations are concerned.
2. As $\theta \to \pi/2$, the frequency ω goes to ∞, which makes sense. And as $\theta \to 0$, ω approaches $\sqrt{3g/2\ell}$, which isn't so obvious.
3. As explained in Problem 2, the instantaneous $\boldsymbol{\omega}$ is not uniquely defined in some situations. At the instant shown in Fig. 8.18, the stick is moving directly into the page. So let's say someone else wants to think of the stick as (instantaneously) rotating around the axis $\boldsymbol{\omega}'$ perpendicular to the stick (the x-axis, from above), instead of the vertical axis, as shown in Fig. 8.20. What is the angular speed ω'?

   Well, if ω is the angular speed of the stick around the vertical axis, then we may view the tip of the stick as instantaneously moving in a circle of radius $\ell\sin\theta$ around the vertical axis $\boldsymbol{\omega}$. So $\omega(\ell\sin\theta)$ is the speed of the tip of the stick. But we may also view the tip of the stick as instantaneously moving in a circle of radius ℓ around $\boldsymbol{\omega}'$. The speed of the tip is still $\omega(\ell\sin\theta)$, so the angular speed about this axis is given by $\omega'\ell = \omega(\ell\sin\theta)$. Hence $\omega' = \omega\sin\theta$, which is simply the x-component of $\boldsymbol{\omega}$ that we found above, right before eq. (8.31). The moment of inertia around $\boldsymbol{\omega}'$ is $m\ell^2/3$, so the angular momentum has magnitude $(m\ell^2/3)(\omega\sin\theta)$, in agreement with eq. (8.31). And the direction is along the x-axis, as it should be.

   Note that although $\boldsymbol{\omega}$ is not uniquely defined at any instant, $\mathbf{L} \equiv \int (\mathbf{r} \times \mathbf{p})\, dm$ certainly is.[9] Choosing $\boldsymbol{\omega}$ to point vertically, as we did in the above solution, is in some sense the natural choice, because this $\boldsymbol{\omega}$ does not change with time. ♣

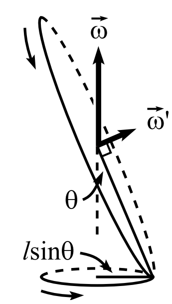

Figure 8.20

[9]The non-uniqueness of $\vec{\omega}$ arises from the fact that $I_y = 0$ here. If all the moments are nonzero, then $(L_x, L_y, L_z) = (I_x\omega_x, I_y\omega_y, I_z\omega_z)$ uniquely determines $\vec{\omega}$, for a given $\mathbf{L}$.

8.5 Euler's equations

Consider a rigid body instantaneously rotating around an axis $\boldsymbol{\omega}$. ($\boldsymbol{\omega}$ may change as time goes on, but all we care about for now is what it is at a given instant.) The angular momentum, $\mathbf{L}$, is given by eq. (8.8) as

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega}, \tag{8.36}$$

where $\mathbf{I}$ is the inertial tensor, calculated with respect to a given set of axes (and $\boldsymbol{\omega}$ is written in the same basis, of course).

As usual, things are much nicer if we use the principal axes (relative to the chosen origin) as the basis vectors of our coordinate system. Since these axes are fixed with respect to the rotating object, they will of course rotate with respect to the fixed reference frame. In this basis, $\mathbf{L}$ takes the nice form,

$$\mathbf{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3), \tag{8.37}$$

where ω_1, ω_2, and ω_3 are the components of $\boldsymbol{\omega}$ along the principal axes. In other words, if you take the vector $\mathbf{L}$ in space and project it onto the instantaneous principal axes, then you get these components.

On one hand, writing $\mathbf{L}$ in terms of the rotating principal axes allows us to write it in the nice form of (8.37). But on the other hand, writing $\mathbf{L}$ in this way makes it nontrivial to determine how it changes in time (since the principal axes themselves are changing). The benefits outweigh the detriments, however, so we will invariably use the principal axes as our basis vectors.

The goal of this section is to find an expression for $d\mathbf{L}/dt$, and to then equate this with the torque. The result will be Euler's equations, eqs. (8.43).

Derivation of Euler's equations

If we write $\mathbf{L}$ in terms of the body frame, then we see that $\mathbf{L}$ can change (relative to the lab frame) due to two effects. $\mathbf{L}$ can change because its coordinates in the body frame may change, and $\mathbf{L}$ can also change because of the rotation of the body frame.

To be precise, let $\mathbf{L}_0$ be the vector $\mathbf{L}$ at a given instant. At this instant, imagine painting the vector $\mathbf{L}_0$ onto the body frame (so that $\mathbf{L}_0$ will then rotate with the body frame). The rate of change of $\mathbf{L}$ with respect to the lab frame may be written in the (identically true) way,

$$\frac{d\mathbf{L}}{dt} = \frac{d(\mathbf{L} - \mathbf{L}_0)}{dt} + \frac{d\mathbf{L}_0}{dt}\,. \tag{8.38}$$

The second term here is simply the rate of change of a body-fixed vector, which we know is $\boldsymbol{\omega} \times \mathbf{L}_0$ (which equals $\boldsymbol{\omega} \times \mathbf{L}$ at this instant). The first term is the rate of change of $\mathbf{L}$ with respect to the body frame, which we will denote by $\delta\mathbf{L}/\delta t$. So we end up with

$$\frac{d\mathbf{L}}{dt} = \frac{\delta\mathbf{L}}{\delta t} + \boldsymbol{\omega} \times \mathbf{L}. \tag{8.39}$$

This is actually a general statement, true for any vector in any rotating frame.[10] There is nothing particular to **L** that we used in the above derivation. Also, there was no need to restrict ourselves to principal axes.

In words, what we've shown is that the total change equals the change relative to the rotating frame, plus the change of the rotating frame relative to the fixed frame. Simply addition of changes.

Let us now be specific and choose our body-axes to be the principal-axes. This will put eq. (8.39) in a very usable form. Using eq. (8.37), we have

$$\frac{d\mathbf{L}}{dt} = \frac{\delta}{\delta t}(I_1\omega_1, I_2\omega_2, I_3\omega_3) + (\omega_1, \omega_2, \omega_3) \times (I_1\omega_1, I_2\omega_2, I_3\omega_3). \tag{8.40}$$

This equation equates two vectors. As is true for any vector, these (equal) vectors have an existence that is independent of what coordinate system we choose to describe them with (eq. (8.39) makes no reference to a coordinate system). But since we've chosen an explicit frame on the right-hand side of eq. (8.40), we should choose the same frame for the left-hand side; we can then equate the components on the left with the components on the right. Projecting $d\mathbf{L}/dt$ onto the instantaneous principal axes, we have

$$\left(\left(\frac{d\mathbf{L}}{dt}\right)_1, \left(\frac{d\mathbf{L}}{dt}\right)_2, \left(\frac{d\mathbf{L}}{dt}\right)_3\right) = \frac{\delta}{\delta t}(I_1\omega_1, I_2\omega_2, I_3\omega_3) + (\omega_1, \omega_2, \omega_3) \times (I_1\omega_1, I_2\omega_2, I_3\omega_3). \tag{8.41}$$

REMARK: The left-hand side looks nastier than it really is. At the risk of belaboring the point, consider the following (this is a remark that has to be read very slowly): We could have written the left-hand side as $(d/dt)(L_1, L_2, L_3)$, but this might cause confusion as to whether the L_i refer to the components with respect to the rotating axes, or the components with respect to the fixed set of axes that coincide with the rotating principal axes at this instant. That is, do we project **L** onto the principal axes, and then take the derivative; or do we take the derivative and then project? The latter is what we mean in eq. (8.41). (The former is $\delta\mathbf{L}/\delta t$, by definition.) The way we've written the left-hand side of eq. (8.41), it's clear that we're taking the derivative first. We are, after all, simply projecting eq. (8.39) onto the principal axes. ♣

The time derivatives on the right-hand side of eq. (8.41) are $\delta(I_1\omega_1)/\delta t = I_1\dot{\omega}_1$ (because I_1 is constant), etc. Performing the cross product and equating the corresponding components on each side yields the three equations,

$$\begin{aligned} \left(\frac{d\mathbf{L}}{dt}\right)_1 &= I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2, \\ \left(\frac{d\mathbf{L}}{dt}\right)_2 &= I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3, \\ \left(\frac{d\mathbf{L}}{dt}\right)_3 &= I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1. \end{aligned} \tag{8.42}$$

[10] We will prove eq. (8.39) in another more mathematical way in Chapter 9.

If we have chosen the origin of our rotating frame to be either a fixed point or the CM (which we will always do), then the results of Section 7.4 tell us that we may equate $d\mathbf{L}/dt$ with the torque, $\boldsymbol{\tau}$. We therefore have

$$\begin{aligned}\tau_1 &= I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2, \\ \tau_2 &= I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3, \\ \tau_3 &= I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1. \end{aligned} \tag{8.43}$$

These are *Euler's equations.* You need only remember one of them, because the other two can be obtained by cyclic permutation of the indices.

REMARKS:

1. We repeat that the left- and right-hand sides of eq. (8.43) are components that are measured with respect to the instantaneous principal axes. Let's say we do a problem, for example, where at all times $\tau_1 = \tau_2 = 0$, and τ_3 equals some nonzero number. This doesn't mean, of course, that $\boldsymbol{\tau}$ is a constant vector. On the contrary, $\boldsymbol{\tau}$ always points along the $\hat{\mathbf{x}}_3$ vector in the rotating frame, but this vector is changing in the fixed frame (unless $\hat{\mathbf{x}}_3$ points along $\boldsymbol{\omega}$).

   The two types of terms on the right-hand sides of eqs. (8.42) are the two types of changes that $\mathbf{L}$ can undergo. $\mathbf{L}$ can change because its components with respect to the rotating frame change, and $\mathbf{L}$ can also change because the body is rotating around $\boldsymbol{\omega}$.

2. Section 8.6.1 on the free symmetric top (viewed from the body frame) provides a good example of the use of Euler's equations. Another interesting application is the famed "tennis racket theorem" (Problem 14).

3. It should be noted that you never *have* to use Euler's equations. You can simply start from scratch and use eq. (8.39) each time you solve a problem. The point is that we've done the calculation of $d\mathbf{L}/dt$ once and for all, so you can just invoke the result in eqs. (8.43). ♣

8.6 Free symmetric top

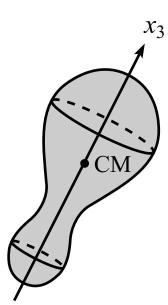

Figure 8.21

The free symmetric top is the classic example of an application of the Euler equations. Consider an object which has two of its principal moments equal (with the CM as the origin). Let the object be in outer space, far from any external forces.[11] We will choose our object to have cylindrical symmetry around some axis (see Fig. 8.21), although this is not necessary (a square cross-section, for example, would yield two equal moments). The principal axes are then the symmetry axis and any two orthogonal axes in the cross-section plane through the CM. Let the symmetry axis be chosen as the $\hat{\mathbf{x}}_3$ axis. Then our moments are $I_1 = I_2 \equiv I$, and I_3.

8.6.1 View from body frame

Plugging $I_1 = I_2 \equiv I$ into Euler's equations, eqs. (8.43), with the τ_i equal to zero (since there are no torques, because the top is "free"), gives

$$0 = I\dot{\omega}_1 + (I_3 - I)\omega_3\omega_2,$$

[11]Equivalently, the object is thrown up in the air, and we are traveling along on the CM.

$$\begin{aligned} 0 &= I\dot{\omega}_2 + (I - I_3)\omega_1\omega_3, \\ 0 &= I_3\dot{\omega}_3. \end{aligned} \tag{8.44}$$

The last equation says that ω_3 is a constant. If we then define

$$\Omega \equiv \left(\frac{I_3 - I}{I}\right)\omega_3, \tag{8.45}$$

the first two equations become

$$\dot{\omega}_1 + \Omega\omega_2 = 0, \qquad \text{and} \qquad \dot{\omega}_2 - \Omega\omega_1 = 0. \tag{8.46}$$

Taking the derivative of the first of these, and then using the second one to eliminate $\dot{\omega}_2$, gives

$$\ddot{\omega}_1 + \Omega^2\omega_1 = 0, \tag{8.47}$$

and likewise for ω_2. This is a nice simple-harmonic equation. The solutions for $\omega_1(t)$ and (by using eq. (8.46)) $\omega_2(t)$ are

$$\omega_1(t) = A\cos(\Omega t + \phi), \qquad \omega_2(t) = A\sin(\Omega t + \phi). \tag{8.48}$$

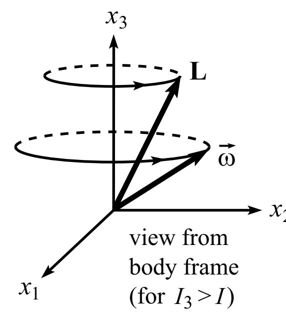

Figure 8.22

Therefore, $\omega_1(t)$ and $\omega_2(t)$ are the components of a circle in the body frame. Hence, the $\boldsymbol{\omega}$ vector traces out a cone around $\hat{\mathbf{x}}_3$ (see Fig. 8.22), with frequency Ω, as viewed by someone standing on the body. The angular momentum is

$$\mathbf{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3) = \Big(IA\cos(\Omega t + \phi), IA\sin(\Omega t + \phi), I_3\omega_3\Big), \tag{8.49}$$

so $\mathbf{L}$ also traces out a cone around $\hat{\mathbf{x}}_3$ (see Fig. 8.22), with frequency Ω, as viewed by someone standing on the body.

The frequency, Ω, in eq. (8.45) depends on the value of ω_3 and on the geometry of the object. But the amplitude, A, of the $\boldsymbol{\omega}$ cone is determined by the initial values of ω_1 and ω_2.

Note that Ω may be negative (if $I > I_3$). In this case, $\boldsymbol{\omega}$ traces out its cone in the opposite direction compared to the $\Omega > 0$ case.

Example (The earth): Let's consider the earth to be our object. Then $\omega_3 \approx 2\pi/(1 \text{ day})$.[12] The bulge at the equator (caused by the spinning of the earth) makes I_3 slightly larger than I, and it turns out that $(I_3 - I)/I \approx 1/300$. Therefore, eq. (8.45) gives $\Omega \approx (1/300)\, 2\pi/(1 \text{ day})$. So the $\boldsymbol{\omega}$ vector should precess around its cone once every 300 days, as viewed by someone on the earth. The true value is more like 400 days. The difference has to do with various things, including the non-rigidity of the earth. But at least we got an answer in the right ballpark.

How do you determine the direction of $\boldsymbol{\omega}$? Simply make an extended-time photograph exposure at night. The stars will form arcs of circles. At the center of all these circles is a point that doesn't move. This is the direction of $\boldsymbol{\omega}$.

[12] This isn't quite correct, since the earth rotates 366 times for every 365 days (due to the motion around the sun), but it's close enough for the purposes here.

How big is the $\boldsymbol{\omega}$ cone, for the earth? Equivalently, what is the value of A in eq. (8.48)? Observation has shown that $\boldsymbol{\omega}$ pierces the earth at a point on the order of 10 m from the north pole. Hence, $A/\omega_3 \approx (10\,\text{m})/R_E$. The half-angle of the $\boldsymbol{\omega}$ cone is therefore found to be only on the order of 10^{-4} degrees. So if you use an extended-time photograph exposure one night to see which point in the sky stands still, and then if you do the same thing 200 nights later, you probably won't be able to tell that they're really two different points.

8.6.2 View from fixed frame

Now let's see what our symmetric top looks like from a fixed frame. In terms of the principal axes, $\hat{\mathbf{x}}_1$,$\hat{\mathbf{x}}_2$,$\hat{\mathbf{x}}_3$, we have

$$\begin{aligned}\boldsymbol{\omega} &= (\omega_1\hat{\mathbf{x}}_1 + \omega_2\hat{\mathbf{x}}_2) + \omega_3\hat{\mathbf{x}}_3, \qquad \text{and} \\ \mathbf{L} &= I(\omega_1\hat{\mathbf{x}}_1 + \omega_2\hat{\mathbf{x}}_2) + I_3\omega_3\hat{\mathbf{x}}_3. \end{aligned} \tag{8.50}$$

Eliminating the $(\omega_1\hat{\mathbf{x}}_1 + \omega_2\hat{\mathbf{x}}_2)$ term from these equations gives (in terms of the Ω defined in eq. (8.45))

$$\mathbf{L} = I(\boldsymbol{\omega} + \Omega\hat{\mathbf{x}}_3), \qquad \text{or} \qquad \boldsymbol{\omega} = \frac{L}{I}\hat{\mathbf{L}} - \Omega\hat{\mathbf{x}}_3, \tag{8.51}$$

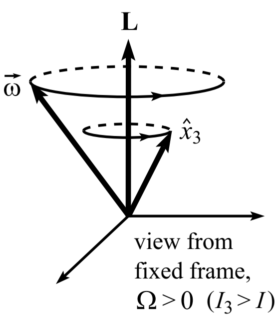

Figure 8.23

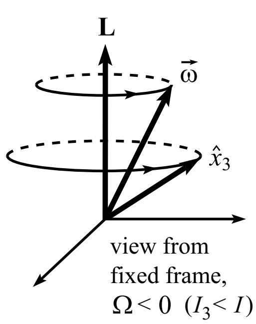

Figure 8.24

where $L = |\mathbf{L}|$, and $\hat{\mathbf{L}}$ is the unit vector in the $\mathbf{L}$ direction. The linear relationship between $\mathbf{L}$, $\boldsymbol{\omega}$, and $\hat{\mathbf{x}}_3$, implies that these three vectors lie in a plane. Since there are no torques on the system, $\mathbf{L}$ remains constant. Therefore, $\boldsymbol{\omega}$ and $\hat{\mathbf{x}}_3$ precess (as we will see below) around $\mathbf{L}$, with the three vectors always coplanar. See Fig. 8.23 for the case $I_3 > I$ (an *oblate* top, such as a coin), and Fig. 8.24 for the case $I_3 < I$ (a *prolate* top, such as a carrot).

What is the frequency of this precession, as viewed from the fixed frame? The rate of change of $\hat{\mathbf{x}}_3$ is $\boldsymbol{\omega} \times \hat{\mathbf{x}}_3$ (because $\hat{\mathbf{x}}_3$ is fixed in the body frame, so its change comes only from rotation around $\boldsymbol{\omega}$). Therefore, eq. (8.51) gives

$$\frac{d\hat{\mathbf{x}}_3}{dt} = \left(\frac{L}{I}\hat{\mathbf{L}} - \Omega\hat{\mathbf{x}}_3\right) \times \hat{\mathbf{x}}_3 = \left(\frac{L}{I}\hat{\mathbf{L}}\right) \times \hat{\mathbf{x}}_3. \tag{8.52}$$

But this is simply the expression for the rate of change of a vector rotating around the fixed vector $\tilde{\boldsymbol{\omega}} \equiv (L/I)\hat{\mathbf{L}}$. The frequency of this rotation is $|\tilde{\boldsymbol{\omega}}| = L/I$. Therefore, $\hat{\mathbf{x}}_3$ precesses around the fixed vector $\mathbf{L}$ with frequency

$$\tilde{\omega} = \frac{L}{I}\,, \tag{8.53}$$

in the fixed frame (and therefore $\boldsymbol{\omega}$ does also, since it is coplanar with $\hat{\mathbf{x}}_3$ and $\mathbf{L}$).

REMARKS:

1. We just said that $\boldsymbol{\omega}$ precesses around $\mathbf{L}$ with frequency L/I. What, then, is wrong with the following reasoning: "Just as the rate of change of $\hat{\mathbf{x}}_3$ equals $\boldsymbol{\omega} \times \hat{\mathbf{x}}_3$, the rate of change of $\boldsymbol{\omega}$ should equal $\boldsymbol{\omega} \times \boldsymbol{\omega}$, which is zero. Hence, $\boldsymbol{\omega}$ should remain constant." The error is that the vector $\boldsymbol{\omega}$ is not fixed in the body frame. A vector $\mathbf{A}$ must be fixed in the body frame in order for its rate of change to be given by $\boldsymbol{\omega} \times \mathbf{A}$.

2. We found in eqs. (8.49) and (8.45) that a person standing on the rotating body sees $\mathbf{L}$ (and $\boldsymbol{\omega}$) precess with frequency $\Omega \equiv \omega_3(I_3 - I)/I$ around $\hat{\mathbf{x}}_3$. But we found in eq. (8.53) that a person standing in the fixed frame sees $\hat{\mathbf{x}}_3$ (and $\boldsymbol{\omega}$) precess with frequency L/I around $\mathbf{L}$. Are these two facts compatible? Should we have obtained the same frequency from either point of view? (Answers: yes, no).

   These two frequencies are indeed consistent, as can be seen from the following reasoning. Consider the plane (call it S) containing the three vectors $\mathbf{L}$, $\boldsymbol{\omega}$, and $\hat{\mathbf{x}}_3$. We know from eq. (8.49) that S rotates with frequency $\Omega\hat{\mathbf{x}}_3$ with respect to the body. Therefore, the body rotates with frequency $-\Omega\hat{\mathbf{x}}_3$ with respect to S. And from eq. (8.53), S rotates with frequency $(L/I)\hat{\mathbf{L}}$ with respect to the fixed frame. Therefore, the total angular velocity of the body with respect to the fixed frame (using the frame S as an intermediate step) is

$$\boldsymbol{\omega}_{\text{total}} = \frac{L}{I}\hat{\mathbf{L}} - \Omega\hat{\mathbf{x}}_3. \tag{8.54}$$

   But from eq. (8.51), this is simply $\boldsymbol{\omega}$, as it should be. So the two frequencies in eqs. (8.45) and (8.53) are indeed consistent.

   For the earth, $\Omega \equiv \omega_3(I_3 - I)/I$ and L/I are much different. L/I is roughly equal to L/I_3, which is essentially equal to ω_3. Ω, on the other hand is about $(1/300)\omega_3$. Basically, an external observer sees $\boldsymbol{\omega}$ precess around its cone at roughly the rate at which the earth spins. But it's not exactly the same rate, and this difference is what causes the earth-based observer to see $\boldsymbol{\omega}$ precess with a nonzero Ω. ♣

8.7 Heavy symmetric top

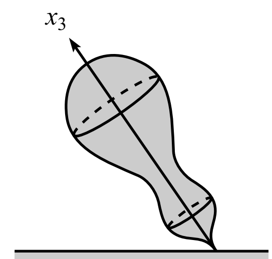

Figure 8.25

Consider now a heavy symmetrical top; that is, one that spins on a table, under the influence of gravity (see Fig. 8.25). Assume that the tip of the top is fixed on the table by a free pivot. We will solve for the motion of the top in two different ways. The first will use $\boldsymbol{\tau} = d\mathbf{L}/dt$. The second will use the Lagrangian method.

8.7.1 Euler angles

For both of these methods, it is very convenient to use the *Euler angles*, θ,ϕ,ψ, which are shown in Fig. 8.26 and are defined as follows.

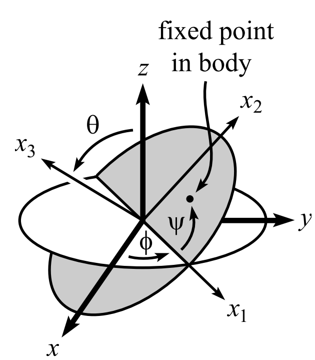

Figure 8.26

- θ**:** Let $\hat{\mathbf{x}}_3$ be the symmetry axis of the top. Define θ to be the angle that $\hat{\mathbf{x}}_3$ makes with the vertical axis $\hat{\mathbf{z}}$ of the fixed frame.

- ϕ**:** Draw the plane orthogonal to $\hat{\mathbf{x}}_3$. Let $\hat{\mathbf{x}}_1$ be the intersection of this plane with the horizontal x-y plane. Define ϕ to be the angle $\hat{\mathbf{x}}_1$ makes with the $\hat{\mathbf{x}}$ axis of the fixed frame.

- ψ**:** Let $\hat{\mathbf{x}}_2$ be orthogonal to $\hat{\mathbf{x}}_3$ and $\hat{\mathbf{x}}_1$, as shown. Let frame S be the frame whose axes are $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, and $\hat{\mathbf{x}}_3$. Define ψ to be the angle of rotation of the body around the $\hat{\mathbf{x}}_3$ axis in frame S. (That is, $\dot{\psi}\hat{\mathbf{x}}_3$ is the angular velocity of the body with respect to S.) Note that the angular velocity of frame S with respect to the fixed frame is $\dot{\phi}\hat{\mathbf{z}} + \dot{\theta}\hat{\mathbf{x}}_1$.

The angular velocity of the body with respect to the fixed frame is equal to the angular velocity of the body with respect to frame S, plus the angular velocity of frame S with respect to the fixed frame. In other words, it is

$$\boldsymbol{\omega} = \dot{\psi}\hat{\mathbf{x}}_3 + (\dot{\phi}\hat{\mathbf{z}} + \dot{\theta}\hat{\mathbf{x}}_1). \tag{8.55}$$

Note that the vector $\hat{\mathbf{z}}$ is not orthogonal to $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_3$. It is often more convenient to rewrite $\boldsymbol{\omega}$ entirely in terms of the orthogonal $\hat{\mathbf{x}}_1$,$\hat{\mathbf{x}}_2$,$\hat{\mathbf{x}}_3$ basis vectors. Since $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{x}}_3 + \sin\theta\hat{\mathbf{x}}_2$, eq. (8.55) gives

$$\boldsymbol{\omega} = (\dot{\psi} + \dot{\phi}\cos\theta)\hat{\mathbf{x}}_3 + \dot{\phi}\sin\theta\hat{\mathbf{x}}_2 + \dot{\theta}\hat{\mathbf{x}}_1. \tag{8.56}$$

This form of $\boldsymbol{\omega}$ is often more useful, because $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, and $\hat{\mathbf{x}}_3$ are principal axes of the body. (We are assuming that we are working with a symmetrical top, with $I_1 = I_2 \equiv I$. Hence, any axes in the $\hat{\mathbf{x}}_1$-$\hat{\mathbf{x}}_2$ plane are principal axes.) Although $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ are not fixed in the object, they are still good principal axes at any instant.

8.7.2 Digression on the components of $\vec{\omega}$

The previous expressions for $\boldsymbol{\omega}$ look rather formidable, but there is a very helpful diagram we can draw (see Fig. 8.27) which makes it easier to see what is going on. Let's talk a bit about this before returning to the original problem of the spinning top. The diagram is rather pithy, so we'll go through it nice and slowly.

In the following discussion, we will simplify things by setting $\dot{\theta} = 0$. All the interesting features of $\boldsymbol{\omega}$ remain. The $\dot{\theta}\hat{\mathbf{x}}_1$ component of $\boldsymbol{\omega}$ in eqs. (8.55) and (8.56) simply arises from the easily-visualizable rising and falling of the top. We will therefore concentrate here on the more complicated issues, namely the components of $\boldsymbol{\omega}$ in the plane of $\hat{\mathbf{x}}_3$, $\hat{\mathbf{z}}$, and $\hat{\mathbf{x}}_2$.

With $\dot{\theta} = 0$, Fig. 8.27 shows the vector $\boldsymbol{\omega}$ in the $\hat{\mathbf{x}}_3$-$\hat{\mathbf{z}}$-$\hat{\mathbf{x}}_2$ plane (the way we've drawn it, $\hat{\mathbf{x}}_1$ points into the page, in contrast with Fig. 8.26). This is an extremely useful diagram, and we will refer to it many times in the problems for this chapter. There are numerous comments to be made on it, so let's just list them out.

1. If someone asks you to "decompose" $\boldsymbol{\omega}$ into pieces along $\hat{\mathbf{z}}$ and $\hat{\mathbf{x}}_3$, what would you do? Would you draw the lines perpendicular to these axes to obtain the lengths shown (which we will label as ω_z and ω_3), or would you draw the lines parallel to these axes to obtain the lengths shown (which we will label as Ω and ω')? There is no "correct" answer to this question. The four quantities, ω_z, ω_3, Ω, ω' simply represent different things. We will interpret each of these below, along with ω_2 (the projection of $\boldsymbol{\omega}$ along $\hat{\mathbf{x}}_2$). It turns out that Ω and ω' are the frequencies that your eye can see the easiest, while ω_2 and ω_3 are what you want to use when you're doing calculations involving the angular momentum. (And as far as I can see, ω_z is not of much use.)

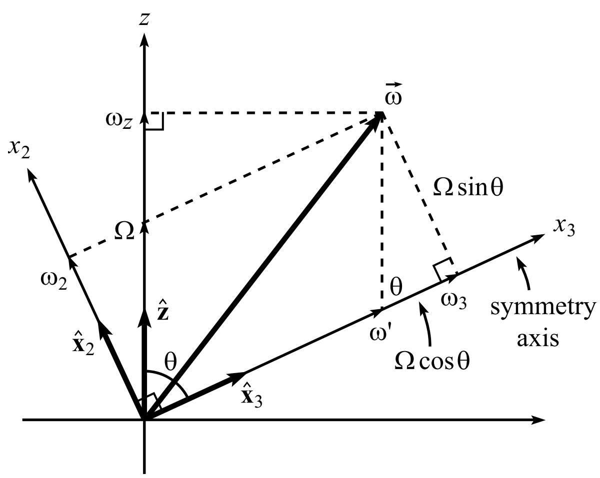

Figure 8.27

2. Note that it is true that
$$\boldsymbol{\omega} = \omega' \hat{\mathbf{x}}_3 + \Omega \hat{\mathbf{z}}, \tag{8.57}$$
but it is *not* true that $\boldsymbol{\omega} = \omega_z \hat{\mathbf{z}} + \omega_3 \hat{\mathbf{x}}_3$. Another true statement is
$$\boldsymbol{\omega} = \omega_3 \hat{\mathbf{x}}_3 + \omega_2 \hat{\mathbf{x}}_2. \tag{8.58}$$

3. In terms of the Euler angles, we see (by comparing eq. (8.57) with eq. (8.55), with $\dot\theta = 0$) that
$$\begin{aligned} \omega' &= \dot\psi, \\ \Omega &= \dot\phi. \end{aligned} \tag{8.59}$$
And we also have (by comparing eq. (8.58) with eq. (8.56), with $\dot\theta = 0$)
$$\begin{aligned} \omega_3 &= \dot\psi + \dot\phi\cos\theta = \omega' + \Omega\cos\theta, \\ \omega_2 &= \dot\phi\sin\theta = \Omega\sin\theta. \end{aligned} \tag{8.60}$$
These are also clear from Fig. 8.27.

   There is therefore technically no need to introduce the new ω_2, ω_3, Ω, ω' definitions in Fig. 8.27, since the Euler angles are quite sufficient. But we will be referring to this figure many times, and it is a little easier to refer to these omega's than to the various combinations of Euler angles.

4. Ω is the easiest of these frequencies to visualize. It is simply the frequency of precession of the top around the vertical $\hat{\mathbf{z}}$ axis.[13] In other words, the

[13] Although we're using the same letter, this Ω doesn't have anything to do with the Ω defined in eq. (8.45), except for the fact that they both represent a precession frequency.

symmetry axis $\hat{\mathbf{x}}_3$ traces out a cone around the $\hat{\mathbf{z}}$ axis with frequency Ω. (Note that this precession frequency is *not* ω_z.) Let's prove this.

The vector $\boldsymbol{\omega}$ is the vector which gives the speed of any point (at position $\mathbf{r}$) fixed in the top as $\boldsymbol{\omega} \times \mathbf{r}$. Therefore, since the vector $\hat{\mathbf{x}}_3$ is fixed in the top, we may write

$$\frac{d\hat{\mathbf{x}}_3}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{x}}_3 = (\omega'\hat{\mathbf{x}}_3 + \Omega\hat{\mathbf{z}}) \times \hat{\mathbf{x}}_3 = (\Omega\hat{\mathbf{z}}) \times \hat{\mathbf{x}}_3. \tag{8.61}$$

But this is precisely the expression for the rate of change of a vector rotating around the $\hat{\mathbf{z}}$ axis, with frequency Ω. (This was exactly the same type of proof as the one leading to eq. (8.52).)

REMARK: In the derivation of eq. (8.61), we've basically just stripped off a certain part of $\boldsymbol{\omega}$ that points along the $\hat{\mathbf{x}}_3$ axis, because a rotation around $\hat{\mathbf{x}}_3$ contributes nothing to the motion of $\hat{\mathbf{x}}_3$. Note, however, that there is in fact an infinite number of ways to strip off a piece along $\hat{\mathbf{x}}_3$. For example, we can also break $\boldsymbol{\omega}$ up as, say, $\boldsymbol{\omega} = \omega_3\hat{\mathbf{x}}_3 + \omega_2\hat{\mathbf{x}}_2$. We then obtain $d\hat{\mathbf{x}}_3/dt = (\omega_2\hat{\mathbf{x}}_2) \times \hat{\mathbf{x}}_3$, which means that $\hat{\mathbf{x}}_3$ is instantaneously rotating around $\hat{\mathbf{x}}_2$ with frequency ω_2. Although this is true, it is not as useful as the result in eq. (8.61), because the $\hat{\mathbf{x}}_2$ axis changes with time. The point here is that the instantaneous angular velocity vector around which the symmetry axis rotates is not well-defined (Problem 2 discusses this issue).[14] But the $\hat{\mathbf{z}}$-axis is the only one of these angular velocity vectors that is fixed. When we look at the top, we therefore see it precessing around the $\hat{\mathbf{z}}$-axis. ♣

5. ω' is also easy to visualize. Imagine that you are at rest in a frame that rotates around the $\hat{\mathbf{z}}$-axis with frequency Ω. Then you will see the symmetry axis of the top remain perfectly still, and the only motion you will see is the top spinning around this axis with frequency ω'. (This is true because $\boldsymbol{\omega} = \omega'\hat{\mathbf{x}}_3 + \Omega\hat{\mathbf{z}}$, and the rotation of your frame causes you to not see the $\Omega\hat{\mathbf{z}}$ part.) If you paint a dot somewhere on the top, then the dot will trace out a fixed tilted circle, and the dot will return to, say, its maximum height at frequency ω'.

   Note that someone in the lab frame will see the dot undergo a rather complicated motion, but she must observe the same frequency at which the dot returns to its highest point. Hence, ω' is something quite physical in the lab frame, also.

6. ω_3 is what you use to obtain the component of $\mathbf{L}$ along $\hat{\mathbf{x}}_3$, because $L_3 = I_3\omega_3$. It is not quite as easy to visualize as Ω and ω', but it is the frequency with which the top instantaneously rotates, as seen by someone at rest in a frame that rotates around the $\hat{\mathbf{x}}_2$ axis with frequency ω_2. (This is true because $\boldsymbol{\omega} = \omega_2\hat{\mathbf{x}}_2 + \omega_3\hat{\mathbf{x}}_3$, and the rotation of the frame causes you to not see the $\omega_2\hat{\mathbf{x}}_2$ part.) This rotation is a little harder to see, because the $\hat{\mathbf{x}}_2$ axis changes with time.

[14] The instantaneous angular velocity of the *whole body* is well defined, of course. But if you just look at the symmetry axis by itself, then there is an ambiguity (see footnote 9).

There is one physical scenario in which ω_3 is the easily observed frequency. Imagine that the top is precessing around the $\hat{\mathbf{z}}$ axis at constant θ, and imagine that the top has a frictionless rod protruding along its symmetry axis. If you grab the rod and stop the precession motion (so that the top is now spinning around its stationary symmetry axis), then this spinning will occur at frequency ω_3. This is true because when you grab the rod, you apply a torque in only the (negative) $\hat{\mathbf{x}}_2$ direction. Therefore, you don't change L_3, and hence you don't change ω_3.

7. ω_2 is similar to ω_3, of course. ω_2 is what you use to obtain the component of $\mathbf{L}$ along $\hat{\mathbf{x}}_2$, because $L_2 = I_2\omega_2$. It is the frequency with which the top instantaneously rotates, as seen by someone at rest in a frame that rotates around the $\hat{\mathbf{x}}_3$ axis with frequency ω_3. (This is true because $\boldsymbol{\omega} = \omega_2\hat{\mathbf{x}}_2 + \omega_3\hat{\mathbf{x}}_3$, and the rotation of the frame causes you to not see the $\omega_3\hat{\mathbf{x}}_3$ part.) Again, this rotation is a little harder to see, because the $\hat{\mathbf{x}}_3$ axis changes with time.

8. ω_z is not very useful (as far as I can see). The most important thing to note about it is that it is *not* the frequency of precession around the $\hat{\mathbf{z}}$-axis, even though it is the projection of $\boldsymbol{\omega}$ onto $\hat{\mathbf{z}}$. The frequency of the precession is Ω, as we found above in eq. (8.61). A true, but somewhat useless, fact about ω_z is that if someone is at rest in a frame that rotates around the $\hat{\mathbf{z}}$ axis with frequency ω_z, then she will see all points in the top instantaneously rotating around the $\hat{\mathbf{x}}$-axis with frequency ω_x, where ω_x is the projection of $\boldsymbol{\omega}$ onto the horizontal $\hat{\mathbf{x}}$ axis. (This is true because $\boldsymbol{\omega} = \omega_x\hat{\mathbf{x}} + \omega_z\hat{\mathbf{z}}$, and the rotation of the frame causes you to not see the $\omega_z\hat{\mathbf{z}}$ part.)

8.7.3 Torque method

This method of solving the heavy top will be straightforward, although a little tedious. We include it here to (1) show that this problem can be done without resorting to Lagrangians, and to (2) get some practice using $\boldsymbol{\tau} = d\mathbf{L}/dt$.

We will make use of the form of $\boldsymbol{\omega}$ given in eq. (8.56), because there it is broken up into the principal-axis components. For convenience, define $\dot\beta = \dot\psi + \dot\phi\cos\theta$, so that

$$\boldsymbol{\omega} = \dot\beta\hat{\mathbf{x}}_3 + \dot\phi\sin\theta\hat{\mathbf{x}}_2 + \dot\theta\hat{\mathbf{x}}_1. \tag{8.62}$$

Note that we've returned to the most general motion, where $\dot\theta$ is not necessarily zero.

We will choose the tip of the top as our origin, which is assumed to be fixed on the table.[15] Let the principal moments relative to this origin be $I_1 = I_2 \equiv I$, and I_3. The angular momentum of the body is then

$$\mathbf{L} = I_3\dot\beta\hat{\mathbf{x}}_3 + I\dot\phi\sin\theta\hat{\mathbf{x}}_2 + I\dot\theta\hat{\mathbf{x}}_1. \tag{8.63}$$

We must now calculate $d\mathbf{L}/dt$. What makes this nontrivial is the fact that the $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, and $\hat{\mathbf{x}}_3$ unit vectors change with time (they change with θ and ϕ). But let's

[15] We could use the CM as our origin, but then we would have to include the complicated forces acting at the pivot point, which is difficult.

forge ahead and take the derivative of eq. (8.63). Using the product rule (which works fine with the product of a scalar and a vector), we have

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= I_3\frac{d\dot\beta}{dt}\hat{\mathbf{x}}_3 + I\frac{d(\dot\phi\sin\theta)}{dt}\hat{\mathbf{x}}_2 + I\frac{d\dot\theta}{dt}\hat{\mathbf{x}}_1 \\ &\quad + I_3\dot\beta\frac{d\hat{\mathbf{x}}_3}{dt} + I\dot\phi\sin\theta\frac{d\hat{\mathbf{x}}_2}{dt} + I\dot\theta\frac{d\hat{\mathbf{x}}_1}{dt}\,. \end{aligned} \tag{8.64}$$

Using a little geometry, you can show

$$\begin{aligned}\frac{d\hat{\mathbf{x}}_3}{dt} &= -\dot\theta\hat{\mathbf{x}}_2 + \dot\phi\sin\theta\hat{\mathbf{x}}_1, \\ \frac{d\hat{\mathbf{x}}_2}{dt} &= \dot\theta\hat{\mathbf{x}}_3 - \dot\phi\cos\theta\hat{\mathbf{x}}_1, \\ \frac{d\hat{\mathbf{x}}_1}{dt} &= -\dot\phi\sin\theta\hat{\mathbf{x}}_3 + \dot\phi\cos\theta\hat{\mathbf{x}}_2. \end{aligned} \tag{8.65}$$

As an exercise, prove these by making use of Fig. 8.26. In the first equation, for example, show that a change in θ causes $\hat{\mathbf{x}}_3$ to move a certain distance in the $\hat{\mathbf{x}}_2$ direction; and show that a change in ϕ causes $\hat{\mathbf{x}}_3$ to move a certain distance in the $\hat{\mathbf{x}}_1$ direction. Plugging eqs. (8.65) into eq. (8.64) gives

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= I_3\ddot\beta\hat{\mathbf{x}}_3 + \Big(I\ddot\phi\sin\theta + 2I\dot\theta\dot\phi\cos\theta - I_3\dot\beta\dot\theta\Big)\hat{\mathbf{x}}_2 \\ &\quad + \Big(I\ddot\theta - I\dot\phi^2\sin\theta\cos\theta + I_3\dot\beta\dot\phi\sin\theta\Big)\hat{\mathbf{x}}_1. \end{aligned} \tag{8.66}$$

The torque on the top arises from gravity pulling down on the CM. $\boldsymbol\tau$ points in the $\hat{\mathbf{x}}_1$ direction and has magnitude $Mg\ell\sin\theta$, where ℓ is the distance from the pivot to CM. Equating $\boldsymbol\tau$ with $d\mathbf{L}/dt$ gives

$$\ddot\beta = 0, \tag{8.67}$$

for the $\hat{\mathbf{x}}_3$ component. Therefore, $\dot\beta$ is a constant, which we will call ω_3 (an obvious label, in view of eq. (8.62)). The other two components of $\boldsymbol\tau = d\mathbf{L}/dt$ then give

$$\begin{aligned} I\ddot\phi\sin\theta + \dot\theta(2I\dot\phi\cos\theta - I_3\omega_3) &= 0, \\ (Mg\ell + I\dot\phi^2\cos\theta - I_3\omega_3\dot\phi)\sin\theta &= I\ddot\theta. \end{aligned} \tag{8.68}$$

We will wait to fiddle with these equations until we have derived them again using the Lagrangian method.

8.7.4 Lagrangian method

Eq. (8.13) gives the kinetic energy of the top as $T = \frac{1}{2}\boldsymbol\omega\cdot\mathbf{L}$. Eqs. (8.62) and (8.63) give (using $\dot\psi + \dot\phi\cos\theta$ instead of the shorthand $\dot\beta$)[16]

$$T = \frac{1}{2}\boldsymbol\omega\cdot\mathbf{L} = \frac{1}{2}I_3(\dot\psi + \dot\phi\cos\theta)^2 + \frac{1}{2}I(\dot\phi^2\sin^2\theta + \dot\theta^2). \tag{8.69}$$

[16] It was ok to use β in Subsection 8.7.3; we introduced it simply because it was quicker to write. But we can't use β here, because it depends on the other coordinates, and the Lagrangian method requires the use of independent coordinates. (The variational proof back in Chapter 5 assumed this independence.)

The potential energy is

$$V = Mg\ell\cos\theta, \tag{8.70}$$

where ℓ is the distance from the pivot to CM. The Lagrangian is $\mathcal{L} = T - V$ (we'll use "$\mathcal{L}$" here to avoid confusion with the angular momentum, "L"), and so the equation of motion obtained from varying ψ is

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\psi} = \frac{\partial\mathcal{L}}{\partial\psi} \Longrightarrow \frac{d}{dt}(\dot\psi + \dot\phi\cos\theta) = 0. \tag{8.71}$$

Therefore, $\dot\psi + \dot\phi\cos\theta$ is a constant. Call it ω_3. The equations of motion obtained from varying ϕ and θ are then

$$\begin{aligned}\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\phi} = \frac{\partial\mathcal{L}}{\partial\phi} \quad &\Longrightarrow \quad \frac{d}{dt}\Big(I_3\omega_3\cos\theta + I\dot\phi\sin^2\theta\Big) = 0,\\ \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta} = \frac{\partial\mathcal{L}}{\partial\theta} \quad &\Longrightarrow \quad I\ddot\theta = (Mg\ell + I\dot\phi^2\cos\theta - I_3\omega_3\dot\phi)\sin\theta. \end{aligned} \tag{8.72}$$

These are equivalent to eqs. (8.68), as you can check. Note that there are two conserved quantities, arising from the facts that $\partial\mathcal{L}/\partial\psi$ and $\partial\mathcal{L}/\partial\phi$ equal zero. The conserved quantities are simply the angular momenta in the $\hat{\mathbf{x}}_3$ and $\hat{\mathbf{z}}$ directions, respectively. (There is no torque in the plane spanned by these vectors, since the torque points in the $\hat{\mathbf{x}}_1$ direction.)

8.7.5 Gyroscope with $\dot\theta = 0$

A special case of eqs. (8.68) occurs when $\dot\theta = 0$. In this case, the first of eqs. (8.68) says that $\dot\phi$ is a constant. The CM of the top therefore undergoes uniform circular motion in a horizontal plane. Let $\Omega \equiv \dot\phi$ be the frequency of this motion (this is the same notation as in eq. (8.59)). Then the second of eqs. (8.68) says that

$$I\Omega^2\cos\theta - I_3\omega_3\Omega + Mg\ell = 0. \tag{8.73}$$

This quadratic equation may be solved to yield two possible precessional frequencies for the top. (Yes, there are indeed two of them, provided that ω_3 is greater than a certain minimum value.)

The previous pages in this "Heavy Symmetric Top" section have been a bit abstract. So let's now pause for a moment, take a breather, and rederive eq. (8.73) from scratch. That is, we'll assume $\dot\theta = 0$ from the start of the solution, and solve things by simply finding $\mathbf{L}$ and using $\boldsymbol{\tau} = d\mathbf{L}/dt$, in the spirit of Section 8.4.2.

The following Gyroscope example is the classic "top" problem. We'll warm up by solving it in an approximate way. Then we'll do it for real.

Example (Gyroscope): A symmetric top of mass M has its CM a distance ℓ from its pivot. The moments of inertia relative to the pivot are $I_1 = I_2 \equiv I$ and I_3. The top spins around its symmetry axis with frequency ω_3 (in the language of Section 8.7.2), and initial conditions have been set up so that the CM precesses in a circle around the vertical axis. The symmetry axis makes a constant angle θ with the vertical (see Fig. 8.28).

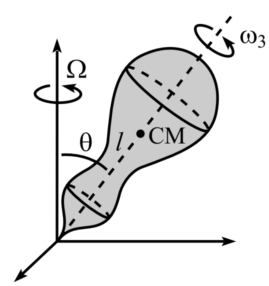

Figure 8.28

(a) Assuming that the angular momentum due to ω_3 is much larger than any other angular momentum in the problem, find an approximate expression for the frequency, Ω, of precession.

(b) Now do the problem exactly. That is, find Ω by considering all of the angular momentum.

Solution:

(a) The angular momentum (relative to the pivot) due to the spinning of the top has magnitude $L_3 = I_3\omega_3$, and it is directed along $\hat{\mathbf{x}}_3$. Let's label this angular momentum vector as $\mathbf{L}_3 \equiv L_3\hat{\mathbf{x}}_3$. As the top precesses, $\mathbf{L}_3$ traces out a cone around the vertical axis. So the tip of $\mathbf{L}_3$ moves in a circle of radius $L_3 \sin\theta$. The frequency of this circular motion is the frequency of precession, Ω. So $d\mathbf{L}_3/dt$, which is the velocity of the tip, has magnitude

$$\Omega(L_3 \sin\theta) = \Omega I_3\omega_3 \sin\theta, \tag{8.74}$$

and is directed into the page.

The torque relative to the pivot point is due to gravity acting on the CM, so it has magnitude $Mg\ell\sin\theta$. It is directed into the page. Therefore, $\boldsymbol{\tau} = d\mathbf{L}/dt$ gives

$$\Omega = \frac{Mg\ell}{I_3\omega_3}\,. \tag{8.75}$$

Note that this is independent of θ. And it is inversely proportional to ω_3.

(b) The error in the above analysis is that we omitted the angular momentum arising from the $\hat{\mathbf{x}}_2$ (defined in Section 8.7.1) component of the angular velocity due to the precession of the top around the $\hat{\mathbf{z}}$-axis. This component has magnitude $\Omega\sin\theta$.[17] The angular momentum due to this angular velocity component has magnitude

$$L_2 = I\Omega\sin\theta, \tag{8.76}$$

and is directed along $\hat{\mathbf{x}}_2$. Let's label this as $\mathbf{L}_2 \equiv L_2\hat{\mathbf{x}}_2$. The total $\mathbf{L} = \mathbf{L}_2 + \mathbf{L}_3$ is shown in Fig. 8.29.

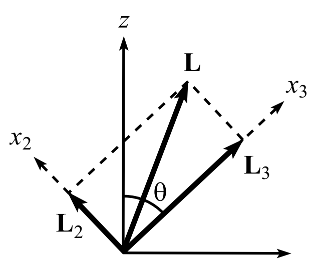

Figure 8.29

Only the horizontal component of $\mathbf{L}$ (call it $L_\perp$) changes. From the figure, $L_\perp$ is the difference in lengths of the horizontal components of $\mathbf{L}_3$ and $\mathbf{L}_2$. Therefore,

$$L_\perp = L_3\sin\theta - L_2\cos\theta = I_3\omega_3\sin\theta - I\Omega\sin\theta\cos\theta. \tag{8.77}$$

The magnitude of the rate of change of $\mathbf{L}$ is simply $\Omega L_\perp = \Omega(I_3\omega_3\sin\theta - I\Omega\sin\theta\cos\theta)$.[18] Equating this with the torque, $Mg\ell\sin\theta$, gives

$$I\Omega^2\cos\theta - I_3\omega_3\Omega + Mg\ell = 0, \tag{8.78}$$

in agreement with eq. (8.73), as we wanted to show. The quadratic formula quickly gives the two solutions for Ω, which may be written as

$$\Omega_\pm = \frac{I_3\omega_3}{2I\cos\theta}\left(1 \pm \sqrt{1 - \frac{4MIg\ell\cos\theta}{I_3^2\omega_3^2}}\right). \tag{8.79}$$

[17] The angular velocity due to the precession is $\Omega\hat{\mathbf{z}}$. We may break this up into components along the orthogonal directions $\hat{\mathbf{x}}_2$ and $\hat{\mathbf{x}}_3$. The $\Omega\cos\theta$ component along $\hat{\mathbf{x}}_3$ was absorbed into the definition of ω_3 (see Fig. 8.27).

[18] This result can also be obtained in a more formal way. Since $\mathbf{L}$ precesses with angular velocity $\Omega\hat{\mathbf{z}}$, the rate of change of $\mathbf{L}$ is $d\mathbf{L}/dt = \Omega\hat{\mathbf{z}} \times \mathbf{L}$. This cross product is easily computed in the x_2-x_3 basis, and gives the same result.

REMARK: Note that if $\theta = \pi/2$, then eq. (8.78) is actually a linear equation, so there is only one solution for Ω, which is the one in eq. (8.75). $\mathbf{L}_2$ points vertically, so it doesn't change. Only $\mathbf{L}_3$ contributes to $d\mathbf{L}/dt$. For this reason, a gyroscope is much easier to deal with when its symmetry axis is horizontal. ♣

The two solutions in eq. (8.79) are known as the *fast-precession* and *slow-precession* frequencies. For large ω_3, you can show that the slow-precession frequency is

$$\Omega_- \approx \frac{Mg\ell}{I_3\omega_3}\,, \tag{8.80}$$

in agreement with the solution found in eq. (8.75).[19] This task, along with many other interesting features of this problem (including the interpretation of the fast-precession frequency, Ω_+), is the subject of Problem 16, which you are strongly encouraged to do.

8.7.6 Nutation

Let us now solve eqs. (8.68) in a somewhat more general case, where θ is allowed to vary slightly. That is, we will consider a slight perturbation to the circular motion associated with eq. (8.73). We will assume ω_3 is large here, and we will assume that the original circular motion corresponds to the slow precession, so that $\dot\phi$ is small. Under these assumptions, we will find that the top will bounce around slightly as it travels (roughly) in a circle. This bouncing is known as *nutation*.

Since $\dot\theta$ and $\dot\phi$ are small, we can (to a good approximation) ignore the quadratic terms in eqs. (8.68) and obtain

$$\begin{aligned} I\ddot\phi\sin\theta - \dot\theta I_3\omega_3 &= 0, \\ (Mg\ell - I_3\omega_3\dot\phi)\sin\theta &= I\ddot\theta. \end{aligned} \tag{8.81}$$

We must somehow solve these equations for $\theta(t)$ and $\phi(t)$. Taking the derivative of the first equation and dropping the quadratic term gives $\ddot\theta = (I\sin\theta/I_3\omega_3)\, d^2\dot\phi/dt^2$. Substituting this into the second equation gives

$$\frac{d^2\dot\phi}{dt^2} + \omega_n^2(\dot\phi - \Omega_s) = 0, \tag{8.82}$$

where

$$\omega_n \equiv \frac{I_3\omega_3}{I} \qquad \text{and} \qquad \Omega_s = \frac{Mg\ell}{I_3\omega_3} \tag{8.83}$$

are, respectively, the frequency of nutation (as we shall soon see), and the slow-precession frequency given in eq. (8.75). Shifting variables to $y \equiv \dot\phi - \Omega_s$ in eq. (8.82) gives us a nice harmonic-oscillator equation. Solving this and then shifting back to $\dot\phi$ yields

$$\dot\phi(t) = \Omega_s + A\cos(\omega_n t + \gamma), \tag{8.84}$$

[19] This is fairly clear. If ω_3 is large enough compared to Ω, then we can ignore the first term in eq. (8.78). That is, we can ignore the effects of $\mathbf{L}_2$, which is exactly what we did in the approximate solution in part (a).

where A and γ are determined from initial conditions. Integrating this gives

$$\phi(t) = \Omega_s t + \left(\frac{A}{\omega_n}\right)\sin(\omega_n t + \gamma), \tag{8.85}$$

plus an irrelevant constant.

Now let's solve for $\theta(t)$. Plugging $\phi(t)$ into the first of eqs. (8.81) gives

$$\dot{\theta}(t) = -\left(\frac{I\sin\theta}{I_3\omega_3}\right)A\omega_n\sin(\omega_n t + \gamma) = -A\sin\theta\sin(\omega_n t + \gamma). \tag{8.86}$$

Since $\theta(t)$ doesn't change much, we may set $\sin\theta \approx \sin\theta_0$, where θ_0 is, say, the initial value of $\theta(t)$. (Any errors here are second-order effects in small quantities.) Integration then gives

$$\theta(t) = B + \left(\frac{A}{\omega_n}\sin\theta_0\right)\cos(\omega_n t + \gamma), \tag{8.87}$$

where B is a constant of integration.

Eqs. (8.85) and (8.87) show that both ϕ (neglecting the uniform $\Omega_s t$ part) and θ oscillate with frequency ω_n, and with amplitudes inversely proportional to ω_n. Note that eq. (8.83) says ω_n grows with ω_3.

Example (Sideways kick): Assume that uniform circular precession is initially taking place with $\theta = \theta_0$ and $\dot{\phi} = \Omega_s$. You then give the top a quick kick along the direction of motion, so that $\dot{\phi}$ is now equal to $\Omega_s + \Delta\Omega$ ($\Delta\Omega$ may be positive or negative). Find $\phi(t)$ and $\theta(t)$.

Solution: This is simply an exercise in initial conditions. We are given the initial values for $\dot{\phi}$, $\dot{\theta}$, and θ. So we will need to solve for the unknowns A, B and γ in eqs. (8.84), (8.86), and (8.87). $\dot{\theta}$ is initially zero, so eq. (8.86) gives $\gamma = 0$. And $\dot{\phi}$ is initially $\Omega_s + \Delta\Omega$, so eq. (8.84) gives $A = \Delta\Omega$. Finally, θ is initially θ_0, so eq. (8.87) gives $B = \theta_0 - (\Delta\Omega/\omega_n)\sin\theta_0$. Putting it all together, we have

$$\begin{aligned}
\phi(t) &= \Omega_s t + \left(\frac{\Delta\Omega}{\omega_n}\right)\sin\omega_n t, \\
\theta(t) &= \left(\theta_0 - \frac{\Delta\Omega}{\omega_n}\sin\theta_0\right) + \left(\frac{\Delta\Omega}{\omega_n}\sin\theta_0\right)\cos\omega_n t.
\end{aligned} \tag{8.88}$$

And for future reference (in the problems for this chapter), we'll also list the derivatives,

$$\begin{aligned}
\dot{\phi}(t) &= \Omega_s + \Delta\Omega\cos\omega_n t, \\
\dot{\theta}(t) &= -\Delta\Omega\sin\theta_0\sin\omega_n t.
\end{aligned} \tag{8.89}$$

REMARKS:

(a) With the initial conditions we have chosen, eq. (8.88) shows that θ always stays on one side of θ_0. If $\Delta\Omega > 0$, then $\theta(t) \le \theta_0$ (that is, the top is always at a higher position, since θ is measured from the vertical). If $\Delta\Omega < 0$, then $\theta(t) \ge \theta_0$ (that is, the top is always at a lower position).

(b) The $\sin\theta_0$ coefficient of the $\cos\omega_n t$ term in eq. (8.88) implies that the amplitude of the θ oscillation is $\sin\theta_0$ times the amplitude of the ϕ oscillation. This is precisely the factor needed to make the CM travel in a circle around its average precessing position (because a change in θ causes a displacement of $\ell\, d\theta$, whereas a change in ϕ causes a displacement of $\ell \sin\theta_0\, d\phi$). ♣

8.8 Exercises

Section 8.1: Preliminaries concerning rotations

1. **Rolling wheel** $*$

   A wheel with spokes rolls on the ground. A stationary camera takes a picture of it, from the side. Do to the nonzero exposure time of the camera, the spokes will generally appear blurred. At what location (locations) in the picture does (do) a spoke (the spokes) *not* appear blurred?

Section 8.2: The inertia tensor

2. **Inertia tensor** $*$

   Calculate the $\mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{r})$ double cross-product in eq. (8.7) by using the vector identity,

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}. \tag{8.90}$$

Section 8.3: Principal axes

3. **Tennis racket theorem** $**$

   Problem 14 gives the statement of the "tennis racket theorem," and the solution there involves Euler's equations.

   Demonstrate the theorem here by using conservation of L^2 and conservation of rotational kinetic energy in the following way. Produce an equation which says that if ω_2 and ω_3 (or ω_1 and ω_2) start small, then they must remain small. And produce the analogous equation which says that if ω_1 and ω_3 start small, then they need *not* remain small.[20]

Section 8.4: Two basic types of problems

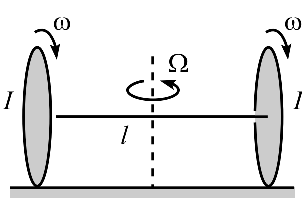

Figure 8.30

4. **Rotating axle** $**$

   Two wheels (with moment of inertia I) are connected by a massless axle of length ℓ, as shown in Fig. 8.30. The system rests on a frictionless surface, and the wheels rotate with frequency ω around the axle. Additionally, the whole system rotates with frequency Ω around the vertical axis through the center of the axle. What is largest value of Ω for which both wheels stay on the ground?

Section 8.7: Heavy symmetric top

5. **Relation between Ω and ω'** $**$

   Initial conditions have been set up so that a symmetric top undergoes precession, with its symmetry axis always making an angle θ with the vertical. The top has mass M, and the principal moments are I_3 and $I \equiv I_1 = I_2$. The CM

[20] It's another matter to show that they actually *won't* remain small. But don't bother with that here.

is a distance ℓ from the pivot. In the language of Fig. 8.27, show that ω' is related to Ω by

$$\omega' = \frac{Mg\ell}{I_3\Omega} + \Omega\cos\theta\left(\frac{I - I_3}{I_3}\right). \tag{8.91}$$

Note: You could just plug $\omega_3 = \omega' + \Omega\cos\theta$ (from eq. (8.60)) into eq. (8.73), and then solve for ω'. But solve this problem from scratch, using $\boldsymbol{\tau} = d\mathbf{L}/dt$.

6. **Sliding lollipop** $***$

Consider a lollipop made of a solid sphere of mass m and radius r, which is radially pierced by massless stick. The free end of the stick is pivoted on the ground, which is frictionless (see Fig. 8.31). The sphere slides along the ground (keeping the same contact point on the sphere), with its center moving in a circle of radius R, with frequency Ω.

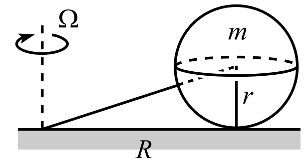

Figure 8.31

Show that the normal force between the ground and the sphere is $F_N = mg + mr\Omega^2$ (which is independent of R). Solve this by:

(a) Using a simple $\mathbf{F} = m\mathbf{a}$ argument.[21]

(b) Using a (more complicated) $\boldsymbol{\tau} = d\mathbf{L}/dt$ argument.

7. **Rolling wheel and axle** $***$

A massless axle has one end attached to a wheel (which is a uniform disc of mass m and radius r), with the other end pivoted on the ground (see Fig. 8.32). The wheel rolls on the ground without slipping, with the axle inclined at an angle θ. The point of contact with the ground traces out a circle with frequency Ω.

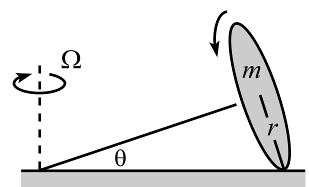

Figure 8.32

(a) Show that $\boldsymbol{\omega}$ points horizontally to the right (at the instant shown), with magnitude $\omega = \Omega/\tan\theta$.

(b) Show that the normal force between the ground and the wheel is

$$N = mg\cos^2\theta + mr\Omega^2\left(\frac{3}{2}\cos^3\theta + \frac{1}{4}\cos\theta\sin^2\theta\right). \tag{8.92}$$

[21] This method happens to work here, due to the unusually nice nature of the sphere's motion. For more general motion (for example, in Problem 21, where the sphere is spinning), you must use $\vec{\tau} = d\mathbf{L}/dt$.

8.9 Problems

Section 8.1: Preliminaries concerning rotations

1. **Fixed points on a sphere** $**$

   Consider a transformation of a rigid sphere into itself. Show that two points on the sphere end up where they started.

2. **Many different $\vec{\omega}$'s** $*$

   Consider a particle at the point $(a, 0, 0)$, with velocity $(0, v, 0)$. This particle may be considered to be rotating around many different $\boldsymbol{\omega}$ vectors passing through the origin. (There is no one "correct" $\boldsymbol{\omega}$.) Find all the possible $\boldsymbol{\omega}$'s. (That is, find their directions and magnitudes.)

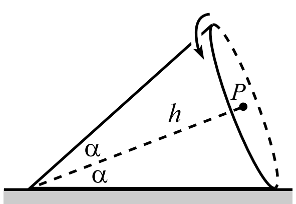

Figure 8.33

3. **Rolling cone** $**$

   A cone rolls without slipping on a table. The half-angle at the vertex is α, and the axis of the cone has length h (see Fig. 8.33). Let the speed of the center of the base (label this as point P) be v. What is the angular velocity of the cone with respect to the lab frame (at the instant shown)?

   There are many ways to do this problem, so you are encouraged to take a look at the three given solutions, even if you solve it.

Section 8.2: The inertia tensor

4. **Parallel-axis theorem**

   Let (X, Y, Z) be the position of an object's CM, and let (x', y', z') be the position relative to the CM. Prove the parallel-axis theorem, eq. (8.17), by setting $x = X + x'$, $y = Y + y'$, and $z = Z + z'$ in eq. (8.8).

Section 8.3: Principal axes

5. **Existence of principal axes for a pancake** $*$

   Given a pancake object in the x-y plane, show that there exist principal axes by considering what happens to the integral $\int xy$ as the coordinate axes are rotated about the origin.

6. **Symmetries and principal axes for a pancake** $**$

   A rotation of the axes in the x-y plane through an angle θ transforms the coordinates according to

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{8.93}$$

   Use this to show that if a pancake object in the x-y plane has a symmetry under a rotation through $\theta \neq \pi$, then all axes (through the origin) in the plane are principal axes.

7. **Rotating square** $*$

Here's an exercise in geometry. Theorem 8.5 says that if the moments of inertia of two principal axes are equal, then any axis in the plane of these axes is a principal axis. This means that the object will rotate happily about any axis in this plane (no torque is needed). Demonstrate this explicitly for four masses m in the shape of a square (which obviously has two moments equal), with the CM as the origin (see Fig. 8.34). Assume that the masses are connected with strings to the axis, as shown. Your task is to show that the tensions in the strings are such that there is no torque about the center of the square.

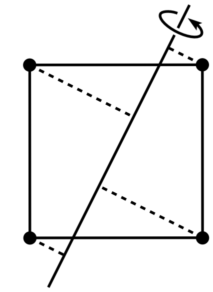

Figure 8.34

8. **A nice cylinder** $*$

What must the ratio of height to diameter of a cylinder be so that every axis is a principal axis (with the CM as the origin)?

Section 8.4: Two basic types of problems

9. **Rotating rectangle** $*$

A flat uniform rectangle with sides of length a and b sits it space (not rotating). You strike the corners at the ends of one diagonal, with equal and opposite forces (see Fig. 8.35). Show that the resulting initial $\boldsymbol{\omega}$ points along the other diagonal.

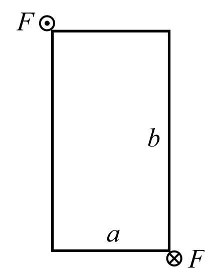

Figure 8.35

10. **Rotating stick** $**$

A stick of mass m and length ℓ spins with frequency ω around an axis, as shown in Fig. 8.36. The stick makes an angle θ with the axis and is pivoted at its center. It is kept in this motion by two strings which are perpendicular to the axis. What is the tension in the strings?

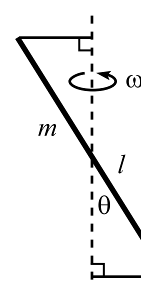

Figure 8.36

11. **Another rotating stick** $**$

A stick of mass m and length ℓ is arranged to have its CM motionless and its top end slide in a circle on a frictionless rail (see Fig. 8.37). The stick makes an angle θ with the vertical. What is the frequency of this motion?

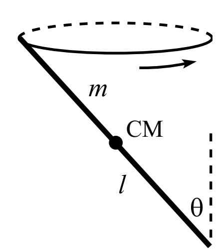

Figure 8.37

12. **Spherical pendulum** $**$

Consider a pendulum made of a massless rod of length ℓ and a point mass m. Assume conditions have been set up so that the mass moves in a horizontal circle. Let θ be the constant angle the rod makes with the vertical. Find the frequency, Ω, of this circular motion in three different ways.

(a) Use $\mathbf{F} = m\mathbf{a}$. (The net force accounts for the centripetal acceleration.)[22]

(b) Use $\boldsymbol{\tau} = d\mathbf{L}/dt$ with the pendulum pivot as the origin.

(c) Use $\boldsymbol{\tau} = d\mathbf{L}/dt$ with the mass as the origin.

[22] This method works only if you have a point mass. With an extended object, you have to use one of the following methods involving torque.

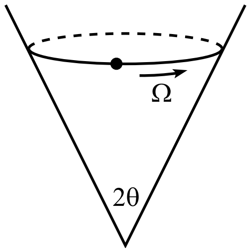

Figure 8.38

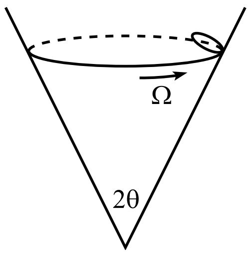

Figure 8.39

13. **Rolling in a cone** $**$

(a) A fixed cone stands on its tip, with its axis in the vertical direction. The half-angle at the vertex is θ. A particle of negligible size slides on the inside frictionless surface of the cone (see Fig. 8.38).

Assume conditions have been set up so that the particle moves in a circle at height h above the tip. What is the frequency, Ω, of this circular motion?

(b) Assume now that the surface has friction, and a small ring of radius r rolls without slipping on the surface. Assume conditions have been set up so that (1) the point of contact between the ring and the cone moves in a circle at height h above the tip, and (2) the plane of the ring is at all times perpendicular to the line joining the point of contact and the tip of the cone (see Fig. 8.39).

What is the frequency, Ω, of this circular motion? (You may work in the approximation where r is much less than the radius of the circular motion, $h \tan\theta$.)

Section 8.5: Euler's equations

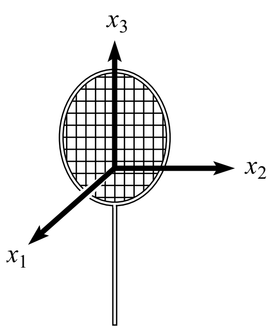

Figure 8.40

14. **Tennis racket theorem** $***$

If you try to spin a tennis racket (or a book, etc.) around any of its three principal axes, you will find that different things happen with the different axes. Assuming that the principal moments (relative to the CM) are labeled according to $I_1 > I_2 > I_3$ (see Fig. 8.40), you will find that the racket will spin nicely around the $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_3$ axes, but it will wobble in a rather messy manner if you try to spin it around the $\hat{\mathbf{x}}_2$ axis.

Verify this claim experimentally with a book (preferably lightweight, and wrapped with a rubber band), or a tennis racket (if you happen to study with one on hand).

Verify this claim mathematically. The main point here is that you can't start the motion off with $\boldsymbol{\omega}$ pointing *exactly* along a principal axis. Therefore, what you want to show is that the motion around the $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_3$ axes is *stable* (that is, small errors in the initial conditions remain small); whereas the motion around the $\hat{\mathbf{x}}_2$ axis is *unstable* (that is, small errors in the initial conditions get larger and larger, until the motion eventually doesn't resemble rotation around the $\hat{\mathbf{x}}_2$ axis).[23] Your task is to use Euler's equations to prove these statements about stability. (Exercise 3 gives another derivation of this result.)

[23]If you try for a long enough time, you will eventually be able to get the initial $\vec{\omega}$ pointing close enough to $\hat{\mathbf{x}}_2$ so that the book will remain rotating (almost) around $\hat{\mathbf{x}}_2$ for the entire time of its flight. There is, however, probably a better use for your time, as well as for the book...

Section 8.6: Free symmetric top

15. **Free-top angles** ∗

In Section 8.6.2, we showed that for a free symmetric top, the angular momentum $\mathbf{L}$, the angular velocity $\boldsymbol{\omega}$, and the symmetry axis $\hat{\mathbf{x}}_3$ all lie in a plane. Let α be the angle between $\hat{\mathbf{x}}_3$ and $\mathbf{L}$, and let β be the angle between $\hat{\mathbf{x}}_3$ and $\boldsymbol{\omega}$ (see Fig. 8.41). Find the relationship between α and β in terms of the principal moments, I and I_3.

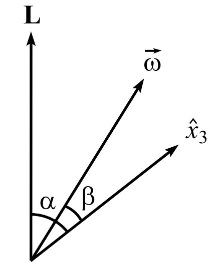

Figure 8.41

Section 8.7: Heavy symmetric top

16. **Gyroscope** ∗∗

This problem deals with the gyroscope example in Section 8.7.5, and uses the result for Ω in eq. (8.79).

(a) What is the minimum ω_3 for which circular precession is possible?

(b) Let ω_3 be very large, and find approximate expressions for $\Omega_\pm$. The phrase "very large" is rather meaningless, however. What mathematical statement should replace it?

17. **Many gyroscopes** ∗∗

N identical plates and massless sticks are arranged as shown in Fig. 8.42. Each plate is glued to the stick on its left. And each plate is attached by a free pivot to the stick on its right. (And the leftmost stick is attached by a free pivot to a pole.) You wish to set up a circular precession with the sticks always forming a straight horizontal line. What should the relative angular speeds of the plates be so that this is possible?

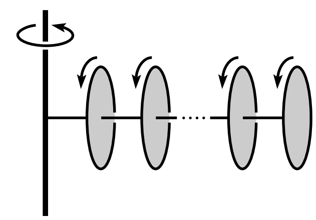
Figure 8.42

18. **Heavy top on slippery table** ∗

Solve the problem of a heavy symmetric top spinning on a frictionless table (see Fig. 8.43). You may do this by simply stating what modifications are needed in the derivation in Section 8.7.3 (or Section 8.7.4).

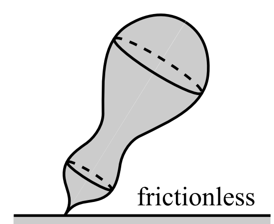

Figure 8.43

19. **Fixed highest point** ∗∗

Consider a top made of a uniform disc of radius R, connected to the origin by a massless stick (which is perpendicular to the disc) of length ℓ. Paint a dot on the top at its highest point, and label this as point P (see Fig. 8.44). You wish to set up uniform circular precession, with the stick making a constant angle θ with the vertical, and with P always being the highest point on the top. What relation between R and ℓ must be satisfied for this motion to be possible? What is the frequency of precession, Ω?

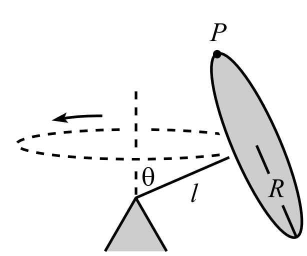

Figure 8.44

20. **Basketball on rim** ∗∗∗

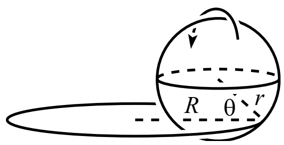

Figure 8.45

A basketball rolls without slipping around a basketball rim in such a way that the contact points trace out a great circle on the ball, and the CM moves around in a horizontal circle with frequency Ω. The radii of the ball and rim are r and R, respectively, and the ball's radius to the contact point makes an angle θ with the horizontal (see Fig. 8.45). Assume that the ball's moment of inertia around its center is $I = (2/3)mr^2$. Find Ω.

21. **Rolling lollipop** ∗∗∗

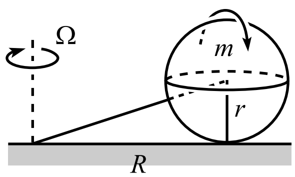

Figure 8.46

Consider a lollipop made of a solid sphere of mass m and radius r, which is radially pierced by massless stick. The free end of the stick is pivoted on the ground (see Fig. 8.46). The sphere rolls on the ground without slipping, with its center moving in a circle of radius R, with frequency Ω.

(a) Find the angular velocity vector, $\boldsymbol{\omega}$.

(b) What is the normal force between the ground and the sphere?

22. **Rolling coin** ∗∗∗

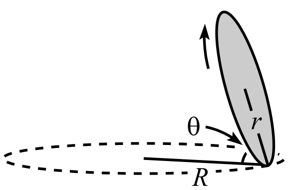

Figure 8.47

Initial conditions have been set up so that a coin of radius r rolls around in a circle, as shown in Fig. 8.47. The contact point on the ground traces out a circle of radius R, and the coin makes a constant angle θ with the horizontal. The coin rolls without slipping. (Assume that the friction with the ground is as large as needed.) What is the frequency of the circular motion of the contact point on the ground? Show that such motion exists only if $R > (5/6)r\cos\theta$.

23. **Wobbling coin** ∗∗∗∗

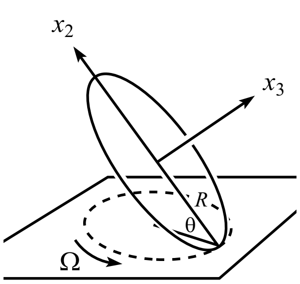

Figure 8.48

If you spin a coin around a vertical diameter on a table, it will slowly lose energy and begin a wobbling motion. The angle between the coin and the table will decrease, and eventually the coin will come to rest. Assume that this process is slow, and consider the motion when the coin makes an angle θ with the table (see Fig. 8.48). You may assume that the CM is essentially motionless. Let R be the radius of the coin, and let Ω be the frequency at which the point of contact on the table traces out its circle. Assume that the coin rolls without slipping.

(a) Show that the angular velocity vector of the coin is $\boldsymbol{\omega} = \Omega\sin\theta\hat{\mathbf{x}}_2$, where $\hat{\mathbf{x}}_2$ points upward along the coin, directly away from the contact point (see Fig. 8.27).

(b) Show that

$$\Omega = 2\sqrt{\frac{g}{R\sin\theta}}\,. \tag{8.94}$$

(c) Show that Abe (or Tom, Franklin, George, John, Dwight, Sue, or Sacagawea) appears to rotate, when viewed from above, with frequency

$$2(1-\cos\theta)\sqrt{\frac{g}{R\sin\theta}}\,. \tag{8.95}$$

24. **Nutation cusps** $**$

    (a) Using the notation and initial conditions of the example in Section 8.7.6, prove that kinks occur in nutation if and only if $\Delta\Omega = \pm\Omega_s$. (A kink is where the plot of $\theta(t)$ vs. $\phi(t)$ has a discontinuity in its slope.)

    (b) Prove that these kinks are in fact cusps. (A cusp is a kink where the plot reverses direction in the ϕ-θ plane.)

25. **Nutation circles** $**$

    (a) Using the notation and initial conditions of the example in Section 8.7.6, and assuming $\omega_3 \gg \Delta\Omega \gg \Omega_s$, find (approximately) the direction of the angular momentum right after the sideways kick takes place.

    (b) Use eqs. (8.88) to then show that the CM travels (approximately) in a circle around $\mathbf{L}$. And show that this "circular" motion is just what you would expect from the reasoning in Section 8.6.2 (in particular, eq. (8.53)), concerning the free top.

Additional problems

26. **Rolling straight?** $**$

    In some situations, for example the rolling-coin setup in Problem 22, the velocity of the CM of a rolling object changes direction as time goes by. Consider now a uniform sphere that rolls on the ground without slipping. Is it possible for the velocity of its CM to change direction? Justify your answer rigorously.

27. **Ball on paper** $***$

    A ball rolls without slipping on a table. It rolls onto a piece of paper. You slide the paper around in an arbitrary (horizontal) manner. (It's fine if there are abrupt, jerky motions, so that the ball slips with respect to the paper.) After you allow the ball to come off the paper, it will eventually resume rolling without slipping on the table. Show that the final velocity equals the initial velocity.

28. **Ball on turntable** $****$

    A ball with uniform mass density rolls without slipping on a turntable. Show that the ball moves in a circle (as viewed from the inertial lab frame), with a frequency equal to 2/7 times the frequency of the turntable.

8.10 Solutions

1. **Fixed points on a sphere**

First solution: For the purposes of Theorem 8.1, we only need to show that two points remain fixed for an *infinitesimal* transformation. But since it's possible to prove this result for a general transformation, we'll consider the general case here.

Consider the point A that ends up farthest away from where it started. (If there is more than one such point, pick any one of them.) Label the ending point B. Draw the great circle, C_{AB}, through A and B. Draw the great circle, C_A, that is perpendicular to C_{AB} at A; and draw the great circle, C_B, that is perpendicular to C_{AB} at B.

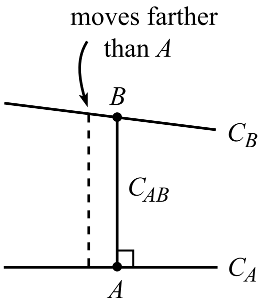

Figure 8.49

We claim that the transformation must take C_A to C_B. This is true for the following reason. The image of C_A is certainly a great circle through B, and this great circle must be perpendicular to C_{AB}, because otherwise there would exist another point that ended up farther away from its starting point than A did (see Fig. 8.49). Since there is only one great circle through B that is perpendicular to C_{AB}, the image of C_A must in fact be C_B.

Now consider the two points, P_1 and P_2, where C_A and C_B intersect. (Any two great circles must intersect.) Let's look at P_1. The distances P_1A and P_1B are equal. Therefore, the point P_1 is not moved by the transformation. This is true because P_1 ends up on C_B (because C_B is the image of C_A, which is where P_1 started), and if it ends up at a point other than P_1, then its final distance from B would be different from its initial distance from A. This would contradict the fact that distances are preserved on a rigid sphere. Likewise for P_2.

Note that for a non-infinitesimal transformation, every point on the sphere may move at some time during the transformation. What we just showed is that two of the points end up back where they started.

Second solution: In the spirit of the above solution, we can give simpler solution, but which is valid only in the case of infinitesimal transformations.

Pick any point, A, that moves during the transformation. Draw the great circle that passes through A and is perpendicular to the direction of A's motion. (Note that this direction is well-defined, because we are considering an infinitesimal transformation.) All points on this great circle must move (if they move at all) perpendicularly to the great circle, because otherwise their distances to A would change. But they cannot all move in the same direction, because then the center of the sphere would move (but it is assumed to remain fixed). Therefore, at least one point on the great circle moves in the direction opposite to the direction in which A moves. Therefore (by continuity), some point (and hence also its diametrically opposite point) on the great circle must remain fixed.

2. **Many different $\vec{\omega}$'s**

We want to find all the vectors, $\boldsymbol{\omega}$, such that $\boldsymbol{\omega} \times a\hat{\mathbf{x}} = v\hat{\mathbf{y}}$. Since $\boldsymbol{\omega}$ is orthogonal to this cross product, $\boldsymbol{\omega}$ must lie in the x-z plane. We claim that if $\boldsymbol{\omega}$ makes an angle θ with the x-axis, and has magnitude $v/(a\sin\theta)$, then it will satisfy $\boldsymbol{\omega} \times a\hat{\mathbf{x}} = v\hat{\mathbf{y}}$. Indeed,

$$\boldsymbol{\omega} \times a\hat{\mathbf{x}} = |\boldsymbol{\omega}||a\hat{\mathbf{x}}|\sin\theta\hat{\mathbf{y}} = v\hat{\mathbf{y}}. \tag{8.96}$$

Alternatively, note that $\boldsymbol{\omega}$ may be written as

$$\boldsymbol{\omega} = \frac{v}{a\sin\theta}(\cos\theta, 0, \sin\theta) = \left(\frac{v}{a\tan\theta}\,,\ 0,\ \frac{v}{a}\right), \tag{8.97}$$

and only the z-component is relevant in the cross product with $a\hat{\mathbf{x}}$.

It makes sense that the magnitude of $\boldsymbol{\omega}$ is $v/(a\sin\theta)$, because the particle is traveling in a circle of radius $a\sin\theta$ around $\boldsymbol{\omega}$, at speed v.

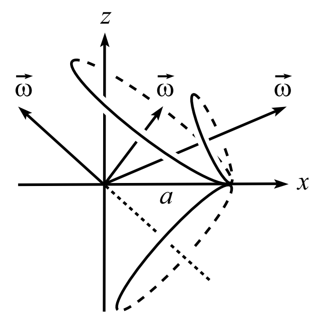

Figure 8.50

A few possible $\boldsymbol{\omega}$'s are drawn in Fig. 8.50. Technically, it is possible to have $\pi < \theta < 2\pi$, but then the $v/(a\sin\theta)$ coefficient in eq. (8.97) is negative, so $\boldsymbol{\omega}$ really points upward in the x-z plane. ($\boldsymbol{\omega}$ must point upward if the particle's velocity is to be in the positive y-direction.) Note that ω_z is independent of θ, so all the possible $\boldsymbol{\omega}$'s look like those in Fig. 8.51.

For $\theta = \pi/2$, we have $\omega = v/a$, which makes sense. If θ is very small, then ω is very large. This makes sense, because the particle is traveling around in a very small circle at the given speed v.

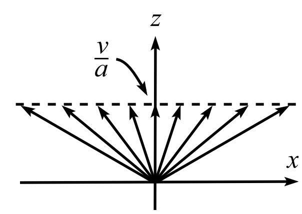

Figure 8.51

Remark: The point of this problem is that the particle may actually be in the process of having its position vector trace out a cone around one of many possible axes (or perhaps it may be undergoing some other complicated motion). If we are handed only the given initial information on position and velocity, then it is impossible to determine which of these motions is happening. And it is likewise impossible to uniquely determine $\boldsymbol{\omega}$. (This is true for a collection of points that lie on at most one line through the origin. If the points, along with the origin, span more than a 1-D line, then $\boldsymbol{\omega}$ is in fact uniquely determined.) ♣

3. **Rolling cone**

At the risk of overdoing it, we'll present three solutions. The second and third solutions are the type that tend to make your head hurt, so you may want to reread them after studying the discussion on the angular velocity vector in Section 8.7.2.

First solution: Without doing any calculations, we know that $\boldsymbol{\omega}$ points along the line of contact of the cone with the table, because these are the points on the cone that are instantaneously at rest. And we know that as time goes by, $\boldsymbol{\omega}$ rotates around in the horizontal plane with angular speed $v/(h\cos\alpha)$, because P travels at speed v in a circle of radius $h\cos\alpha$ around the z-axis.

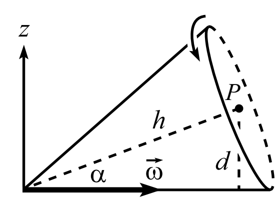

Figure 8.52

The magnitude of $\boldsymbol{\omega}$ can be found as follows. At a given instant, P may be considered to be rotating in a circle of radius $d = h\sin\alpha$ around $\boldsymbol{\omega}$. (see Fig. 8.52). Since P moves with speed v, the angular speed of this rotation is v/d. Therefore,

$$\omega = \frac{v}{h\sin\alpha}\,. \tag{8.98}$$

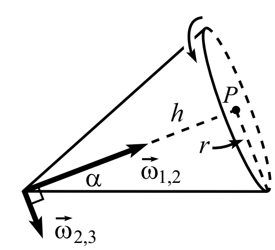

Figure 8.53

Second solution: We can use Theorem 8.3 with the following frames. S_1 is fixed in the cone; S_3 is the lab frame; and S_2 is the frame that (instantaneously) rotates around the tilted $\boldsymbol{\omega}_{2,3}$ axis shown in Fig. 8.53, at the speed such that the axis of the cone remains fixed in it. (The tip of $\boldsymbol{\omega}_{2,3}$ will trace out a circle as it precesses around the z-axis, so after the cone moves a little, we will need to use a new S_2 frame. But at any moment, S_2 instantaneously rotates around the axis perpendicular to the axis of the cone.) In the language of Theorem 8.3, $\boldsymbol{\omega}_{1,2}$ and $\boldsymbol{\omega}_{2,3}$ point in the directions shown. We must find their magnitudes and then add the vectors to find the angular velocity of S_1 with respect to S_3.

First, we have

$$|\boldsymbol{\omega}_{2,3}| = \frac{v}{h}\,, \tag{8.99}$$

because point P moves (instantaneously) with speed v in a circle of radius h around $\boldsymbol{\omega}_{2,3}$.

We now claim that

$$|\boldsymbol{\omega}_{1,2}| = \frac{v}{r}, \tag{8.100}$$

where $r = h\tan\alpha$ is the radius of the base of the cone. This is true because someone fixed in S_2 will see the endpoint of the radius (the one drawn) moving "backwards" at speed v, because it is stationary with respect to the table. Hence, the cone must be spinning with frequency v/r in S_2.

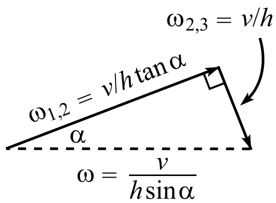

Figure 8.54

The addition of $\boldsymbol{\omega}_{1,2}$ and $\boldsymbol{\omega}_{2,3}$ is shown in Fig. 8.54. The result has magnitude $v/(h\sin\alpha)$, and it points horizontally (because $|\boldsymbol{\omega}_{2,3}|/|\boldsymbol{\omega}_{1,2}| = \tan\alpha$).

Third solution: We can use Theorem 8.3 with the following frames. S_1 is fixed in the cone; and S_3 is the lab frame (as in the second solution). But now let S_2 be the frame that rotates around the (negative) z-axis, at the speed such that the axis of the cone remains fixed in it. (Note that we can keep using this same S_2 frame as time goes by, unlike the S_2 frame in the second solution.) $\boldsymbol{\omega}_{1,2}$ and $\boldsymbol{\omega}_{2,3}$ point in the directions shown in Fig. 8.55. As above, we must find their magnitudes and then add the vectors to find the angular velocity of S_1 with respect to S_3.

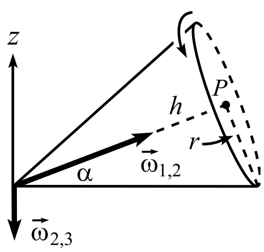

Figure 8.55

First, we have

$$|\boldsymbol{\omega}_{2,3}| = \frac{v}{h\cos\alpha}, \tag{8.101}$$

because point P moves with speed v in a circle of radius $h\cos\alpha$ around $\boldsymbol{\omega}_{2,3}$.

It's a little tricker to find $|\boldsymbol{\omega}_{1,2}|$. Consider the circle of contact points on the table where the base of the cone touches it. This circle has a radius $h/\cos\alpha$. From the point of view of someone spinning around with S_2, the table rotates backwards with frequency $|\boldsymbol{\omega}_{2,3}| = v/(h\cos\alpha)$, so this person sees the circle of contact points move with speed $v/\cos^2\alpha$ around the vertical. Since there is no slipping, the contact point on the cone must also move with this speed around the axis of the cone (which is fixed in S_2). And since the radius of the base is r, this means that the cone rotates with angular speed $v/(r\cos^2\alpha)$ with respect to S_1. Therefore,

$$|\boldsymbol{\omega}_{1,2}| = \frac{v}{r\cos^2\alpha} = \frac{v}{h\sin\alpha\cos\alpha}. \tag{8.102}$$

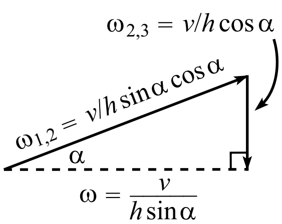

Figure 8.56

The addition of $\boldsymbol{\omega}_{1,2}$ and $\boldsymbol{\omega}_{2,3}$ is shown in Fig. 8.56. The result has magnitude $v/(h\sin\alpha)$, and it points horizontally (because $|\boldsymbol{\omega}_{2,3}|/|\boldsymbol{\omega}_{1,2}| = \sin\alpha$).

4. **Parallel-axis theorem**

Consider one of the diagonal entries in $\mathbf{I}$, say $\mathbf{I}_{11} = \int(y^2+z^2)$. In terms of the new variables, this equals

$$\begin{aligned}\mathbf{I}_{11} &= \int\Big((Y+y')^2 + (Z+z')^2\Big)\\ &= \int(Y^2+Z^2) + \int(y'^2+z'^2)\\ &= M(Y^2+Z^2) + \int(y'^2+z'^2),\end{aligned} \tag{8.103}$$

where we have used the fact that the cross terms vanish because, for example, $\int Yy' = Y\int y' = 0$, by definition of the CM.

Similarly, consider an off-diagonal entry in $\mathbf{I}$, say $\mathbf{I}_{12} = -\int xy$. We have

$$\mathbf{I}_{12} = -\int(X+x')(Y+y')$$

$$= -\int XY - \int x'y'$$
$$= -M(XY) - \int x'y', \tag{8.104}$$

where the cross terms have likewise vanished. We therefore see that all of the terms in **I** take the form of those in eq. (8.17), as desired.

5. **Existence of principal axes for a pancake**

For a pancake object, the inertia tensor **I** takes the form in eq. (8.8), with $z = 0$. Therefore, if we can find a set of axes for which $\int xy = 0$, then **I** will be diagonal, and we will have found our principal axes. We can prove, using a continuity argument, that such a set of axes exists.

Pick a set of axes, and write down the integral $\int xy \equiv I_0$. If $I_0 = 0$, then we are done. If $I_0 \neq 0$, then rotate these axes by an angle $\pi/2$, so that the new $\hat{\mathbf{x}}$ is the old $\hat{\mathbf{y}}$, and the new $\hat{\mathbf{y}}$ is the old $-\hat{\mathbf{x}}$ (see Fig. 8.57). Write down the new integral $\int xy \equiv I_{\pi/2}$. Since the new and old coordinates are related by $x_{\text{new}} = y_{\text{old}}$ and $y_{\text{new}} = -x_{\text{old}}$, we have $I_{\pi/2} = -I_0$. Therefore, since $\int xy$ switched sign during the rotation of the axes, there must exist some intermediate angle for which the integral $\int xy$ is zero.

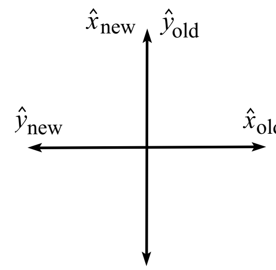

Figure 8.57

6. **Symmetries and principal axes for a pancake**

First Solution: In view of the form of the inertia tensor given in eq. (8.8), we want to show that if a pancake object has a symmetry under a rotation through $\theta \neq \pi$, then $\int xy = 0$ for any set of axes (through the origin).

Take an arbitrary set of axes and rotate them through an angle $\theta \neq \pi$. The new coordinates are $x' = (x\cos\theta + y\sin\theta)$ and $y' = (-x\sin\theta + y\cos\theta)$, so the new matrix entries, in terms of the old ones, are

$$\begin{aligned}
I'_{xx} \equiv \int x'^2 &= I_{xx}\cos^2\theta + 2I_{xy}\sin\theta\cos\theta + I_{yy}\sin^2\theta, \\
I'_{yy} \equiv \int y'^2 &= I_{xx}\sin^2\theta - 2I_{xy}\sin\theta\cos\theta + I_{yy}\cos^2\theta, \\
I'_{xy} \equiv \int x'y' &= -I_{xx}\sin\theta\cos\theta + I_{xy}(\cos^2\theta - \sin^2\theta) + I_{yy}\sin\theta\cos\theta.
\end{aligned} \tag{8.105}$$

If the object looks exactly like it did before the rotation, then $I'_{xx} = I_{xx}$, $I'_{yy} = I_{yy}$, and $I'_{xy} = I_{xy}$. The first two of these are actually equivalent statements, so we'll just use the first and third. Using $\cos^2\theta - 1 = -\sin^2\theta$, these give

$$\begin{aligned}
0 &= -I_{xx}\sin^2\theta + 2I_{xy}\sin\theta\cos\theta + I_{yy}\sin^2\theta, \\
0 &= -I_{xx}\sin\theta\cos\theta - 2I_{xy}\sin^2\theta + I_{yy}\sin\theta\cos\theta.
\end{aligned} \tag{8.106}$$

Multiplying the first of these by $\cos\theta$ and the second by $\sin\theta$, and subtracting, gives

$$2I_{xy}\sin\theta = 0. \tag{8.107}$$

Under the assumption $\theta \neq \pi$ (and $\theta \neq 0$, of course), we must therefore have $I_{xy} = 0$. Our initial axes were arbitrary; hence, any set of axes (through the origin) in the plane is a set of principal axes.

REMARK: If you don't trust this result, then you may want to show explicitly that the moments around two orthogonal axes are equal for, say, an equilateral triangle centered at the origin (which implies, by Theorem 8.5, all axes in the plane are principal axes). ♣

Second Solution: If an object is invariant under a rotation through an angle θ, then θ must be of the form $\theta = 2\pi/N$, for some integer N (convince yourself of this).[24] Consider a regular N-gon with "radius" R, with point-masses m located at the vertices. Any object that is invariant under a rotation through $\theta = 2\pi/N$ can be considered to be built out of regular point-mass N-gons of various sizes. Therefore, if we can show that any axis in the plane of a regular point-mass N-gon is a principal axis, then we're done. We can do this as follows.

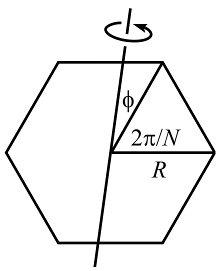

Figure 8.58

In Fig. 8.58, let ϕ be the angle between the axis and the nearest mass to its right. Label the N masses clockwise from 0 to $N-1$, starting with this one. Then the angle between the axis and mass k is $\phi + 2\pi k/N$. And the distance from mass k to the axis is $r_k = |R\sin(\phi + 2\pi k/N)|$.

The moment of inertia around the axis is $I_\phi = \sum_{k=0}^{N-1} mr_k^2$. In view of Theorem 8.5, if we can show that $I_\phi = I_{\phi'}$ for some $\phi \neq \phi'$ (with $\phi \neq \phi' + \pi$), then we have shown that every axis is a principal axis. We will do this by demonstrating that I_ϕ is independent of ϕ. We'll use a nice math trick here, which involves writing a trig function as the real part of a complex exponential. We have

$$\begin{aligned}
I_\phi &= mR^2 \sum_{k=0}^{N-1} \sin^2\left(\phi + \frac{2\pi k}{N}\right) \\
&= \frac{mR^2}{2} \sum_{k=0}^{N-1} \left(1 - \cos\left(2\phi + \frac{4\pi k}{N}\right)\right) \\
&= \frac{NmR^2}{2} - \frac{mR^2}{2} \sum_{k=0}^{N-1} \cos\left(2\phi + \frac{4\pi k}{N}\right) \\
&= \frac{NmR^2}{2} - \frac{mR^2}{2} \sum_{k=0}^{N-1} \mathrm{Re}\left(e^{i(2\phi + 4\pi k/N)}\right) \\
&= \frac{NmR^2}{2} - \frac{mR^2}{2} \mathrm{Re}\left(e^{2i\phi}\left(1 + e^{4\pi i/N} + e^{8\pi i/N} + \cdots + e^{4(N-1)\pi i/N}\right)\right) \\
&= \frac{NmR^2}{2} - \frac{mR^2}{2} \mathrm{Re}\left(e^{2i\phi}\left(\frac{e^{4N\pi i/N} - 1}{e^{4\pi i/N} - 1}\right)\right), \qquad (8.108)
\end{aligned}$$

where we have summed the geometric series to obtain the last line. The numerator in the parentheses equals $e^{4\pi i} - 1 = 0$. And if $N \neq 2$, the denominator is not zero. Therefore, if $N \neq 2$ (which is equivalent to the $\theta \neq \pi$ restriction), then

$$I_\phi = \frac{NmR^2}{2}, \qquad (8.109)$$

which is independent of ϕ. Hence, the moments around all axes in the plane are equal, so every axis in the plane is a principal axis, by Theorem 8.5

REMARKS: Given that the moments around all the axes in the plane are equal, they must be equal to $NmR^2/2$, because the perpendicular-axis theorem says that they all must be one-half of the moment around the axis perpendicular to the plane (which is NmR^2). ♣

7. **Rotating square**

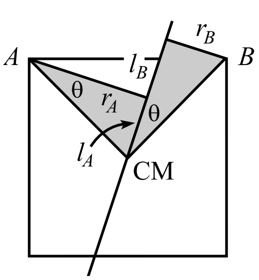

Figure 8.59

Label two of the masses A and B, as shown in Fig. 8.59. Let ℓ_A be the distance along

[24] If N is divisible by 4, then a quick application of Theorem 8.5 shows that any axis in the plane is a principal axis. But if N isn't divisible by 4, this isn't so obvious.

the axis from the CM to A's string, and let r_A be the length of A's string. Likewise for B. The force, F_A, in A's string must account for the centripetal acceleration of A. Hence, $F_A = mr_A\omega^2$. The torque around the CM due to F_A is therefore

$$\tau_A = mr_A\ell_A\omega^2. \tag{8.110}$$

Likewise, the torque around the CM due to B's string is $\tau_B = mr_B\ell_B\omega^2$, in the opposite direction.

But the two shaded triangles in Fig. 8.59 are congruent (they have the same hypotenuse and the same angle θ). Therefore, $\ell_A = r_B$ and $\ell_B = r_A$. Hence, $\tau_A = \tau_B$, and the torques cancel. The torques from the other two masses likewise cancel. (Note that a uniform square is made up of many sets of these squares of point masses, so we've also shown that no torque is needed for a uniform square.)

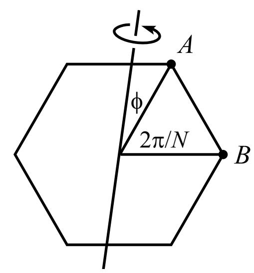

Figure 8.60

REMARK: For a general N-gon of point masses, Problem 6 shows that any axis in the plane is a principal axis. We should be able to use the above torque argument to prove this. This can be done as follows. (It's time for a nice math trick, involving the imaginary part of a complex exponential.) Using eq. (8.110), we see that the torque from mass A in Fig. 8.60 is $\tau_A = m\omega^2R^2\sin\phi\cos\phi$. Likewise, the torque from mass B is $\tau_B = m\omega^2R^2\sin(\phi+2\pi/N)\cos(\phi+2\pi/N)$, and so on. The total torque around the CM is therefore

$$\begin{aligned}
\tau &= mR^2\omega^2\sum_{k=0}^{N-1}\sin\left(\phi+\frac{2\pi k}{N}\right)\cos\left(\phi+\frac{2\pi k}{N}\right)\\
&= \frac{mR^2\omega^2}{2}\sum_{k=0}^{N-1}\sin\left(2\phi+\frac{4\pi k}{N}\right)\\
&= \frac{mR^2\omega^2}{2}\sum_{k=0}^{N-1}\mathrm{Im}\left(e^{i(2\phi+4\pi k/N)}\right)\\
&= \frac{mR^2\omega^2}{2}\mathrm{Im}\Big(e^{2i\phi}\left(1+e^{4\pi i/N}+e^{8\pi i/N}+\cdots++e^{4(N-1)\pi i/N}\right)\Big)\\
&= \frac{mR^2\omega^2}{2}\mathrm{Im}\left(e^{2i\phi}\left(\frac{e^{4N\pi i/N}-1}{e^{4\pi i/N}-1}\right)\right)\\
&= 0,
\end{aligned} \tag{8.111}$$

provided that $N \neq 2$ (but it's hard to have a 2-gon, anyway). This was essentially another proof of Problem 6. To prove that the torque was zero (which is one of the definitions of a principal axis), we showed here that $\sum r_i\ell_i = 0$. In terms of the chosen axes, this is equivalent to showing that $\sum xy = 0$, that is, showing that the off-diagonal terms in the inertia tensor vanish (which is simply another definition of the principal axes). ♣

8. **A nice cylinder**

Three axes that are certainly principal axes are the symmetry axis and two orthogonal diameters. The moments around the latter two are equal (call them I). Therefore, by Theorem 8.5, if the moment around the symmetry axis also equals I, then every axis is a principal axis.

Let the mass of the cylinder be M. Let its radius be R and its height be h. Then the moment around the symmetry axis is $MR^2/2$.

Let D be a diameter through the CM. The moment around D can be calculated as follows. Slice the cylinder into horizontal disks of thickness dy. Let ρ be the mass per unit height (so $\rho = M/h$). The mass of each disk is then $\rho\,dy$, so the moment around a diameter through the disk is $(\rho\,dy)R^2/4$. Therefore, by the parallel-axis theorem, the moment of a disk at height y (where $-h/2 \le y \le h/2$) around D is $(\rho\, dy)R^2/4 + (\rho\, dy)y^2$. Hence, the moment of the entire cylinder around D is

$$I = \int_{-h/2}^{h/2} \left(\frac{\rho R^2}{4} + \rho y^2 \right) dy = \frac{\rho R^2 h}{4} + \frac{\rho h^3}{12} = \frac{MR^2}{4} + \frac{Mh^2}{12}. \tag{8.112}$$

We want this to equal $MR^2/2$. Therefore,

$$h = \sqrt{3}R. \tag{8.113}$$

You can show that if the origin was instead taken to be the center of one of the circular faces, then the answer would be $h = \sqrt{3}R/2$.

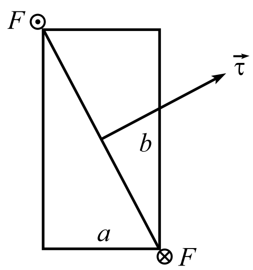

Figure 8.61

9. **Rotating rectangle**

If the force is out of the page at the upper left corner and into the page at the lower right corner, then the torque $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ points upward to the right, as shown in Fig. 8.61, with $\boldsymbol{\tau} \propto (b, a)$. The angular momentum equals $\int \boldsymbol{\tau}\, dt$. Therefore, immediately after the strike, $\mathbf{L}$ is proportional to (b, a).

The principal moments are $I_x = mb^2/12$ and $I_y = ma^2/12$. The angular momentum may be written as $\mathbf{L} = (I_x\omega_x, I_y\omega_y)$. Therefore, since we know $\mathbf{L} \propto (b, a)$, we have

$$(\omega_x, \omega_y) \propto \left(\frac{b}{I_x}, \frac{a}{I_y} \right) \propto \left(\frac{b}{b^2}, \frac{a}{a^2} \right) \propto (a, b), \tag{8.114}$$

which is the direction of the other diagonal. This answer checks in the special case $a = b$, and also in the limit where either a or b goes to zero.

10. **Rotating stick**

The angular momentum around the CM may be found as follows. Break $\boldsymbol{\omega}$ up into its components along the principal axes of the stick (which are parallel and perpendicular to the stick). The moment of inertia around the stick is zero. Therefore, to compute $\mathbf{L}$, we need to know only the component of $\boldsymbol{\omega}$ perpendicular to the stick. This component is $\omega \sin\theta$, and the associated moment of inertia is $m\ell^2/12$. Hence, the angular momentum at any time has magnitude

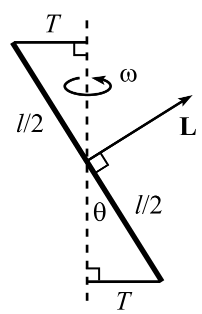

Figure 8.62

$$L = \frac{1}{12} m\ell^2 \omega \sin\theta, \tag{8.115}$$

and it points as shown in Fig. 8.62. The tip of the vector $\mathbf{L}$ traces out a circle in a horizontal plane, with frequency ω. The radius of this circle is the horizontal component of $\mathbf{L}$, which is $L_\perp \equiv L\cos\theta$. The rate of change of $\mathbf{L}$ therefore has magnitude

$$\left| \frac{d\mathbf{L}}{dt} \right| = \omega L_\perp = \omega L \cos\theta = \omega \left(\frac{1}{12} m\ell^2 \omega \sin\theta \right) \cos\theta, \tag{8.116}$$

and it is directed into the page at the instant shown.

Let the tension in the strings be T. Then the torque due to the strings is $\boldsymbol{\tau} = 2T(\ell/2)\cos\theta$, directed into the page at the instant shown. Therefore, $\boldsymbol{\tau} = d\mathbf{L}/dt$ gives

$$T\ell \cos\theta = \omega \left(\frac{1}{12} m\ell^2 \omega \sin\theta \right) \cos\theta, \tag{8.117}$$

and so

$$T = \frac{1}{12} m\ell\omega^2 \sin\theta. \tag{8.118}$$

REMARKS: For $\theta \to 0$, this goes to zero, which makes sense. For $\theta \to \pi/2$, it goes to the finite value $m\ell\omega^2/12$, which isn't entirely obvious.

Note that if we instead had a massless stick with equal masses of $m/2$ on the ends (so that the relevant moment of inertia is now $m\ell^2/4$), then our answer would be $T = m\ell\omega^2 \sin\theta/4$. This makes sense if we write it as $T = (m/2)(\ell \sin\theta/2)\omega^2$, because each tension is simply responsible for keeping a mass of $m/2$ moving in a circle of radius $(\ell/2)\sin\theta$ at frequency ω. ♣

11. **Another rotating stick**

As in Problem 10, the angular momentum around the CM may be found by breaking $\boldsymbol{\omega}$ up into its components along the principal axes of the stick (which are parallel and perpendicular to the stick). The moment of inertia around the stick is zero. Therefore, to compute $\mathbf{L}$, we need to know only the component of $\boldsymbol{\omega}$ perpendicular to the stick. This component is $\omega \sin\theta$, and the associated moment of inertia is $m\ell^2/12$. Hence, the angular momentum at any time has magnitude

$$L = \frac{1}{12} m\ell^2 \omega \sin\theta, \tag{8.119}$$

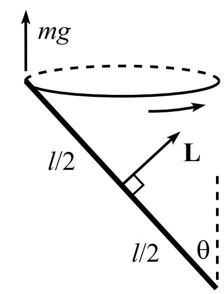

Figure 8.63

and it points as shown in Fig. 8.63. The change in $\mathbf{L}$ comes from the horizontal component. This has length $L\cos\theta$ and travels in a circle at frequency ω. Hence, $|d\mathbf{L}/dt| = \omega L \cos\theta$, and it is directed into the page at the instant shown.

The torque around the CM has magnitude $mg(\ell/2)\sin\theta$, and it points into the page at the instant shown. (This torque arises from the vertical force from the rail. There is no horizontal force from the rail, because the CM does not move.) Therefore, $\boldsymbol{\tau} = d\mathbf{L}/dt$ gives

$$\frac{mg\ell \sin\theta}{2} = \omega \left(\frac{m\ell^2 \omega \sin\theta}{12} \right) \cos\theta, \tag{8.120}$$

and so

$$\omega = \sqrt{\frac{6g}{\ell \cos\theta}}\,. \tag{8.121}$$

REMARKS: For $\theta \to \pi/2$, this goes to infinity, which makes sense. For $\theta \to 0$, it goes to the constant $\sqrt{6g/\ell}$, which isn't so obvious.

The motion in this problem is *not* possible if the bottom end of the stick, instead of the top end, slides along a rail. The magnitudes of all quantities are the same as in the original problem, but the direction of the torque (as you can check) is in the wrong direction.

If we instead had a massless stick with equal masses of $m/2$ on the ends (so that the relevant moment of inertia is now $m\ell^2/4$), then our answer would be $\omega = \sqrt{2g/(\ell\cos\theta)}$. This is simply the $\omega = \sqrt{g/[(\ell/2)\cos\theta]}$ answer for a point-mass spherical pendulum of length $\ell/2$ (see Problem 12), because the middle of the stick is motionless.

Note that the original massive stick *cannot* be treated like two sticks of length $\ell/2$. That is, the answer in eq. (8.121) is not obtained by using $\ell/2$ for the length in eq. (8.35). This is because there are internal forces in the stick that provide torques; if a free pivot were placed at the CM of the stick in this problem, the stick would not remain straight. ♣

12. **Spherical pendulum**

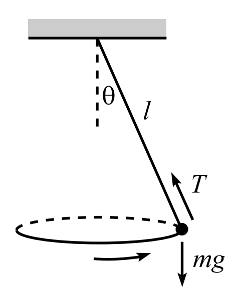

Figure 8.64

(a) The forces on the mass are gravity and the tension from the rod (see Fig. 8.64). Since there is no vertical acceleration, we have $T\cos\theta = mg$. The unbalanced horizontal force from the tension is therefore $T\sin\theta = mg\tan\theta$. This force accounts for the centripetal acceleration, $m(\ell\sin\theta)\Omega^2$. Hence,

$$\Omega = \sqrt{\frac{g}{\ell\cos\theta}}\,. \tag{8.122}$$

REMARK: For $\theta \approx 0$, this is the same as the $\sqrt{g/\ell}$ frequency for a simple pendulum. For $\theta \approx \pi/2$, it goes to infinity, which makes sense. Note that θ must be less than $\pi/2$ for circular motion to be possible. (This restriction does not hold for a gyroscope with extended mass.) ♣

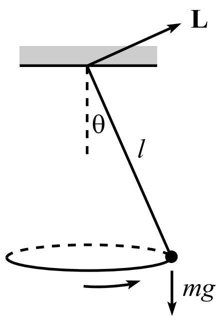

Figure 8.65

(b) The only force that applies a torque relative to the pivot is the gravitational force. The torque is $\tau = mg\ell\sin\theta$, directed into the page (see Fig. 8.65).
At this instant in time, the mass has a speed $(\ell\sin\theta)\Omega$, directed into the page. Therefore, $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ has magnitude $m\ell^2\Omega\sin\theta$, and is directed upward to the right, as shown.
The tip of $\mathbf{L}$ traces out a circle of radius $L\cos\theta$, at frequency Ω. Therefore, $d\mathbf{L}/dt$ has magnitude $\Omega L\cos\theta$, and is directed into the page.
Hence, $\boldsymbol{\tau} = d\mathbf{L}/dt$ gives $mg\ell\sin\theta = \Omega(m\ell^2\Omega\sin\theta)\cos\theta$. This yields eq. (8.122).

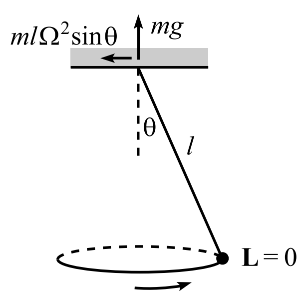

Figure 8.66

(c) The only force that applies a torque relative to the mass is that from the pivot. There are two components to this force (see Fig. 8.66).
There is the vertical piece, which is mg. Relative to the mass, this provides a torque of $mg(\ell\sin\theta)$, which is directed into the page.
There is also the horizontal piece, which accounts for the centripetal acceleration of the mass. This equals $m(\ell\sin\theta)\Omega^2$. Relative to the mass, this provides a torque of $m\ell\Omega^2\sin\theta(\ell\cos\theta)$, which is directed out of the page.
Relative to the mass, there is no angular momentum. Therefore, $d\mathbf{L}/dt = 0$. Hence, there must be no torque; so the above two torques cancel. This implies that $mg(\ell\sin\theta) = m\ell\Omega^2\sin\theta(\ell\cos\theta)$, which yields eq. (8.122).

REMARK: In problems that are more complicated than this one, it is often easier to work with a fixed pivot as the origin (if there is one) instead of the CM, because then you don't have to worry about messy pivot forces contributing to the torque. ♣

13. **Rolling in a cone**

(a) The forces on the particle are gravity (mg) and the normal force (N) from the cone. Since there is no net force in the vertical direction, we have

$$N\sin\theta = mg. \tag{8.123}$$

The inward horizontal force is therefore $N\cos\theta = mg/\tan\theta$. This force accounts for the centripetal acceleration of the particle moving in a circle of radius $h\tan\theta$. Hence, $mg/\tan\theta = m(h\tan\theta)\Omega^2$, and so

$$\Omega = \frac{1}{\tan\theta}\sqrt{\frac{g}{h}}\,. \tag{8.124}$$

(b) The forces on the ring are gravity (mg), the normal force (N) from the cone, and a friction force (F) pointing up along the cone. Since there is no net force in the vertical direction, we have

$$N\sin\theta + F\cos\theta = mg. \tag{8.125}$$

The fact that the inward horizontal force accounts for the centripetal acceleration yields

$$N\cos\theta - F\sin\theta = m(h\tan\theta)\Omega^2. \tag{8.126}$$

The previous two equations may be solved to yield F. The result is

$$F = mg\cos\theta - m\Omega^2(h\tan\theta)\sin\theta. \tag{8.127}$$

The torque on the ring (relative to the CM) is due solely to this F (because gravity provides no torque, and N points though the center of the ring, by the second assumption in the problem). Therefore, $\tau = rF$ equals

$$\tau = r\big(mg\cos\theta - m\Omega^2(h\tan\theta)\sin\theta\big). \tag{8.128}$$

and $\boldsymbol{\tau}$ points horizontally.

We must now find $d\mathbf{L}/dt$. Since are assuming $r \ll h\tan\theta$, the frequency of the spinning of the ring (call it ω) is much greater than the frequency of precession, Ω. We will therefore neglect the latter in finding $\mathbf{L}$. In this approximation, L is simply $mr^2\omega$, and $\mathbf{L}$ points upward (or downward, depending on the direction of the precession) along the cone. The horizontal component of $\mathbf{L}$ has magnitude $L_\perp \equiv L\sin\theta$, and it traces out a circle at frequency Ω. Therefore, $d\mathbf{L}/dt$ has magnitude

$$\left|\frac{d\mathbf{L}}{dt}\right| = \Omega L_\perp = \Omega L\sin\theta = \Omega(mr^2\omega)\sin\theta, \tag{8.129}$$

and it points horizontally, in the same direction as $\boldsymbol{\tau}$.

The non-slipping condition is $r\omega = (h\tan\theta)\Omega$,[25] which gives $\omega = (h\tan\theta)\Omega/r$. Using this in eq. (8.129) yields

$$\left|\frac{d\mathbf{L}}{dt}\right| = \Omega^2 mrh\tan\theta\sin\theta, \tag{8.130}$$

Equating this $|d\mathbf{L}/dt|$ with the torque in eq. (8.128) gives

$$\Omega = \frac{1}{\tan\theta}\sqrt{\frac{g}{2h}}\,. \tag{8.131}$$

This frequency is $1/\sqrt{2}$ times the frequency found in part (a).

Remark: If you consider an object with moment of inertia ηmr^2 (our ring has $\eta = 1$), then you can show by the above reasoning that the "2" in eq. (8.131) is simply replaced by $(1+\eta)$. ♣

14. **Tennis racket theorem**

Presumably the experiment worked out as it was supposed to, without too much harm to the book. Now let's demonstrate the result mathematically.

Rotation around $\hat{\mathbf{x}}_1$: If the racket is rotated (nearly) around the $\hat{\mathbf{x}}_1$ axis, then the initial ω_2 and ω_3 are much smaller than ω_1. To emphasize this, relabel $\omega_2 \to \epsilon_2$ and

[25] This is technically not quite correct, for the same reason that the earth spins around 366 times instead of 365 times in a year. But it's valid enough in the limit of small r.

$\omega_3 \to \epsilon_3$. Then eqs. (8.43) become (with the torque equal to zero, because only gravity acts on the racket)

$$\begin{aligned} 0 &= \dot{\omega}_1 - A\epsilon_2\epsilon_3, \\ 0 &= \dot{\epsilon}_2 + B\omega_1\epsilon_3, \\ 0 &= \dot{\epsilon}_3 - C\omega_1\epsilon_2, \end{aligned} \tag{8.132}$$

where we have defined (for convenience)

$$A \equiv \frac{I_2 - I_3}{I_1}, \qquad B \equiv \frac{I_1 - I_3}{I_2}, \qquad C \equiv \frac{I_1 - I_2}{I_3}. \tag{8.133}$$

Note that A, B, and C are all positive (this fact will be very important).

Our goal here is to show that if the ϵ's start out small, then they remain small. Assuming that they are small (which is true initially), the first equation says that $\dot{\omega}_1 \approx 0$ (to first order in the ϵ's). Therefore, we may assume that ω_1 is essentially constant (when the ϵ's are small). Taking the derivative of the second equation then gives $0 = \ddot{\epsilon}_2 + B\omega_1\dot{\epsilon}_3$. Plugging the value of $\dot{\epsilon}_3$ from the third equation into this yields

$$\ddot{\epsilon}_2 = -(BC\omega_1^2)\epsilon_2. \tag{8.134}$$

Because of the negative coefficient on the right-hand side, this equation describes simple harmonic motion. Therefore, ϵ_2 oscillates sinusoidally around zero. Hence, if it starts small, it remains small. By the same reasoning, ϵ_3 remains small.

We therefore see that $\boldsymbol{\omega} \approx (\omega_1, 0, 0)$ at all times, which implies that $\mathbf{L} \approx (I_1\omega_1, 0, 0)$ at all times. That is, $\mathbf{L}$ always points (nearly) along the $\hat{\mathbf{x}}_1$ direction (which is fixed in the racket frame). But the direction of $\mathbf{L}$ is fixed in the lab frame (because there is no torque). Therefore, the direction of $\hat{\mathbf{x}}_1$ must also be (nearly) fixed in the lab frame. In other words, the racket doesn't wobble.

Rotation around $\hat{\mathbf{x}}_3$: The calculation goes through exactly as above, except with "1" and "3" interchanged. We find that if ϵ_1 and ϵ_2 start small, they remain small. And $\boldsymbol{\omega} \approx (0, 0, \omega_3)$ at all times.

Rotation around $\hat{\mathbf{x}}_2$: If the racket is rotated (nearly) around the $\hat{\mathbf{x}}_2$ axis, then the initial ω_1 and ω_3 are much smaller than ω_2. As above, let's emphasize this by relabeling $\omega_1 \to \epsilon_1$ and $\omega_3 \to \epsilon_3$. Then as above, eqs. (8.43) become

$$\begin{aligned} 0 &= \dot{\epsilon}_1 - A\omega_2\epsilon_3, \\ 0 &= \dot{\omega}_2 + B\epsilon_1\epsilon_3, \\ 0 &= \dot{\epsilon}_3 - C\omega_2\epsilon_1, \end{aligned} \tag{8.135}$$

Our goal here is to show that if the ϵ's start out small, then they do *not* remain small. Assuming that they are small (which is true initially), the second equation says that $\dot{\omega}_2 \approx 0$ (to first order in the ϵ's). So we may assume that ω_2 is essentially constant (when the ϵ's are small). Taking the derivative of the first equation then gives $0 = \ddot{\epsilon}_1 - A\omega_2\dot{\epsilon}_3$. Plugging the value of $\dot{\epsilon}_3$ from the third equation into this yields

$$\ddot{\epsilon}_1 = (AC\omega_2^2)\epsilon_1. \tag{8.136}$$

Because of the positive coefficient on the right-hand side, this equation describes an exponentially growing motion, instead of an oscillatory one. Therefore, ϵ_1 grows quickly from its initial small value. Hence, even if it starts small, it becomes large. By the same reasoning, ϵ_3 becomes large. (Of course, once the ϵ's become large, then our assumption of $\dot{\omega}_2 \approx 0$ isn't valid anymore. But once the ϵ's become large, we've shown what we wanted to.)

We see that $\boldsymbol{\omega}$ does *not* remain (nearly) equal to $(0, \omega_2, 0)$ at all times, which implies that $\mathbf{L}$ does *not* remain (nearly) equal to $(0, I_2\omega_2, 0)$ at all times. That is, $\mathbf{L}$ does not always point (nearly) along the $\hat{\mathbf{x}}_2$ direction (which is fixed in the racket frame). But the direction of $\mathbf{L}$ is fixed in the lab frame (because there is no torque). Therefore, the direction of $\hat{\mathbf{x}}_2$ must change in the lab frame. In other words, the racket wobbles.

15. **Free-top angles**

In terms of the principal axes, $\hat{\mathbf{x}}_1$,$\hat{\mathbf{x}}_2$,$\hat{\mathbf{x}}_3$, we have

$$\begin{aligned} \boldsymbol{\omega} &= (\omega_1\hat{\mathbf{x}}_1 + \omega_2\hat{\mathbf{x}}_2) + \omega_3\hat{\mathbf{x}}_3, \qquad \text{and} \\ \mathbf{L} &= I(\omega_1\hat{\mathbf{x}}_1 + \omega_2\hat{\mathbf{x}}_2) + I_3\omega_3\hat{\mathbf{x}}_3. \end{aligned} \tag{8.137}$$

Let $(\omega_1\hat{\mathbf{x}}_1 + \omega_2\hat{\mathbf{x}}_2) \equiv \omega_\perp\hat{\boldsymbol{\omega}}_\perp$ be the component of $\boldsymbol{\omega}$ orthogonal to $\boldsymbol{\omega}_3$. Then, by definition, we have

$$\tan\beta = \frac{\omega_\perp}{\omega_3}, \qquad \text{and} \qquad \tan\alpha = \frac{I\omega_\perp}{I_3\omega_3}. \tag{8.138}$$

Therefore,

$$\frac{\tan\alpha}{\tan\beta} = \frac{I}{I_3}. \tag{8.139}$$

If $I > I_3$, then $\alpha > \beta$, and we have the situation shown in Fig. 8.67. A top with this property is called a "prolate top". An example is a football or a pencil.

If $I < I_3$, then $\alpha < \beta$, and we have the situation shown in Fig. 8.68. A top with this property is called an "oblate top". An example is a coin or a Frisbee.

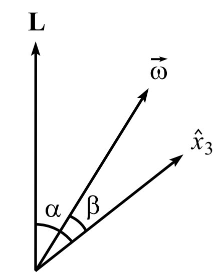

Figure 8.67

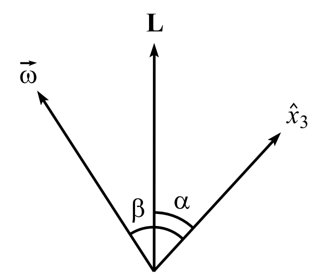

Figure 8.68

16. **Gyroscope**

(a) In order for there to exist real solutions for Ω in eq. (8.79), the discriminant must be non-negative. If $\theta \geq \pi/2$, then $\cos\theta \leq 0$, so the discriminant is automatically positive, and any value of ω_3 is allowed. But if $\theta < \pi/2$, then the lower limit on ω_3 is

$$\omega_3 \geq \frac{\sqrt{4MIg\ell\cos\theta}}{I_3} \equiv \tilde{\omega}_3. \tag{8.140}$$

Note that at this critical value, eq. (8.79) gives

$$\Omega_+ = \Omega_- = \frac{I_3\tilde{\omega}_3}{2I\cos\theta} = \sqrt{\frac{Mg\ell}{I\cos\theta}} \equiv \Omega_0. \tag{8.141}$$

(b) Since ω_3 has units, "large ω_3" is a meaningless description. What we really mean is that the fraction in the square root in eq. (8.79) is very small compared to 1. That is, $\epsilon \equiv (4MIg\ell\cos\theta)/(I_3^2\omega_3^2) \ll 1$. In this case, we may use $\sqrt{1-\epsilon} \approx 1 - \epsilon/2 + \cdots$ to write

$$\Omega_\pm \approx \frac{I_3\omega_3}{2I\cos\theta}\left(1 \pm \left(1 - \frac{2MIg\ell\cos\theta}{I_3^2\omega_3^2}\right)\right). \tag{8.142}$$

Therefore, the two solutions for Ω are (to leading order in ω_3)

$$\Omega_+ \approx \frac{I_3\omega_3}{I\cos\theta}, \qquad \text{and} \qquad \Omega_- \approx \frac{Mg\ell}{I_3\omega_3}. \tag{8.143}$$

These are known as the "fast" and "slow" frequencies of precession, respectively. Ω_- is the approximate answer we found in eq. (8.75), and it was obtained here under the assumption $\epsilon \ll 1$, which is equivalent to

$$\omega_3 \gg \frac{\sqrt{4MIg\ell\cos\theta}}{I_3} \qquad \text{(that is, } \omega_3 \gg \tilde{\omega}_3\text{)}. \tag{8.144}$$

This, therefore, is the condition for the result in eq. (8.75) to be a good approximation. Note that if I is of the same order as I_3 (so that they are both of the order $M\ell^2$), and if $\cos\theta$ is of order 1, then this condition may be written as $\omega \gg \sqrt{g/\ell}$, which is the frequency of a pendulum of length ℓ.

REMARKS: The Ω_+ solution is a fairly surprising result. Two strange features of Ω_+ are that it grows with ω_3, and that it is independent of g. To see what is going on with this precession, note that Ω_+ is the value of Ω that makes the $L_\perp$ in eq. (8.77) essentially equal to zero. So **L** points nearly along the vertical axis. The rate of change of **L** is the product of a very small radius (of the circle the tip traces out) and a very large Ω (if we've picked ω_3 to be large). The product of these equals the "medium sized" torque $Mg\ell\sin\theta$.

In the limit of large ω_3, the fast precession should look basically like the motion of a free top (because **L** is essentially constant), discussed in Section 8.6.2. And indeed, Ω_+ is independent of g. We'll leave it to you to show that $\Omega_+ \approx L/I$, which is the precession frequency of a free top (eq. (8.53)), as viewed from a fixed frame.

We can plot the $\Omega_\pm$ of eq. (8.79) as functions of ω_3. With the definitions of $\tilde{\omega}_3$ and Ω_0 in eqs. (8.140) and (8.141), we can rewrite eq. (8.79) as

$$\Omega_\pm = \frac{\omega_3\Omega_0}{\tilde{\omega}_3}\left(1 \pm \sqrt{1 - \frac{\tilde{\omega}_3^2}{\omega_3^2}}\right). \tag{8.145}$$

It is easier to work with dimensionless quantities, so let's rewrite this as

$$y_\pm = x \pm \sqrt{x^2 - 1}, \qquad \text{with } y_\pm \equiv \frac{\Omega_\pm}{\Omega_0}, \quad x \equiv \frac{\omega_3}{\tilde{\omega}_3}. \tag{8.146}$$

A rough plot of $y_\pm$ vs. x is shown in Fig. 8.69. ♣

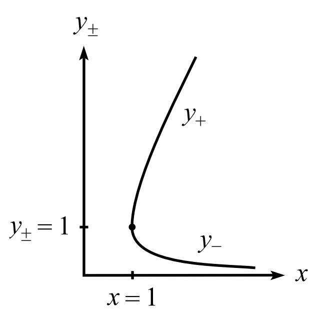

Figure 8.69

17. **Many gyroscopes**

The system is made up of N rigid bodies, each consisting of a plate and the massless stick glued to it on its left (see Fig. 8.70). Label these sub-systems as S_i, with S_1 being the one closest to the pole.

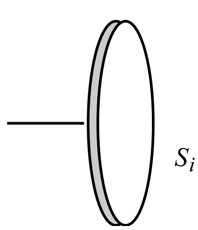

Figure 8.70

Let each plate have mass m and moment of inertia I, and let each stick have length ℓ. Let the angular speeds be ω_i. The relevant angular momentum of S_i is then $L_i = I\omega_i$, and it points horizontally.[26] Let the desired precession frequency be Ω. Then the magnitude of $d\mathbf{L}_i/dt$ is $L_i\Omega = (I\omega_i)\Omega$, and this points perpendicularly to $\mathbf{L}_i$.

Consider the torque $\boldsymbol{\tau}_i$ on S_i, around its CM. Let's first look at S_1. The pole provides an upward force of Nmg (this force is what keeps all the gyroscopes up), so it provides a torque of $Nmg\ell$ around the CM of S_1. The downward force from the stick to the right provides no torque around the CM (because it acts at the CM). Therefore, $\boldsymbol{\tau}_1 = d\mathbf{L}_1/dt$ gives $Nmg\ell = (I\omega_1)\Omega$, and so

$$\omega_1 = \frac{Nmg\ell}{I\Omega}. \tag{8.147}$$

[26]We are ignoring the angular momentum arising from the precession. This part of **L** points vertically (because the gyroscopes all point horizontally) and therefore does not change. Hence, it does not enter into $\vec{\tau} = d\mathbf{L}/dt$.

Now look at S_2. S_1 provides an upward force of $(N-1)mg$ (this force is what keeps S_2 through S_N up), so it provides a torque of $(N-1)mg\ell$ around the CM of S_2. The downward force from the stick to the right provides no torque around the CM of S_2. Therefore, $\boldsymbol{\tau}_2 = d\mathbf{L}_2/dt$ gives $(N-1)mg\ell = (I\omega_2)\Omega$, and so

$$\omega_2 = \frac{(N-1)mg\ell}{I\Omega}\,. \tag{8.148}$$

Similar reasoning applies to the other S_i, and we arrive at

$$\omega_i = \frac{(N+1-i)mg\ell}{I\Omega}\,. \tag{8.149}$$

The ω_i are therefore in the ratio

$$\omega_1 : \omega_2 : \cdots : \omega_{N-1} : \omega_N = N : (N-1) : \cdots : 2 : 1. \tag{8.150}$$

Note that we needed to apply $\boldsymbol{\tau} = d\mathbf{L}/dt$ many times, using each CM as an origin. Using only the pivot point on the pole as the origin would have given only one piece of information, whereas we needed N pieces.

Remarks: As a double-check, we can verify that these ω's make $\vec{\tau} = d\mathbf{L}/dt$ true, where $\vec{\tau}$ and $\mathbf{L}$ are the total torque and angular momentum relative to the pivot on the pole. (Using the CM of the entire system as the origin would give the same equation.) The CM of the entire system is $(N+1)\ell/2$ from the wall, so the torque due to gravity is

$$\tau = Nmg\frac{(N+1)\ell}{2}\,. \tag{8.151}$$

The total angular momentum is, using eq. (8.149),

$$\begin{aligned} L &= I(\omega_1 + \omega_2 + \cdots + \omega_N) \\ &= \frac{mg\ell}{\Omega}\Big(N + (N-1) + (N-2) + \cdots + 2 + 1\Big) \\ &= \frac{mg\ell}{\Omega}\frac{N(N+1)}{2}\,. \end{aligned} \tag{8.152}$$

So indeed, $\tau = L\Omega = |d\mathbf{L}/dt|$.

We can also pose this problem for the setup where all the ω_i are equal (call them ω), and the goal is the find the lengths of the sticks that will allow the desired motion. We can use the same reasoning as above, and eq. (8.149) takes the modified form

$$\omega = \frac{(N+1-i)mg\ell_i}{I\Omega}\,, \tag{8.153}$$

where ℓ_i is the length of the ith stick. Therefore, the ℓ_i are in the ratio

$$\ell_1 : \ell_2 : \cdots : \ell_{N-1} : \ell_N = \frac{1}{N} : \frac{1}{N-1} : \cdots : \frac{1}{2} : 1. \tag{8.154}$$

Note that since the sum $\sum 1/n$ diverges, it is possible to make the setup extend arbitrarily far from the pole.

Again, we can verify that these ℓ's make $\vec{\tau} = d\mathbf{L}/dt$ true, where $\vec{\tau}$ and $\mathbf{L}$ are the total torque and angular momentum relative to the pivot on the pole. As an exercise, you can show that the CM happens to be a distance ℓ_N from the pole. So the torque due to gravity is, using eq. (8.153) to obtain ℓ_N,

$$\tau = Nmg\ell_N = Nmg(\omega I\Omega/mg) = NI\omega\Omega. \tag{8.155}$$

The total angular momentum is simply $L = NI\omega$. So indeed, $\tau = L\Omega = |d\mathbf{L}/dt|$. ♣

18. **Heavy top on slippery table**

In Section 8.7.3, we looked at $\boldsymbol{\tau}$ and $\mathbf{L}$ relative to the pivot point. Such quantities are of no use here, because we can't use $\boldsymbol{\tau} = d\mathbf{L}/dt$ relative to the pivot point (because it is accelerating). We will therefore look at $\boldsymbol{\tau}$ and $\mathbf{L}$ relative to the CM, which is always a legal origin around which we can apply $\boldsymbol{\tau} = d\mathbf{L}/dt$.

The only force the floor applies is the normal force Mg. Therefore, the torque relative to the CM has the same magnitude, $Mg\ell\sin\theta$, and the same direction as in Section 8.7.3. If we choose the CM as the origin of our coordinate system, then all the Euler angles are the same as before. The only change in the whole analysis is the change in the $I_1 = I_2 \equiv I$ moment of inertia. We are now measuring these moments with respect to the CM, instead of the pivot point. By the parallel axis theorem, they are now equal to

$$I' = I - M\ell^2. \tag{8.156}$$

Therefore, changing I to $I - M\ell^2$ is the only modification needed.

19. **Fixed highest point**

For the desired motion, the important thing to note is that every point in the top moves in a fixed circle around the $\hat{\mathbf{z}}$-axis. Therefore, $\boldsymbol{\omega}$ points vertically. Hence, if Ω is the frequency of precession, we have $\boldsymbol{\omega} = \Omega\hat{\mathbf{z}}$.

(Another way to see that $\boldsymbol{\omega}$ points vertically is to view things in the frame that rotates with angular velocity $\Omega\hat{\mathbf{z}}$. In this frame, the top has no motion whatsoever. It is not even spinning, because the point P is always the highest point. In the language of Fig. 8.27, we therefore have $\omega' = 0$. Hence, $\boldsymbol{\omega} = \Omega\hat{\mathbf{z}} + \omega'\hat{\mathbf{x}}_3 = \Omega\hat{\mathbf{z}}$.)

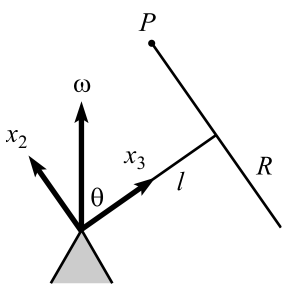

Figure 8.71

The principal moments are (with the pivot as the origin; see Fig. 8.71)

$$I_3 = \frac{MR^2}{2}, \qquad \text{and} \qquad I \equiv I_1 = I_2 = M\ell^2 + \frac{MR^2}{4}, \tag{8.157}$$

where we have used the parallel-axis theorem to obtain the latter. The components of $\boldsymbol{\omega}$ along the principal axes are $\omega_3 = \Omega\cos\theta$, and $\omega_2 = \Omega\sin\theta$. Therefore (keeping things in terms of the general moments, I_3 and I),

$$\mathbf{L} = I_3\Omega\cos\theta\hat{\mathbf{x}}_3 + I\Omega\sin\theta\hat{\mathbf{x}}_2. \tag{8.158}$$

The horizontal component of $\mathbf{L}$ is then $L_\perp = (I_3\Omega\cos\theta)\sin\theta - (I\Omega\sin\theta)\cos\theta$, and so $d\mathbf{L}/dt$ has magnitude

$$\left|\frac{d\mathbf{L}}{dt}\right| = L_\perp\Omega = \Omega^2\sin\theta\cos\theta(I_3 - I), \tag{8.159}$$

and is directed into the page (or out of the page, if this quantity is negative). This must equal the torque, which has magnitude $|\boldsymbol{\tau}| = Mg\ell\sin\theta$, and is directed into the page. Therefore,

$$\Omega = \sqrt{\frac{Mg\ell}{(I_3 - I)\cos\theta}}\,. \tag{8.160}$$

We see that for a general symmetric top, such precessional motion (where the same "side" always points up) is possible only if

$$I_3 > I. \tag{8.161}$$

Note that this condition is independent of θ. For the problem at hand, I_3 and I are given in eq. (8.157), and we find

$$\Omega = \sqrt{\frac{4g\ell}{(R^2 - 4\ell^2)\cos\theta}}\,, \tag{8.162}$$

and the necessary condition for such motion is $R > 2\ell$.

REMARKS:

(a) It is intuitively clear that Ω should become very large as $\theta \to \pi/2$, although it is by no means intuitively clear that such motion should exist at all for angles near $\pi/2$.

(b) Ω approaches a non-zero constant as $\theta \to 0$, which isn't entirely obvious.

(c) If both R and ℓ are scaled up by the same factor, then Ω decreases. (This also follows from dimensional analysis.)

(d) The condition $I_3 > I$ can be understood in the following way. If $I_3 = I$, then $\mathbf{L} \propto \vec{\omega}$, and so $\mathbf{L}$ points vertically along $\vec{\omega}$. If $I_3 > I$, then $\mathbf{L}$ points somewhere to the right of the $\hat{\mathbf{z}}$-axis (at the instant shown in Fig. 8.71). This means that the tip of $\mathbf{L}$ is moving into the page, along with the top. This is what we need, because $\vec{\tau}$ points into the page. If, however, $I_3 < I$, then $\mathbf{L}$ points somewhere to the left of the $\hat{\mathbf{z}}$-axis, so $d\mathbf{L}/dt$ points out of the page, and hence cannot be equal to $\vec{\tau}$. ♣

20. **Basketball on rim**

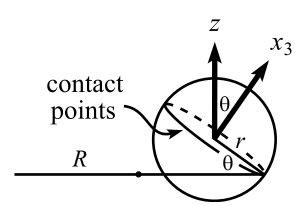

Figure 8.72

Consider the setup in the frame rotating with angular velocity $\Omega\hat{\mathbf{z}}$. In this frame, the center of the ball is at rest. Therefore, if the contact points are to form a great circle, the ball must be spinning around the (negative) $\hat{\mathbf{x}}_3$ axis shown in Fig. 8.72. Let the frequency of this spinning be ω' (in the language of Fig. 8.27). Then the nonslipping condition says that $\omega' r = \Omega R$, and so $\omega' = \Omega R/r$. Therefore, the total angular velocity vector of the ball in the lab frame is

$$\boldsymbol{\omega} = \Omega\hat{\mathbf{z}} - \omega'\hat{\mathbf{x}}_3 = \Omega\hat{\mathbf{z}} - (R/r)\Omega\hat{\mathbf{x}}_3. \tag{8.163}$$

Let us choose the center of the ball as the origin around which $\boldsymbol{\tau}$ and $\mathbf{L}$ are calculated. Then every axis in the ball is a principal axis, with moment of inertia $I = (2/3)mr^2$. The angular momentum is therefore

$$\mathbf{L} = I\boldsymbol{\omega} = I\Omega\hat{\mathbf{z}} - I(R/r)\Omega\hat{\mathbf{x}}_3. \tag{8.164}$$

Only the $\hat{\mathbf{x}}_3$ piece has a horizontal component which will contribute to $d\mathbf{L}/dt$. This component has length $L_\perp = I(R/r)\Omega\sin\theta$. Therefore, $d\mathbf{L}/dt$ has magnitude

$$\left|\frac{d\mathbf{L}}{dt}\right| = \Omega L_\perp = \frac{2}{3}\Omega^2 mrR\sin\theta, \tag{8.165}$$

and points out of the page.

The torque (relative to the center of the ball) comes from the force at the contact point. There are two components of this force. The vertical component is mg, and the horizontal component is $m(R-r\cos\theta)\Omega^2$ (pointing to the left), because the center of the ball moves in a circle of radius $(R - r\cos\theta)$. We then find the torque to have magnitude

$$|\boldsymbol{\tau}| = mg(r\cos\theta) - m(R - r\cos\theta)\Omega^2(r\sin\theta), \tag{8.166}$$

with outward from the page taken to be positive. Using the previous two equations, $\boldsymbol{\tau} = d\mathbf{L}/dt$ gives

$$\Omega^2 = \frac{g\cos\theta}{\sin\theta\left(\frac{5}{3}R - r\cos\theta\right)}\,. \qquad (8.167)$$

REMARKS:

(a) $\Omega \to \infty$ as $\theta \to 0$, which makes sense.

(b) Also, $\Omega \to \infty$ when $R = (3/5)r\cos\theta$. This case, however, is not physical, because we must have $R > r\cos\theta$ in order for the other side of the rim to be outside the basketball.

(c) You can also work out the problem for the case where the contact points trace out a circle other than a great circle (say, one that makes an angle β with respect to the great circle). The expression for the torque in eq. (8.166) is unchanged, but the value of ω' and the angle of the $\hat{\mathbf{x}}_3$-axis both change. Eq. (8.165) is therefore modified. The resulting Ω, however, is a bit messy. ♣

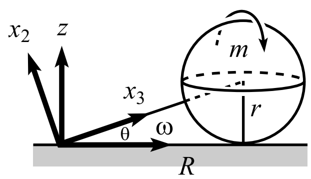

Figure 8.73

21. **Rolling lollipop**

(a) We claim that $\boldsymbol{\omega}$ points horizontally to the right (at the instant shown in Fig. 8.73), with magnitude $(R/r)\Omega$. This can be seen in (at least) two ways.

The first method is to note that we essentially have the same scenario as in the "Rolling cone" setup of Problem 3. The sphere's contact point with the ground is at rest (the non-slipping condition), so $\boldsymbol{\omega}$ must pass through this point (horizontally). The center of the sphere moves with speed ΩR. But since the center may also be considered to be instantaneously moving with frequency ω in a circle of radius r around the horizontal axis, we have $\omega r = \Omega R$. Therefore, $\omega = (R/r)\Omega$.

The second method is to write $\boldsymbol{\omega}$ as $\boldsymbol{\omega} = -\Omega\hat{\mathbf{z}} + \omega'\hat{\mathbf{x}}_3$ (in the language of Fig. 8.27), where ω' is the frequency of the spinning as viewed by someone rotating around the (negative) $\hat{\mathbf{z}}$ axis with frequency Ω. The contact points form a circle of radius R on the ground. But they also form a circle of radius $r\cos\theta$ on the sphere (where θ is the angle between the stick and the ground). The non-slipping condition then implies $\Omega R = \omega'(r\cos\theta)$. Therefore, $\omega' = \Omega R/(r\cos\theta)$, and

$$\boldsymbol{\omega} = -\Omega\hat{\mathbf{z}} + \omega'\hat{\mathbf{x}}_3 = -\Omega\hat{\mathbf{z}} + \left(\frac{\Omega R}{r\cos\theta}\right)(\cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{z}}) = (R/r)\Omega\hat{\mathbf{x}}, \qquad (8.168)$$

where we have used $\tan\theta = r/R$.

(b) Choose the pivot as the origin. The principal axes are then $\hat{\mathbf{x}}_3$ along the stick, along with any two directions orthogonal to the stick. Choose $\hat{\mathbf{x}}_2$ to be in the plane of the paper. Then the components of $\boldsymbol{\omega}$ along the principal axes are

$$\omega_3 = (R/r)\Omega\cos\theta, \qquad \text{and} \qquad \omega_2 = -(R/r)\Omega\sin\theta. \qquad (8.169)$$

The principal moments are

$$I_3 = (2/5)mr^2, \qquad \text{and} \qquad I_2 = (2/5)mr^2 + m(r^2 + R^2), \qquad (8.170)$$

where we have used the parallel-axis theorem. The angular momentum is $\mathbf{L} = I_3\omega_3\hat{\mathbf{x}}_3 + I_2\omega_2\hat{\mathbf{x}}_2$, so its horizontal component has length $L_\perp = I_3\omega_3\cos\theta - I_2\omega_2\sin\theta$. Therefore, the magnitude of $d\mathbf{L}/dt$ is

$$\begin{aligned}\left|\frac{d\mathbf{L}}{dt}\right| &= \Omega L_\perp \\ &= \Omega(I_3\omega_3\cos\theta - I_2\omega_2\sin\theta) \\ &= \Omega\bigg(\Big(\frac{2}{5}mr^2\Big)\Big(\frac{R}{r}\Omega\cos\theta\Big)\cos\theta \\ &\qquad -\Big(\frac{2}{5}mr^2 + m(r^2+R^2)\Big)\Big(-\frac{R}{r}\Omega\sin\theta\Big)\sin\theta\bigg) \\ &= \Omega^2 m\frac{R}{r}\left(\frac{2}{5}r^2 + (r^2+R^2)\sin^2\theta\right) \\ &= \frac{7}{5}mrR\Omega^2, \end{aligned} \tag{8.171}$$

where we have used $\sin\theta = r/\sqrt{r^2+R^2}$. The direction of $d\mathbf{L}/dt$ is out of the page.

REMARK: There is actually a quicker way to calculate $d\mathbf{L}/dt$. At a given instant, the sphere is rotating around the horizontal x-axis with frequency $\omega = (R/r)\Omega$. The moment of inertia around this axis is $I_x = (7/5)mr^2$, from the parallel-axis theorem. Therefore, the horizontal component of $\mathbf{L}$ has magnitude

$$L_x = I_x\omega = \frac{7}{5}mrR\Omega. \tag{8.172}$$

Multiplying this by frequency (namely Ω) at which $\mathbf{L}$ swings around the z-axis gives the result for $|d\mathbf{L}/dt|$ in eq. (8.171). Note that there is also a vertical component of $\mathbf{L}$ relative to the pivot,[27] but this component doesn't change, so it doesn't come into $d\mathbf{L}/dt$. ♣

The torque (relative to the pivot) is due to the gravitational force acting at the CM, along with the normal force, N, acting at the contact point. (Any horizontal friction at the contact point will yield zero torque relative to the pivot.) Therefore, $\boldsymbol{\tau}$ points out of the page with magnitude $|\boldsymbol{\tau}| = (N - mg)R$. Equating this with the $|d\mathbf{L}/dt|$ in eq. (8.171) gives

$$N = mg + \frac{7}{5}mr\Omega^2. \tag{8.173}$$

This has the interesting property of being independent of R (and hence θ).

REMARK: The pivot must provide a downward force of $N - mg = (7/5)mr\Omega^2$, to make the net vertical force on the lollipop equal to zero. This result is slightly larger than the $mr\Omega^2$ result for the "sliding" situation in Exercise 6.

The sum of the horizontal forces at the pivot and the contact point must equal the required centripetal force of $mR\Omega^2$. But it is impossible to say how this force is divided up, without being given more information. ♣

22. **Rolling coin**

Choose the CM as the origin. The principal axes are then $\hat{\mathbf{x}}_2$ and $\hat{\mathbf{x}}_3$ (as shown in Fig. 8.74), along with $\hat{\mathbf{x}}_1$ pointing into the paper. Let Ω be the desired frequency.

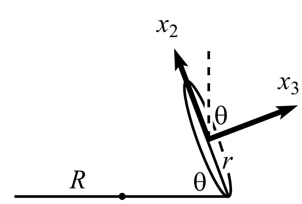

Figure 8.74

[27]This vertical component equals $-mR^2\Omega$, which makes intuitive sense. It can be obtained via the inertia tensor relative to the pivot, or by calculating $L_y = I_3\omega_3\sin\theta + I_2\omega_2\cos\theta$.

Look at things in the frame rotating around the $\hat{\mathbf{z}}$-axis with frequency Ω. In this frame, the CM remains fixed, and the coin rotates with frequency ω' (in the language of Fig. 8.27) around the (negative) $\hat{\mathbf{x}}_3$-axis. The non-slipping condition says that $\omega' r = \Omega R$, and so $\omega' = \Omega R/r$. Therefore, the total angular velocity vector of the coin in the lab frame is

$$\boldsymbol{\omega} = \Omega\hat{\mathbf{z}} - \omega'\hat{\mathbf{x}}_3 = \Omega\hat{\mathbf{z}} - \frac{R}{r}\Omega\hat{\mathbf{x}}_3. \tag{8.174}$$

But $\hat{\mathbf{z}} = \sin\theta\hat{\mathbf{x}}_2 + \cos\theta\hat{\mathbf{x}}_3$, so we may write $\boldsymbol{\omega}$ in terms of the principal axes as

$$\boldsymbol{\omega} = \Omega\sin\theta\hat{\mathbf{x}}_2 - \Omega\left(\frac{R}{r} - \cos\theta\right)\hat{\mathbf{x}}_3. \tag{8.175}$$

The principal moments are

$$I_3 = (1/2)mr^2, \qquad \text{and} \qquad I_2 = (1/4)mr^2. \tag{8.176}$$

The angular momentum is $\mathbf{L} = I_2\omega_2\hat{\mathbf{x}}_2 + I_3\omega_3\hat{\mathbf{x}}_3$, so its horizontal component has length $L_\perp = I_2\omega_2\cos\theta - I_3\omega_3\sin\theta$, with leftward taken to be positive. Therefore, the magnitude of $d\mathbf{L}/dt$ is

$$\begin{aligned}
\left|\frac{d\mathbf{L}}{dt}\right| &= \Omega L_\perp \\
&= \Omega(I_2\omega_2\cos\theta - I_3\omega_3\sin\theta) \\
&= \Omega\left(\left(\frac{1}{4}mr^2\right)\Big(\Omega\sin\theta\Big)\cos\theta - \left(\frac{1}{2}mr^2\right)\Big(-\Omega(R/r - \cos\theta)\Big)\sin\theta\right) \\
&= \frac{1}{4}mr\Omega^2\sin\theta(2R - r\cos\theta),
\end{aligned} \tag{8.177}$$

with a positive quantity corresponding to $d\mathbf{L}/dt$ pointing out of the page (at the instant shown).

The torque (relative to the CM) comes from the force at the contact point. There are two components of this force. The vertical component is mg, and the horizontal component is $m(R - r\cos\theta)\Omega^2$ (pointing to the left), because the CM moves in a circle of radius $(R - r\cos\theta)$. We then find the torque to have magnitude

$$|\boldsymbol{\tau}| = mg(r\cos\theta) - m(R - r\cos\theta)\Omega^2(r\sin\theta), \tag{8.178}$$

with outward from the page taken to be positive. Equating this $|\boldsymbol{\tau}|$ with the $|d\mathbf{L}/dt|$ from eq. (8.177) gives

$$\Omega^2 = \frac{g}{\frac{3}{2}R\tan\theta - \frac{5}{4}r\sin\theta}. \tag{8.179}$$

The right-hand side must be positive if a solution for Ω is to exist. Therefore, the condition for the desired motion to be possible is

$$R > \frac{5}{6}r\cos\theta. \tag{8.180}$$

REMARKS:

(a) For $\theta \to \pi/2$, eq. (8.179) gives $\Omega \to 0$, as it should. And for $\theta \to 0$, we obtain $\Omega \to \infty$, which also makes sense.

(b) Note that for $(5/6)r\cos\theta < R < r\cos\theta$, the CM of the coin lies to the *left* of the center of the contact-point circle (at the instant shown). The centripetal force, $m(R - r\cos\theta)\Omega^2$, is therefore negative (which means that it is directed radially outward, to the right), but the motion is still possible. As R gets close to $(5/6)r\cos\theta$, the frequency Ω goes to infinity, which means that the radially outward force also goes to infinity. The coefficient of friction between the coin and the ground must therefore be correspondingly large.

(c) We may consider a more general coin, whose density depends on only the distance from the center, and which has $I_3 = \eta m r^2$. (For example, a uniform coin has $\eta = 1/2$, and a coin with all its mass on the edge has $\eta = 1$.) By the perpendicular axis theorem, $I_1 = I_2 = (1/2)\eta m r^2$, and you can show that the above methods yield

$$\Omega^2 = \frac{g}{(1+\eta)R\tan\theta - (1+\eta/2)r\sin\theta}. \tag{8.181}$$

The condition for such motion to exist is then

$$R > \left(\frac{1+\eta/2}{1+\eta}\right) r\cos\theta. \;\clubsuit \tag{8.182}$$

23. **Wobbling coin**

(a) Look at the setup in the frame rotating with angular velocity $\Omega\hat{\mathbf{z}}$. In this frame, the location of the contact point remains fixed, and the coin rotates with frequency ω' (in the language of Fig. 8.27) around the negative $\hat{\mathbf{x}}_3$ axis. The radius of the circle of contact points on the table is $R\cos\theta$. Therefore, the non-slipping condition says that $\omega' R = \Omega(R\cos\theta)$, and so $\omega' = \Omega\cos\theta$. Hence, the total angular velocity vector of the coin in the lab frame is

$$\boldsymbol{\omega} = \Omega\hat{\mathbf{z}} - \omega'\hat{\mathbf{x}}_3 = \Omega(\sin\theta\hat{\mathbf{x}}_2 + \cos\theta\hat{\mathbf{x}}_3) - (\Omega\cos\theta)\hat{\mathbf{x}}_3 = \Omega\sin\theta\hat{\mathbf{x}}_2. \tag{8.183}$$

In retrospect, it makes sense that $\boldsymbol{\omega}$ must point in the $\hat{\mathbf{x}}_2$ direction. Both the CM and the instantaneous contact point on the coin are at rest, so $\boldsymbol{\omega}$ must lie along the line containing these two points (that is, along the $\hat{\mathbf{x}}_2$-axis).

(b) Choose the CM as the origin. The principal moment around the $\hat{\mathbf{x}}_2$-axis is $I = mR^2/4$. The angular momentum is $\mathbf{L} = I\omega_2\hat{\mathbf{x}}_2$, so its horizontal component has length $L_\perp = L\cos\theta = (I\omega_2)\cos\theta$. Therefore, $d\mathbf{L}/dt$ has magnitude

$$\left|\frac{d\mathbf{L}}{dt}\right| = \Omega L_\perp = \Omega\left(\frac{mR^2}{4}\right)(\Omega\sin\theta)\cos\theta, \tag{8.184}$$

and it points out of the page.

The torque (relative to the CM) is due to the normal force at the contact point (there is no sideways friction force at the contact point, because the CM is motionless), so it has magnitude

$$|\boldsymbol{\tau}| = mgR\cos\theta, \tag{8.185}$$

and it also points out of the page. Using the previous two equations, $\boldsymbol{\tau} = d\mathbf{L}/dt$ gives

$$\Omega = 2\sqrt{\frac{g}{R\sin\theta}}. \tag{8.186}$$

REMARKS:

i. $\Omega \to \infty$ as $\theta \to 0$. This is quite evident if you do the experiment; the contact point travels very quickly around the circle.

ii. $\Omega \to 2\sqrt{g/R}$ as $\theta \to \pi/2$. This isn't so intuitive (to me, at least). In this case, **L** points nearly vertically, and it traces out a tiny cone, due to a tiny torque.

iii. In this limit $\theta \to \pi/2$, Ω is also the frequency at which the plane of the coin spins around the vertical axis. If you spin a coin very fast about a vertical diameter, it will initially undergo a pure spinning motion with only one contact point. It will then gradually lose energy due to friction, until the spinning frequency slows down to $2\sqrt{g/R}$, at which time it will begin to wobble. (We're assuming, of course, that the coin is very thin, so that it can't balance on its edge.) In the case where the coin is a quarter (with $R \approx .012\,\text{m}$), this critical frequency of $2\sqrt{g/R}$ turns out to be $\Omega \approx 57\,\text{rad/s}$, which corresponds to about 9 Hertz.

iv. The result in eq. (8.186) is a special case of the result in eq. (8.179) of Problem 22. The CM of the coin in Problem 22 will be motionless if $R = r\cos\theta$. Plugging this into eq. (8.179) gives $\Omega^2 = 4g/(r\sin\theta)$, which agrees with eq. (8.186), because r was the coin's radius in Problem 22. ♣

(c) Consider one revolution of the point of contact around the $\hat{\mathbf{z}}$-axis. Since the radius of the circle on the table is $R\cos\theta$, the contact point moves a distance $2\pi R\cos\theta$ around the coin during this time. Hence, the new point of contact on the coin is a distance $2\pi R - 2\pi R\cos\theta$ away from the original point of contact. The coin therefore appears to have rotated by a fraction $(1-\cos\theta)$ of a full turn during this time. The frequency with which you see it turn is therefore

$$(1-\cos\theta)\Omega = \frac{2(1-\cos\theta)}{\sqrt{\sin\theta}}\sqrt{\frac{g}{R}}\,. \tag{8.187}$$

REMARKS:

i. If $\theta \approx \pi/2$, then the frequency of Abe's rotation is essentially equal to Ω. This makes sense, because the top of Abe's head will be, say, always near the top of the coin, and this point will trace out a small circle around the $\hat{\mathbf{z}}$-axis, with nearly the same frequency as the contact point.

ii. As $\theta \to 0$, Abe appears to rotate with frequency $\theta^{3/2}\sqrt{g/R}$ (using $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \theta^2/2$). Therefore, although the contact point moves infinitely quickly in this limit, we nevertheless see Abe rotating infinitely slowly.

iii. All of the results for frequencies in this problem have to look like some multiple of $\sqrt{g/R}$, by dimensional analysis. But whether the multiplication factor is zero, infinite, or something in between, is not at all obvious.

iv. An incorrect answer for the frequency of Abe's turning (when viewed from above) is that it equals the vertical component of $\vec{\omega}$, which is $\omega_z = \omega\sin\theta = (\Omega\sin\theta)\sin\theta = 2(\sin\theta)^{3/2}\sqrt{g/R}$. This does not equal the result in eq. (8.187). (It agrees at $\theta = \pi/2$, but is off by a factor of 2 for $\theta \to 0$.) This answer is incorrect because there is simply no reason why the vertical component of $\vec{\omega}$ should equal the frequency of revolution of, say, Abe's nose, around the vertical axis. For example, at moments when $\vec{\omega}$ passes through the nose, then the nose isn't moving at all, so it certainly cannot be described as moving around the vertical axis with frequency $\omega_z = \omega\sin\theta$.
The result in eq. (8.187) is a sort of average measure of the frequency of rotation. Even though any given point on the coin is not undergoing uniform circular motion, your eye will see the whole coin (approximately) rotating uniformly. ♣

24. **Nutation cusps**

(a) Because both $\dot\phi$ and $\dot\theta$ are continuous functions of time, we must have $\dot\phi = \dot\theta = 0$ at a kink. (Otherwise, either $d\theta/d\phi = \dot\theta/\dot\phi$ or $d\phi/d\theta = \dot\phi/\dot\theta$ would be well-defined at the kink.) Let the kink occur at $t = t_0$. Then the second of eqs. (8.89) gives $\sin(\omega_n t_0) = 0$. Therefore, $\cos(\omega_n t_0) = \pm 1$, and the first of eqs. (8.89) gives

$$\Delta\Omega = \mp\Omega_s, \tag{8.188}$$

as we wanted to show.

REMARK: Note that if $\cos(\omega_n t_0) = 1$, then $\Delta\Omega = -\Omega_s$, so eq. (8.88) says that the kink occurs at the smallest value of θ, that is, at the highest point of the top's motion. And if $\cos(\omega_n t_0) = -1$, then $\Delta\Omega = \Omega_s$, so eq. (8.88) again says that the kink occurs at the highest point of the top's motion. ♣

(b) To show that these kinks are cusps, we will show that the slope of the θ vs. ϕ plot is infinite on either side of the kink. That is, we will show that $d\theta/d\phi = \dot\theta/\dot\phi = \pm\infty$. For simplicity, we will look at the case where $\cos(\omega_n t_0) = 1$ and $\Delta\Omega = -\Omega_s$. (The $\cos(\omega_n t_0) = -1$ case proceeds the same.) With $\Delta\Omega = -\Omega_s$, eqs. (8.89) give

$$\frac{\dot\theta}{\dot\phi} = \frac{\sin\theta_0 \sin\omega_n t}{1 - \cos\omega_n t}\,. \tag{8.189}$$

Letting $t = t_0 + \epsilon$, we have (using $\cos(\omega_n t_0) = 1$ and $\sin(\omega_n t_0) = 0$, and expanding to lowest order in ϵ)

$$\frac{\dot\theta}{\dot\phi} = \frac{\sin\theta_0 \omega_n \epsilon}{\omega_n^2 \epsilon^2/2} = \frac{2\sin\theta_0}{\omega_n \epsilon}\,. \tag{8.190}$$

For infinitesimal ϵ, this switches from $-\infty$ to $+\infty$ as ϵ passes through zero.

25. **Nutation circles**

(a) Note that a change in angular speed of $\Delta\Omega$ around the fixed $\hat{\mathbf{z}}$-axis corresponds to a change in angular speed of $\sin\theta_0\Delta\Omega$ around the $\hat{\mathbf{x}}_2$-axis. The kick therefore produces an angular momentum component (relative to the pivot) of $I\sin\theta_0\Delta\Omega$ in the $\hat{\mathbf{x}}_2$ direction.

The original $\mathbf{L}$ pointed along the $\hat{\mathbf{x}}_3$ direction, with magnitude $I_3\omega_3$. (These two statements hold to a good approximation if $\omega_3 \gg \Omega_s$.) By definition, $\mathbf{L}$ made an angle θ_0 with the vertical $\hat{\mathbf{z}}$-axis. Therefore, from Fig. 8.75, the angle that the new $\mathbf{L}$ makes with the $\hat{\mathbf{z}}$-axis is (using $\Delta\Omega \ll \omega_3$)

$$\theta_0' = \theta_0 - \frac{I\sin\theta_0\Delta\Omega}{I_3\omega_3} \equiv \theta_0 - \frac{\sin\theta_0\Delta\Omega}{\omega_n}\,, \tag{8.191}$$

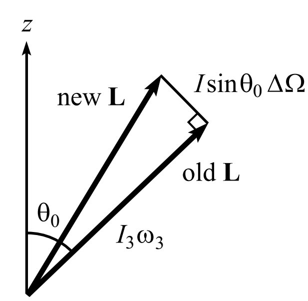

Figure 8.75

where we have used the definition of ω_n from eq. (8.83). We see that the effect of the kick is to make $\mathbf{L}$ quickly change its θ value. (It changes by only a small amount, because we are assuming $\omega_n \sim \omega_3 \gg \Delta\Omega$). The ϕ value doesn't immediately change.

(b) The torque (relative to the pivot) has magnitude $Mg\ell\sin\theta$ and is directed horizontally. Because θ doesn't change appreciably, the magnitude of the torque is essentially constant, so $\mathbf{L}$ traces out a circle at a constant rate. This rate is simply Ω_s. (None of the relevant quantities in $\boldsymbol{\tau} = d\mathbf{L}/dt$ changed much from the original circular-precession case, so the precession frequency remains basically the same.) Therefore, the new $\mathbf{L}$ has its (ϕ, θ) coordinates given by

$$(\phi(t), \theta(t))_{\mathbf{L}} = \left(\Omega_s t,\ \theta_0 - \frac{\sin\theta_0 \Delta\Omega}{\omega_n} \right). \tag{8.192}$$

Looking at eqs. (8.88), we see that the coordinates of the CM relative to $\mathbf{L}$ are

$$(\phi(t), \theta(t))_{\text{CM}-\mathbf{L}} = \left(\left(\frac{\Delta\Omega}{\omega_n} \right) \sin\omega_n t, \left(\frac{\Delta\Omega}{\omega_n} \sin\theta_0 \right) \cos\omega_n t \right). \tag{8.193}$$

The $\sin\theta_0$ factor in $\theta(t)$ is exactly what is needed for the CM to trace out a circle around $\mathbf{L}$, because a change in ϕ corresponds to a CM spatial change of $\ell\Delta\phi\sin\theta_0$, whereas a change in θ corresponds to a CM spatial change of $\ell\Delta\theta$.

This circular motion is exactly what we expect from the results in Section 8.6.2, for the following reason. For ω_n very large and Ω_s very small, $\mathbf{L}$ is essentially motionless, and the CM traces out a circle around it at frequency ω_n. Since $\mathbf{L}$ is essentially constant, the top should therefore behave very much like a free top, as viewed from a fixed frame.

Eq. (8.53) in Section 8.6.2 says that the frequency of the precession of $\hat{\mathbf{x}}_3$ around $\mathbf{L}$ for a free top is L/I. But the frequency of the precession of $\hat{\mathbf{x}}_3$ around $\mathbf{L}$ in the present problem is ω_n, so this had better be equal to L/I. And indeed, L is essentially equal to $I_3\omega_3$, so $L/I = I_3\omega_3/I \equiv \omega_n$.

Therefore, for short enough time scales (short enough so that $\mathbf{L}$ doesn't move much), a nutating top with $\omega_3 \gg \Delta\Omega \gg \Omega_s$ looks very much like a free top.

Remark: We need the $\Delta\Omega \gg \Omega_s$ condition so that the nutation motion looks like circles. This can be seen by the following reasoning. The time for one period of the nutation motion is $2\pi/\omega_n$. From eq. (8.88), $\phi(t)$ increases by $\Delta\phi = 2\pi\Omega_s/\omega_n$ in this time. And also from eq. (8.88), the width, w, of the "circle" along the ϕ axis is roughly $w = 2\Delta\Omega/\omega_n$. The motion looks like basically like a circle if $w \gg \Delta\phi$, that is, if $\Delta\Omega \gg \Omega_s$. ♣

26. **Rolling straight?**

Intuitively, it is fairly clear that the sphere cannot change direction. But it is a little tricky to prove. Qualitatively, we can reason as follows. Assume there is a nonzero friction force at the contact point. (The normal force is irrelevant here, because it doesn't provide a torque about the CM. This is what is special about a sphere.) Then the ball will accelerate in the direction of this force. However, you can show with the right-hand rule that this force will produce a torque that will cause the angular momentum to change in a way that corresponds to the ball accelerating in the direction *opposite* to the friction force. There is thus a contradiction, unless the friction force equals zero.

Let's now be rigorous. Let the angular velocity of the ball be $\boldsymbol{\omega}$. The non-slipping condition says that the ball's velocity equals

$$\mathbf{v} = \boldsymbol{\omega} \times (a\hat{\mathbf{z}}). \tag{8.194}$$

where a is the radius of the sphere. The ball's angular momentum is

$$\mathbf{L} = I\boldsymbol{\omega}. \tag{8.195}$$

The friction force from the ground (if it exists) at the contact point will change both the momentum and the angular momentum of the ball. $\mathbf{F} = d\mathbf{p}/dt$ gives

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt}\,, \tag{8.196}$$

and $\boldsymbol{\tau} = d\mathbf{L}/dt$ (relative to the center of the ball) gives

$$(-a\hat{\mathbf{z}}) \times \mathbf{F} = \frac{d\mathbf{L}}{dt}\,, \tag{8.197}$$

because the force is applied at position $-a\hat{\mathbf{z}}$ relative to the ball's center.

We will now show that the preceding four equations imply $\dot{\boldsymbol{\omega}} = \mathbf{0}$, that is, $\mathbf{v}$ is constant. Use the $\mathbf{v}$ from eq. (8.194) in eq. (8.196), and then plug the resulting $\mathbf{F}$ into eq. (8.197). Also, use the $\mathbf{L}$ from eq. (8.195) in eq. (8.197). The result is

$$(-a\hat{\mathbf{z}}) \times \big(m\dot{\boldsymbol{\omega}} \times (a\hat{\mathbf{z}})\big) = I\dot{\boldsymbol{\omega}}. \tag{8.198}$$

Because the vector $\dot{\boldsymbol{\omega}}$ lies in the horizontal plane, you can work out that $\hat{\mathbf{z}}\times(\dot{\boldsymbol{\omega}}\times\hat{\mathbf{z}}) = \dot{\boldsymbol{\omega}}$. Therefore, we have

$$-ma^2\dot{\boldsymbol{\omega}} = \frac{2}{5}ma^2\dot{\boldsymbol{\omega}}, \tag{8.199}$$

and hence $\dot{\boldsymbol{\omega}} = \mathbf{0}$, as we wanted to show.

27. **Ball on paper**

Our strategy will be to produce (and equate) two different expressions for the total change in the angular momentum of the ball (relative to its center). The first comes from the effects of the friction force on the ball. The second comes from looking at the initial and final motion.

To produce our first expression for $\Delta\mathbf{L}$, note that the normal force provides no torque, so we may ignore it. The friction force, $\mathbf{F}$, from the paper changes both $\mathbf{p}$ and $\mathbf{L}$, according to,

$$\begin{aligned} \Delta\mathbf{p} &= \int \mathbf{F}\,dt, \\ \Delta\mathbf{L} &= \int \boldsymbol{\tau}\,dt = \int(-R\hat{\mathbf{z}}) \times \mathbf{F}\,dt = (-R\hat{\mathbf{z}}) \times \int \mathbf{F}\,dt. \end{aligned} \tag{8.200}$$

Both of these integrals run over the entire slipping time, which may include time on the table after the ball leaves the paper. In the second line above, we have used the fact that the friction force always acts at the same location, namely $(-R\hat{\mathbf{z}})$, relative to the center of the ball. The two above equations yield

$$\Delta\mathbf{L} = (-R\hat{\mathbf{z}}) \times \Delta\mathbf{p}. \tag{8.201}$$

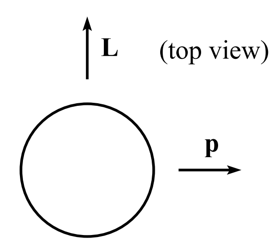

Figure 8.76

To produce our second equation for $\Delta\mathbf{L}$, let's examine how $\mathbf{L}$ is related to $\mathbf{p}$ when the ball is rolling without slipping, which is the case at both the start and the finish. When the ball is not slipping, we have the situation show in Fig. 8.76 (assuming that the ball is rolling to the right). The magnitudes of p and L are given by

$$\begin{aligned} p &= mv, \\ L &= I\omega = \frac{2}{5}mR^2\omega = \frac{2}{5}Rm(R\omega) = \frac{2}{5}Rmv = \frac{2}{5}Rp, \end{aligned} \tag{8.202}$$

where we have used the non-slipping condition, $v = R\omega$. (The actual $I = (2/5)mR^2$ value for a solid sphere will not be important for the final result.) We now note that the directions of $\mathbf{L}$ and $\mathbf{p}$ can be combined with the above $L = 2Rp/5$ scalar relation to give

$$\mathbf{L} = \frac{2}{5}R\hat{\mathbf{z}} \times \mathbf{p}, \tag{8.203}$$

where $\hat{\mathbf{z}}$ points out of the page. Since this relation is true at both the start and the finish, it must also be true for the differences in $\mathbf{L}$ and $\mathbf{p}$. That is,

$$\Delta\mathbf{L} = \frac{2}{5}R\hat{\mathbf{z}} \times \Delta\mathbf{p}. \tag{8.204}$$

Eqs. (8.201) and (8.204) give

$$\begin{aligned} (-R\hat{\mathbf{z}}) \times \Delta\mathbf{p} &= \frac{2}{5}R\hat{\mathbf{z}} \times \Delta\mathbf{p} \\ \Longrightarrow \quad 0 &= \hat{\mathbf{z}} \times \Delta\mathbf{p}. \end{aligned} \tag{8.205}$$

There are three ways this cross product can be zero:

- $\Delta\mathbf{p}$ is parallel to $\hat{\mathbf{z}}$. But it isn't, because $\Delta\mathbf{p}$ lies in the horizontal plane.
- $\hat{\mathbf{z}} = 0$. Not true.
- $\Delta\mathbf{p} = 0$. This one must be true. Therefore, $\Delta\mathbf{v} = 0$, as we wanted to show.

REMARKS:

(a) As stated in the problem, it's fine if you move the paper in a jerky motion, so that the ball slips around on it. We assumed nothing about the nature of the friction force in the above reasoning. And we used the non-slipping condition only at the initial and final times. The intermediate motion is arbitrary.

(b) As a special case, if you start a ball at rest on a piece of paper, then no matter how you choose to (horizontally) slide the paper out from underneath the ball, the ball will be at rest in the end.

(c) You are encouraged to experimentally verify that all these crazy claims are true. Make sure that the paper doesn't wrinkle (this would allow a force to be applied at a point other than the contact point). And balls that don't squish are much better, of course (for the same reason). ♣

28. **Ball on turntable**

Let the angular velocity of the turntable be $\Omega\hat{\mathbf{z}}$, and let the angular velocity of the ball be $\boldsymbol{\omega}$. If the ball is at position $\mathbf{r}$ (with respect to the lab frame), then its velocity (with respect to the lab frame) may be broken up into the velocity of the turntable (at position $\mathbf{r}$) plus the ball's velocity with respect to the turntable. The non-slipping condition says that this latter velocity is given by $\boldsymbol{\omega} \times (a\hat{\mathbf{z}})$, where a is the radius of the ball. The ball's velocity with respect to the lab frame is therefore

$$\mathbf{v} = (\Omega\hat{\mathbf{z}}) \times \mathbf{r} + \boldsymbol{\omega} \times (a\hat{\mathbf{z}}). \tag{8.206}$$

The important point to realize in this problem is that the friction force from the turntable is responsible for changing both the ball's linear momentum and its angular momentum. In particular, $\mathbf{F} = d\mathbf{p}/dt$ gives

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt}\,. \tag{8.207}$$

And the angular momentum of the ball is $\mathbf{L} = I\boldsymbol{\omega}$, so $\boldsymbol{\tau} = d\mathbf{L}/dt$ (relative to the center of the ball) gives

$$(-a\hat{\mathbf{z}}) \times \mathbf{F} = I\frac{d\boldsymbol{\omega}}{dt}\,, \tag{8.208}$$

because the force is applied at position $-a\hat{\mathbf{z}}$ relative to the center.

We will now use the previous three equations to demonstrate that the ball undergoes circular motion. Our goal will be to produce an equation of the form,

$$\frac{d\mathbf{v}}{dt} = \Omega'\hat{\mathbf{z}} \times \mathbf{v}, \tag{8.209}$$

since this describes circular motion, with frequency Ω' (to be determined). Plugging the expression for $\mathbf{F}$ from eq. (8.207) into eq. (8.208) gives

$$\begin{aligned} (-a\hat{\mathbf{z}}) \times \left(m\frac{d\mathbf{v}}{dt}\right) &= I\frac{d\boldsymbol{\omega}}{dt} \\ \Longrightarrow \qquad \frac{d\boldsymbol{\omega}}{dt} &= -\left(\frac{am}{I}\right)\hat{\mathbf{z}} \times \frac{d\mathbf{v}}{dt}\,. \end{aligned} \tag{8.210}$$

Taking the derivative of eq. (8.206) gives

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= \Omega\hat{\mathbf{z}} \times \frac{d\mathbf{r}}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times (a\hat{\mathbf{z}}) \\ &= \Omega\hat{\mathbf{z}} \times \mathbf{v} - \left(\left(\frac{am}{I}\right)\hat{\mathbf{z}} \times \frac{d\mathbf{v}}{dt}\right) \times (a\hat{\mathbf{z}}). \end{aligned} \tag{8.211}$$

Since the vector $d\mathbf{v}/dt$ lies in the horizontal plane, it is easy to work out the cross-product in the right term (or just use the identity $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}$) to obtain

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= \Omega\hat{\mathbf{z}} \times \mathbf{v} - \left(\frac{ma^2}{I}\right)\frac{d\mathbf{v}}{dt} \\ \Longrightarrow \qquad \frac{d\mathbf{v}}{dt} &= \left(\frac{\Omega}{1 + (ma^2/I)}\right)\hat{\mathbf{z}} \times \mathbf{v}. \end{aligned} \tag{8.212}$$

For a uniform sphere, $I = (2/5)ma^2$, so we obtain

$$\frac{d\mathbf{v}}{dt} = \left(\frac{2}{7}\Omega\right)\hat{\mathbf{z}} \times \mathbf{v}. \tag{8.213}$$

Therefore, in view of eq. (8.209), we see that the ball undergoes circular motion, with a frequency equal to 2/7 times the frequency of the turntable. This result for the frequency does not depend on initial conditions.

Remarks:

(a) Integrating eq. (8.213) from the initial time to some later time gives

$$\mathbf{v} - \mathbf{v}_0 = \left(\frac{2}{7}\Omega\right)\hat{\mathbf{z}} \times (\mathbf{r} - \mathbf{r}_0). \tag{8.214}$$

This may be written (as you can show) in the more suggestive form,

$$\mathbf{v} = \left(\frac{2}{7}\Omega\right)\hat{\mathbf{z}} \times \left(\mathbf{r} - \left(\mathbf{r}_0 + \frac{7}{2\Omega}(\hat{\mathbf{z}} \times \mathbf{v}_0)\right)\right). \tag{8.215}$$

This equation describes circular motion, with the center located at the point,

$$\mathbf{r}_c = \mathbf{r}_0 + \frac{7}{2\Omega}(\hat{\mathbf{z}} \times \mathbf{v}_0), \tag{8.216}$$

and with radius,

$$R = |\mathbf{r}_0 - \mathbf{r}_c| = \frac{7}{2\Omega}|\hat{\mathbf{z}} \times \mathbf{v}_0| = \frac{7v_0}{2\Omega}\,. \tag{8.217}$$

(b) There are a few special cases to consider:

- If $v_0 = 0$ (that is, if the spinning motion of the ball exactly cancels the rotational motion of the turntable), then $R = 0$ and the ball remains in the same place (of course).
- If the ball is initially not spinning, and just moving along with the turntable, then $v_0 = \Omega r_0$. The radius of the circle is therefore $R = (7/2)r_0$, and its center is located at (from eq. (8.216))

$$\mathbf{r}_c = \mathbf{r}_0 + \frac{7}{2\Omega}(-\Omega \mathbf{r}_0) = -\frac{5\mathbf{r}_0}{2}\,. \tag{8.218}$$

  The point on the circle diametrically opposite to the initial point is therefore at a radius $r_c + R = (5/2)r_0 + (7/2)r_0 = 6r_0$.
- If we want the center of the circle be the center of the turntable, then eq. (8.216) says that we need $(7/2\Omega)\hat{\mathbf{z}} \times \mathbf{v}_0 = -\mathbf{r}_0$. This implies that $\mathbf{v}_0$ has magnitude $v_0 = (2/7)\Omega r_0$ and points tangentially in the same direction as the turntable moves. (That is, the ball moves at 2/7 times the velocity of the turntable beneath it.)

(c) The fact that the frequency $(2/7)\Omega$ is a rational multiple of Ω means that the ball will eventually return to the same point on the turntable. In the lab frame, the ball will trace out two circles in the time it takes the turntable to undergo seven revolutions. From the point of view of someone on the turntable, the ball will "spiral" around five times before returning to the original position.

(d) If we look at a ball with moment of inertia $I = \eta m a^2$ (so a uniform sphere has $\eta = 2/5$), then you can show that the "2/7" in eq. (8.213) gets replaced by "$\eta/(1+\eta)$". If a ball has most of its mass concentrated at its center (so that $\eta \to 0$), then the frequency of the circular motion goes to 0, and the radius goes to ∞ (as long as $v_0 \neq 0$). ♣