ASYMPTOTIC STABILITY FOR 2 × 2 LINEAR DYNAMIC SYSTEMS ON TIME SCALES

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Abstract. We prove asymptotical stability and instability theorems for 2×2 system of first-order linear dynamic equations on a time scale with complex-valued functions as coefficients. To prove stability estimates and asymptotic stability for a 2×2 system we use the integral representations of the fundamental matrix via asymptotic solutions, the error estimates, and the time scales calculus.

1. Main Result

In this paper we study asymptotic stability of a system of linear dynamic equations on a time scale $\mathbb{T}_\infty = \mathbb{T} \bigcap (t_0, \infty)$:

$$u^\nabla(t) = A(t)u(t), \tag{1.1}$$

where u^∇ is the nabla derivative (see [6]), $u(t)$ is a 2-vector function, and

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} \tag{1.2}$$

is a 2×2 matrix-function ld-differentiable on $\mathbb{T}_\infty$.

Exponential decay and stability of solutions of dynamic equations on time scales were investigated in recent papers [1, 10, 17, 9, 11, 16, 8] by using Lyapunov method. We use different approach based on integral representations of solutions via asymptotic solutions and error estimates developed in [3, 15, 12, 13].

Denote

$$TrA(t) = a_{11}(t) + a_{22}(t), \quad |A(t)| = det(A(t)). \tag{1.3}$$

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. We assume $sup\mathbb{T} = \infty$.

For $t \in \mathbb{T}$ we define the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad \text{for all} \quad t \in \mathbb{T}. \tag{1.4}$$

The backward graininess function $\nu : \mathbb{T} \to [0, \infty]$ is defined by $\nu(t) = t - \rho(t)$. If $\rho(t) < t$ or $\nu > 0$ we say that t is left-scattered. If $t > \inf(\mathbb{T})$and $\rho(t) = t$ then t is called left dense. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_k = \mathbb{T} - \{m\}$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_k$ define the nabla derivative of f at t denoted $f^\nabla(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)(\rho - s)| \le \varepsilon |\rho(t) - s| \tag{1.5}$$

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for all $s \in U$.

The rest state $u(t) = 0$ of the system (1.1) is called stable if for any $\varepsilon > 0$ there exists $\delta(T, \varepsilon) > 0$ such that if $|u(T)| \leq \delta(T, \varepsilon)$ then $|u(t)| \leq \varepsilon$ for all $t \geq T$.

The rest state $u(t) = 0$ of the system (1.1) is called asymptotically stable if it is stable, and attractive:

$$\lim_{t \to \infty} u(t) = 0. \tag{1.6}$$

To prove asymptotic stability we establish stability estimates for dynamic system (1.1) by using integral representations of the fundamental matrix of (1.1) via asymptotic solutions, and calculus on time scales [5, 6].

A function $f :\in \mathbb{T} \to \mathbb{R}$ is called ld-continuous ($C_{ld}(\mathbb{T})$) provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right dense points in $\mathbb{T}$. $C^k_{ld}(\mathbb{T})$ is the class of functions for which nabla derivatives of order k exist and are ld-continuous on $\mathbb{T}$.

Denote by $L_{ld}(\mathbb{T})$ the class of functions $f : \mathbb{T} \to \mathbb{R}$ that are ld-continuous on $\mathbb{T}$ and Lebesgue nabla integrable on $\mathbb{T}$.

$$\mathbb{R}^+_{ld} = \{K : \mathbb{T} \to R, \quad K(t) \geq 0, \quad 1 - \nu K(t) > 0, \quad and \quad K \in C_{ld}(\mathbb{T})\}. \tag{1.7}$$

We assume that $A \in C_{ld}(\mathbb{T}_\infty)$ and $a_{12}(t) \neq 0$ for all $t \in \mathbb{T}_\infty$.

The main idea of this paper is a special construction of the phase functions $\theta_{1,2}$ of asymptotic solutions of non autonomous system (1.1).

From a given non-trivial function $\theta \in C^2_{ld}(\mathbb{T}_\infty)$ we construct the function

$$k(t) = \frac{a_{12}(t)}{2\theta^2(t)} \left(\frac{\theta(t)}{a_{12}(t)} \right)^\nabla. \tag{1.8}$$

Here and further in the text we often suppressed dependance on t for simplicity.

Assuming $1 - 2k(t)\theta(t)\nu(t) \neq 0$ for all $t \in \mathbb{T}_\infty$ we choose a phase function $\theta_1(t)$ as a solution of the equation:

$$\nu\theta_1^2 - 2\theta_1(1 + \nu\theta) + 2\theta + \frac{TrA - \nu|A| - 2k\theta}{1 - 2k\theta\nu} = 0, \tag{1.9}$$

which is the version of Liouville's formula.

If $\nu > 0$ then θ_1 is the solution of the quadratic equation:

$$\theta_1 = \frac{1}{\nu} + \theta + \sqrt{D}, \quad D = \theta^2 + \frac{1 - \nu TrA + \nu^2|A|}{(1 - 2k\theta\nu)\nu^2}, \quad \nu > 0. \tag{1.10}$$

If $\nu = 0$ then (1.9) turns to the linear equation:

$$2\theta_1 - TrA + 2\theta(k - 1) = \theta_1 + \theta_2 - TrA + \frac{a_{12}}{\theta} \left(\frac{\theta}{a_{12}} \right)' = 0,$$

and in this case the function $\theta_1(t)$ is defined from the formula

$$\theta_1(t) = \theta(t) - \frac{a_{12}(t)}{2\theta(t)} \left(\frac{\theta(t)}{a_{12}(t)} \right)' + \frac{TrA(t)}{2}, \quad \nu(t) = 0. \tag{1.11}$$

Define auxiliary functions

$$\theta_2(t) = \theta_1(t) - 2\theta(t), \quad \Psi(t) = \begin{pmatrix} \widehat{e}_{\theta_1}(t, t_0) & \widehat{e}_{\theta_2}(t, t_0) \\ \frac{(\theta_1 - a_{11})\widehat{e}_{\theta_1}(t,t_0)}{a_{12}(t)} & \frac{(\theta_2 - a_{11})\widehat{e}_{\theta_2}(t,t_0)}{a_{12}(t)} \end{pmatrix}, \tag{1.12}$$

$$Hov_j = \theta_j^2 - \theta_j Tr(A) + |A| - a_{12}(1 - \nu\theta_j)\left(\frac{a_{11} - \theta_j}{a_{12}}\right)^{\nabla}, \quad j = 1, 2, \tag{1.13}$$

$$Q_0(t) = \frac{Hov_1(t) - Hov_2(t)}{2\theta(t)}, \tag{1.14}$$

$$M_j = \|(1 - \nu\Psi^{-1}\Psi^{\nabla})^{-1}\| \cdot \left|\frac{\widehat{e}_j Hov_j}{2\theta \cdot \widehat{e}_{3-j}}\right|, \quad j = 1, 2. \tag{1.15}$$

$$K(t) = c\left(\left|\frac{Hov_1}{\theta}\right| + \sigma\left|\frac{a_{12}Q_0}{\theta}\right|\right)\left[1 + \nu\left(\|A\| + \left|\frac{Hov_1}{a_{12}}\right| + \sigma|Q_0|\right)\right](t), \tag{1.16}$$

where $\|\cdot\|$ is the Euclidean matrix norm: $\|A\| = \sqrt{\sum_{k,j=1}^{n} A_{kj}^2}$, and $\widehat{e}_\theta(t, t_0)$ is the nabla exponential function on a time scale (see [10, 6]).

Note that θ_1 and θ_2 can be used to form the approximate fundamental matrix Ψ of system (1.1) in form (1.12).

Theorem 1.1. *Assume $a_{12}(t) \neq 0$, and there exists a non-trivial function $\theta \in C_{ld}^2(\mathbb{T}_\infty)$ such that $M_j \in R_{ld}^+$, $\quad 1 - \nu TrA + \nu^2|A|(t) \neq 0, \quad 1 - 2k\nu\theta(t) \neq 0$ for all $t \in \mathbb{T}_\infty$, and*

$$\lim_{t\to\infty} \widehat{e}_{M_j}(t, t_0) < \infty, \quad j = 1, 2. \tag{1.17}$$

Then equation (1.1) is asymptotically stable if and only if the condition

$$\lim_{t\to\infty}\left(\frac{\theta_j - a_{11}}{a_{12}}\right)^{k-1} \widehat{e}_{\theta_j}(t, t_0) = 0, \quad k, j = 1, 2, \tag{1.18}$$

is satisfied.

Remark 1.1. *If one can find two different phase functions $\theta_j, j = 1, 2$ such that generalized characteristic equations $Hov_j(t) = 0$ are satisfied, then from (1.15) we get $M_j \equiv 0$, condition (1.17) disappears, and formula (1.12) with mentioned above phase functions defines the exact fundamental solution of (1.1). Note also that for a constant matrix A equations $Hov_j(t) = 0$ turn to the usual characteristic equations of system (1.1).*

Condition (1.17) of Theorem 1.1 is complicated and it is very restrictive when one of functions $\left|\frac{\widehat{e}_{\theta_j}(t,t_0)}{\widehat{e}_{\theta_{3-j}}(t,t_0)}\right|$ has exponential growth as $t \to \infty$. In the next Theorem 1.2 we replace condition (1.17) by less restrictive and simple condition (1.19) under some additional conditions.

Theorem 1.2. *Assume $a_{12}(t) \neq 0$, and exists a non-trivial function $\theta(t) \in C_{ld}^2(\mathbb{T}_\infty)$ such that $K \in \mathbb{R}_{ld}^+$, $\quad 1 - \nu TrA + \nu^2|A|(t) \neq 0, \quad 1 - 2k\nu\theta(t) \neq 0$, for all $t \in \mathbb{T}_\infty$, and there exist some constants $\beta > 0$ and $\sigma > 1$ such that*

$$\lim_{t\to\infty} \widehat{e}_K(t, t_0) < \infty, \tag{1.19}$$

$$2\Re[\theta_j(t)] \leq \nu(t)|\theta_j(t)|^2, \quad j = 1, 2, \quad t \in \mathbb{T}_\infty, \tag{1.20}$$

$$\left|1 - \nu\left(TrA + Q_0\right) + \nu^2(|A| + \theta_1 Q_0 - Hov_1)\right| \geq \beta > 0, \tag{1.21}$$

$$1 + \left|\frac{(\theta_j - a_{11})(t)}{a_{12}}\right| \leq \sigma, \quad j = 1, 2, \quad t \in \mathbb{T}_\infty, \tag{1.22}$$

$$\lim_{t\to\infty} |\widehat{e}_{\theta_j}(t,t_0)| = 0, \quad j = 1,2. \tag{1.23}$$

Then equation (1.1) is asymptotically stable.

Note that if $\nu = 0$, then condition (1.20) turns to the classical stability condition $\Re[\theta_j(t)] \leq 0$. Condition (1.19) means that the error of the chosen asymptotic solution is small enough (compare with well known Levinson's integrability condition from [15]).
The next three lemmas from [1],[10],[17] are useful tools for checking condition (1.23) .

Lemma 1.3. *(*[1]*,*[10]*) Let $\theta(t)$ be a complex valued function from $C_{ld}(\mathbb{T})$ such that $1 - \theta(t)\nu(t) \neq 0$ for all $t \in \mathbb{T}_\infty$.*
Then

$$\lim_{t\to\infty} \widehat{e}_{\theta(t)}(t,t_0) = 0 \tag{1.24}$$

if and only if

$$\lim_{T\to\infty} \int_{t_0}^{T} \lim_{p\searrow\nu(s)} \frac{Log|1 - p\theta(s)|}{-p} \nabla s = -\infty. \tag{1.25}$$

The following lemma gives a simpler sufficient conditions of decay of nabla exponential function

Lemma 1.4. *(*[1]*,*[10]*) Assume $\theta(t) \in C_{ld}(\mathbb{T})$, and for some $\varepsilon > 0$*

$$\lim_{t\to\infty} \int_{t_0}^{t} \Re[\theta(s)]\nabla s = -\infty, \quad \text{if} \quad \nu = 0, \tag{1.26}$$

$$|1 - \theta\nu(t)| \geq e^{\varepsilon} > 1, \qquad \int_{t_0}^{\infty} \frac{\nabla s}{\nu(s)} = \infty, \quad \text{if} \quad \nu > 0. \tag{1.27}$$

Then (1.24) is satisfied

Remark 1.2. [1] *The first condition (1.27) for $\nu > 0$ means that values of $\theta(t)$ are located in the the exterior of the ball with the center $\frac{1}{\nu_*}$ and the radius $\frac{e^{\varepsilon}}{\nu_*}$:*

$$\left\{ z : \left| z - \frac{1}{\nu_*} \right| > \frac{e^{\varepsilon}}{\nu_*} \right\}, \quad \nu_* = \inf[\nu(t)], \tag{1.28}$$

and it may be written in the form

$$2\Re[\theta(t)] < \nu(t)|\theta(t)|^2. \tag{1.29}$$

Remark 1.3. *In view of Lemma 1.4 conditions (1.20),(1.23) of theorem 1.2 can be replaced by*

$$\int_{t_0}^{\infty} \frac{ds}{\nu(s)} = \infty, \quad \text{for} \quad \nu > 0. \tag{1.30}$$

$$2\Re[\theta_j(t)] < \nu(t)|\theta_j(t)|^2, \quad t \in \mathbb{T}_\infty, \quad j = 1,2. \tag{1.31}$$

The scalar equation

$$x^{\nabla}(t) = \theta(t)x(t) \tag{1.32}$$

is called *exponentially stable if there exists a constant* $\alpha > 0$ *such that for every* $t_0 \in \mathbb{T}$ *there exist a* $N = N(t_0) \geq 1$ *with*

$$\|\widehat{e}_\theta(t, t_0)\| \leq N(t_0)e^{-\alpha(t-t_0)}, \quad \text{for} \quad t \geq t_0. \tag{1.33}$$

If the constant $N(t_0)$ *from (1.33) can be chosen independent of* t_0*, then equation (1.32) is called uniformly exponentially stable.*

Lemma 1.5 ([17])**.** *Equation (1.32) is exponentially stable if and only if one of following conditions is satisfied for arbitrary* $t_1 \in \mathbb{T}$:

$$\gamma(\theta) := \limsup_{T\to\infty} \frac{1}{T - t_1} \int_{t_1}^{T} \lim_{p \searrow \nu(s)} \frac{(\log|1 - p\theta(s)|)\nabla s}{-p} < 0, \tag{1.34}$$

$$\text{for every } \tau \in \mathbb{T}: \text{ there exist } t \in \mathbb{T} \text{ with } t > \tau \text{ such that } 1 - \nu(t)\theta(t) = 0, \tag{1.35}$$

where we use the convention $\log 0 = -\infty$ *in (1.34).*

Remark 1.4. *In order to apply Theorem 1.2 for the study exponential stability of dynamic system (1.1) one can replace condition (1.23) by the necessary and sufficient condition of exponential stability of an exponential function on a time scale given in Lemma 1.5.*

2. Fundamental Matrix and Error Estimates

If we seek a solution of (1.1) in the form

$$u = \Psi v, \tag{2.1}$$

then from (1.1) we get

$$\Psi^{\nabla} v + \Psi v^{\nabla} - \nu \Psi^{\nabla} v^{\nabla} = A\Psi v,$$
$$\Psi(1 - \nu\Psi^{-1}\Psi^{\nabla})v^{\nabla} = (A\Psi - \Psi^{\nabla})v,$$

or

$$v^{\nabla}(t) = H(t)v(t), \tag{2.2}$$

where

$$H(t) = (1 - \nu\Psi^{-1}\Psi^{\nabla})^{-1}\Psi^{-1}(A\Psi - \Psi^{\nabla})(t). \tag{2.3}$$

Assume we can find an exact solution of an auxiliary system

$$\psi^{\nabla}(t) = A_1(t)\psi(t), \quad t \in \mathbb{T}_\infty, \tag{2.4}$$

with a matrix-function A_1 *close to the matrix-function* A*, which means that condition (2.6) below is satisfied. Note that if* $A = A_1$ *then* $H \equiv 0$ *and (2.6) is satisfied.*

Let $\Psi(t)$ *be the fundamental matrix of the auxiliary system (2.4). If the matrix-function* A_1 *is regressive and ld-continuous then* $\Psi(t)$ *exists ([6]). The solutions of (1.1) can be represented in the form*

$$u(t) = \Psi(t)(C + \varepsilon(t)), \tag{2.5}$$

where $u(t), \varepsilon(t), C$ *are the 2-vector columns:* $u(t) = \text{column}(u_1(t), u_2(t))$*,* $\varepsilon(t) = \text{colomn}(\varepsilon_1(t), \varepsilon_2(t))$*,* $C = \text{colomn}(C_1, C_2), C_j$ *are arbitrary constants. We can consider (2.5) as a definition of the error vector-function* $\varepsilon(t)$*.*

In [12, 14] *was proved the following*

Theorem 2.1. *Assume there exists a matrix function $\Psi(t) \in C^1_{ld}(\mathbb{T}_\infty)$ such that $\|H\| \in R^+_{ld}$, the matrix function $\Psi - \nu\Psi^\nabla$ is invertible, and the following exponential function on a time scale is bounded:*

$$\widehat{e}_{\|H\|}(\infty, t) = exp\int_t^\infty \lim_{p\searrow\nu(s)} \frac{Log(1-p\|H(s)\|)\nabla s}{-p} < \infty. \tag{2.6}$$

Then every solution of (1.1) can be represented in form (2.5) and the error vector-function $\varepsilon(t)$ can be estimated as

$$\|\varepsilon(t)\| \le \|C\| \left(\widehat{e}_{\|H\|}(\infty, t) - 1\right), \tag{2.7}$$

where $\|\cdot\|$ is the Euclidean vector (or matrix) norm.

To find the fundamental matrix function let us seek solutions of equation (1.1)

$$u_1^\nabla = a_{11}u_1 + a_{12}u_2, \quad u_2^\nabla = a_{21}u_1 + a_{22}u_2, \tag{2.8}$$

in the form

$$u_1(t) = C_1\widehat{e}_{\theta_1}(t,t_0) + C_2\widehat{e}_{\theta_2}(t,t_0), \tag{2.9}$$

where

$$\widehat{e}_{\theta_j}(t,t_0) = exp\left(\int_{t_0}^t \lim_{p\searrow\nu(\tau)} \frac{Log(1-p\theta_j(\tau))}{-p}\nabla\tau\right), \quad j=1,2. \tag{2.10}$$

By differentiation

$$u_1^\nabla = C_1\theta_1\widehat{e}_{\theta_1}(t,t_0) + C_2\theta_2\widehat{e}_{\theta_2}(t,t_0), \tag{2.11}$$

and from (2.8) assuming $a_{12} \neq 0$ we get

$$u_2 = \frac{u_1^\nabla - a_{11}u_1}{a_{12}} = C_1U_1\widehat{e}_{\theta_1}(t,t_0) + C_2U_2\widehat{e}_{\theta_2}(t,t_0), \tag{2.12}$$

$$U_1 = \frac{\theta_1 - a_{11}}{a_{12}}, \quad U_2 = \frac{\theta_2 - a_{11}}{a_{12}}, \tag{2.13}$$

and

$$u(t) = \Psi(t)C, \tag{2.14}$$

where the fundamental matrix $\Psi(t)$ of system (1.1) is defined as

$$\Psi(t) = \begin{pmatrix} \widehat{e}_{\theta_1}(t,t_0) & \widehat{e}_{\theta_2}(t,t_0) \\ U_1\widehat{e}_{\theta_1}(t,t_0) & U_2\widehat{e}_{\theta_2}(t,t_0) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ U_1 & U_2 \end{pmatrix} \begin{pmatrix} \widehat{e}_{\theta_1}(t,t_0) & 0 \\ 0 & \widehat{e}_{\theta_2}(t,t_0) \end{pmatrix}. \tag{2.15}$$

Denote

$$B_j = (a_{11} - \theta_j)(1-\nu\theta_j)\frac{U_j^\nabla}{U_j} = a_{12}(1-\nu\theta_j)\left(\frac{a_{11}-\theta_j}{a_{12}}\right)^\nabla, \quad j=1,2, \tag{2.16}$$

then from (1.13) we get

$$Hov_j = E_j - B_j, \quad E_j = \theta_j^2 - \theta_j Tr(A) + |A|, \quad j=1,2. \tag{2.17}$$

where E_j is usual characteristic polynomial of (1.1).

Lemma 2.2. *Assume $a_{12} \neq 0, \quad 1 - \nu TrA + \nu^2|A| \neq 0, \quad 1 - 2k\nu\theta \neq 0$ for all $t \in \mathbb{T}_\infty, \quad \Psi \in C_{ld}(\mathbb{T})$ is invertible and nabla differentiable. Then following formulas are true:*

$$|\Psi(t)| = det[\Psi(t)] = -\frac{2\theta}{a_{12}}\widehat{e}_{\theta_1}(t,t_0)\widehat{e}_{\theta_2}(t,t_0), \tag{2.18}$$

$$\theta_1+\theta_2-TrA+\frac{B_2-B_1}{2\theta}=\frac{Hov_1-Hov_2}{2\theta}, \tag{2.19}$$

$$\Psi^\nabla(t)\Psi^{-1}(t)=\begin{pmatrix} a_{11} & a_{12}\\ Q_1+a_{21} & Q_0+a_{22}\end{pmatrix}, \tag{2.20}$$

where

$$Q_0=\frac{\Lambda_2U_2-\Lambda_1U_1}{U_2-U_1}-a_{22},\quad Q_1=\frac{(\Lambda_1-\Lambda_2)U_1U_2}{U_2-U_1}-a_{21}, \tag{2.21}$$

or

$$Q_0=\frac{Hov_1-Hov_2}{2\theta},\quad Q_1=\frac{U_1Hov_2-U_2Hov_1}{2\theta}=\frac{Hov_1}{a_{12}}-U_1Q_0, \tag{2.22}$$

$$\Psi^{-1}(A\Psi-\Psi^\nabla)(t)=\frac{1}{2\theta}\begin{pmatrix} -Hov_1 & -\frac{\widehat{e}_{\theta_2}}{\widehat{e}_{\theta_1}}Hov_2\\ \frac{\widehat{e}_{\theta_1}}{\widehat{e}_{\theta_2}}Hov_1 & Hov_1\end{pmatrix}, \tag{2.23}$$

$$TrA-\nu|A|=2k\theta+(\theta_1+\theta_2-\nu\theta_1\theta_2)(1-2k\theta\nu). \tag{2.24}$$

Note that (2.24) is the version of Liouville's formula.

Proof. From (2.13) we have

$$U_2-U_1=\frac{\theta_2-\theta_1}{a_{12}}=-\frac{2\theta}{a_{12}}, \tag{2.25}$$

and formula (2.18) follows from (2.15) and (2.25).

From (2.15) we get the inverse matrix

$$\Psi^{-1}(t)=\frac{1}{U_2-U_1}\begin{pmatrix} 1/\widehat{e}_{\theta_1}(t,t_0) & 0\\ 0 & 1/\widehat{e}_{\theta_2}(t,t_0)\end{pmatrix}\begin{pmatrix} U_2 & -1\\ -U_1 & 1\end{pmatrix}.$$

Formula (2.19) follows from (2.17).

From the time scales calculus we have

$$(ab)^\nabla=a^\nabla b+b^\nabla a-\nu a^\nabla b^\nabla,\quad \widehat{e}^\nabla_{\theta_j}(t,t_0)=\theta_j\widehat{e}_{\theta_j}(t,t_0).$$

Nabla derivative of the Ψ matrix function is given by the formula

$$\Psi^\nabla(t)=\begin{pmatrix} \theta_1 & \theta_2\\ \Lambda_1U_1 & \Lambda_2U_2\end{pmatrix}\begin{pmatrix} \widehat{e}_{\theta_1}(t,t_0) & 0\\ 0 & \widehat{e}_{\theta_2}(t,t_0)\end{pmatrix}, \tag{2.26}$$

where

$$\Lambda_j=\frac{(U_j\widehat{e}_{\theta_j})^\nabla}{U_j\widehat{e}_{\theta_j}}=\theta_j+\frac{U_j^\nabla}{U_j}(1-\nu\theta_j)=\theta_j-\frac{B_j}{a_{12}U_j},\quad j=1,2. \tag{2.27}$$

Formulas (2.20),(2.21) are proved by direct calculations:

$$\Psi^\nabla(t)\Psi^{-1}(t)=\frac{1}{U_2-U_1}\begin{pmatrix} \theta_1 & \theta_2\\ \Lambda_1U_1 & \Lambda_2U_2\end{pmatrix}\begin{pmatrix} U_2 & -1\\ -U_1 & 1\end{pmatrix}=$$

$$\frac{1}{U_2-U_1}\begin{pmatrix} \theta_1U_2-\theta_2U_1 & \theta_2-\theta_1\\ (\Lambda_1-\Lambda_2)U_1U_2 & \Lambda_2U_2-\Lambda_1U_1\end{pmatrix}=\begin{pmatrix} a_{11} & a_{12}\\ Q_1+a_{21} & Q_0+a_{22}\end{pmatrix}.$$

From (2.13),(2.17) we get

$$\frac{U_2E_1-U_1E_2}{2\theta}=\frac{(\theta_2-a_{11})(\theta_1^2-\theta_1TrA+|A|)-(\theta_1-a_{11})(\theta_2^2-\theta_2TrA+|A|)}{2\theta a_{12}}=$$

$$\frac{a_{11}(\theta_2^2-\theta_1^2+(\theta_1-\theta_2)TrA)+\theta_1\theta_2(\theta_1-\theta_2)+|A|(\theta_2-\theta_1)}{2\theta a_{12}}=-\frac{P}{a_{12}},$$

where

$$P=|A|+a_{11}(\theta_1+\theta_2-TrA)-\theta_1\theta_2. \tag{2.28}$$

Further we prove formulas (2.22):

$$Q_1+a_{21}=\frac{(\Lambda_1-\Lambda_2)U_1U_2}{U_2-U_1}=\left(2\theta+\frac{B_2U_1-B_1U_2}{a_{12}U_1U_2}\right)\frac{U_1U_2a_{12}}{-2\theta}=$$

$$-U_1U_2a_{12}-\frac{B_2U_1-B_1U_2}{2\theta}=\frac{(a_{11}-\theta_1)(\theta_2-a_{11})}{a_{12}}-\frac{B_2U_1-B_1U_2}{2\theta}=$$

$$\frac{|A|-\theta_1\theta_2+a_{11}(\theta_1+\theta_2-a_{11}-a_{22})}{a_{12}}+a_{21}+\frac{U_2(E_1-Hov_1)-U_1(E_2-Hov_2)}{2\theta}=$$

$$=\frac{P}{a_{12}}+a_{21}+\frac{U_1Hov_2-U_2Hov_1}{2\theta}+\frac{U_2E_1-U_1E_2}{2\theta}=\frac{U_1Hov_2-U_2Hov_1}{2\theta}+a_{21},$$

and

$$Q_0+a_{22}=\frac{\Lambda_2U_2-\Lambda_1U_1}{U_2-U_1}=\frac{a_{11}-\theta_2}{2\theta}\left[\theta_2-\frac{B_2}{U_2a_{12}}\right]-\frac{a_{11}-\theta_1}{2\theta}\left[\theta_1-\frac{B_1}{U_1a_{12}}\right]=$$

$$\frac{a_{11}\theta_2-\theta_2^2-a_{11}\theta_1+\theta_1^2}{2\theta}+\frac{B_2-B_1}{2\theta}=$$

$$=a_{22}+\theta_1+\theta_2-TrA+\frac{B_2-B_1}{2\theta}=\frac{Hov_1-Hov_2}{2\theta}+a_{22},$$

in view of (2.19).

For $j=1,2$ we have

$$\frac{a_{12}}{2\theta}(Q_1+Q_0U_j)=\frac{a_{12}}{2\theta}\left(\frac{U_1Hov_2-U_2Hov_1}{2\theta}+\frac{(Hov_1-Hov_2)U_j}{2\theta}\right)=\frac{Hov_j}{2\theta},$$

from which formula (2.23) is deduced:

$$\Psi^{-1}(A\Psi-\Psi^{\nabla})(t)=\frac{a_{12}}{2\theta}\begin{pmatrix}-Q_1-Q_0U_1 & -(Q_1+Q_0U_2)\frac{\widehat{e}_{\theta_2}}{\widehat{e}_{\theta_1}}\\ (Q_1+Q_0U_1)\frac{\widehat{e}_{\theta_1}}{\widehat{e}_{\theta_2}} & Q_1+Q_0U_2\end{pmatrix}=$$

$$\frac{1}{2\theta}\begin{pmatrix}-Hov_1 & -\frac{\widehat{e}_{\theta_2}}{\widehat{e}_{\theta_1}}Hov_2\\ \frac{\widehat{e}_{\theta_1}}{\widehat{e}_{\theta_2}}Hov_1 & Hov_2\end{pmatrix}.$$

Formula (2.24) is proved by using the well known Liouville's formula for (1.1) (see [6], Theorem 3.9.4):

$$\frac{|\Psi(t)|^{\nabla}}{|\Psi(t)|}=TrA(t)-\nu|A(t)|, \tag{2.29}$$

or in view of (2.18):

$$\frac{a_{12}}{\theta e_1e_2}\left(\frac{\theta e_1e_2}{a_{12}}\right)^{\nabla}=TrA-\nu|A|.$$

From this formula using notation (1.8) we get formula (2.24):

$$TrA-\nu|A|=\left(\frac{\theta}{a_{12}}\right)^{\nabla}\frac{a_{12}}{\theta}+\frac{(\widehat{e}_{\theta_1}\widehat{e}_{\theta_2})^{\nabla}}{\widehat{e}_{\theta_1}\widehat{e}_{\theta_2}}-\nu\left(\frac{\theta}{a_{12}}\right)^{\nabla}\frac{a_{12}}{\theta}\frac{(\widehat{e}_{\theta_1}\widehat{e}_{\theta_2})^{\nabla}}{\widehat{e}_{\theta_1}\widehat{e}_{\theta_2}}=$$

$$2k\theta+\frac{(\widehat{e}_{\theta_1}\widehat{e}_{\theta_2})^{\nabla}}{\widehat{e}_{\theta_1}\widehat{e}_{\theta_2}}(1-2k\theta\nu)=2k\theta+(\theta_1+\theta_2-\nu\theta_1\theta_2)(1-2k\theta\nu).$$

□

Proof of Theorem 1.1. First note that from the assumption $M_j \in R_{ld}^+$ it follows that the exponential functions $\widehat{e}_{M_j}(t,t_0)$ exist ([6]). From assumptions $1-\nu TrA+\nu^2|A|(t) \neq 0, \quad 1-2k\nu\theta(t) \neq 0$ it follows that $1-\nu\theta_j \neq 0$ and the exponential functions $\widehat{e}_{\theta_j}(t,t_0)$ exist.

Consider system (1.1) or equivalent system (2.2). From (2.3), (1.15) and (2.23) it follows that $\|H\| \leq \|(1-\nu\Psi^{-1}\Psi^\nabla)^{-1}\Psi^{-1}(A\Psi-\Psi^\nabla)\| \leq max_{j=1,2}|M_j|$, and from condition (1.17) of Theorem 1.1 it follows that condition (2.6) of Theorem 2.1 is satisfied. Applying Theorem 2.1 we obtain representation (2.5) for the solutions of (1.1) and estimate (2.7) for the error function ε. From (2.5),(2.7) we get the stability inequality

$$\|u(t)\| \leq const\|\Psi(t)\|. \tag{2.30}$$

From (1.18) it follows $\|\Psi(t)\| \to 0$ as $t \to \infty$, and using (2.30) we obtain asymptotic stability of (1.1).

□

Lemma 2.3. *If conditions of Lemma 2.2 are satisfied and for some number* $\sigma > 1$

$$1+\left|\frac{\theta_j-a_{11}}{a_{12}}\right| \leq \sigma, \quad j=1,2, \quad t \in \mathbb{T}_\infty, \tag{2.31}$$

$$|1-\nu\theta_j| = \sqrt{(1-\nu\Re[\theta_j])^2+(\nu\Im[\theta_j])^2} \geq 1, \quad j=1,2, \quad t \in \mathbb{T}_\infty. \tag{2.32}$$

Then

$$\|\Psi(s)\| \leq const, \tag{2.33}$$

$$\|\Psi(t)\Psi^{-1}(s)\| \leq C\left|\frac{a_{12}(s)}{\theta(s)}\right|, \quad s \leq t, \tag{2.34}$$

$$\|A-\Psi^\nabla\Psi^{-1}\| \leq |Q_0|+|Q_1| \leq \left|\frac{Hov_1}{a_{12}}\right|+\sigma|Q_0|. \tag{2.35}$$

Proof. From (2.32) it follows that the functions $|\widehat{e}_{\theta_j}(t,t_0)|, \quad j=1,2$, are non-increasing.

Indeed, if $\nu > 0$ then from (2.32) it follows that

$$\frac{Log|1-\nu(t)\theta_j(t)|}{-\nu(t)} \leq 0, \tag{2.36}$$

so the functions

$$\widehat{e}_{\theta_j}(t,t_0)| = exp\left(\int_{t_0}^t \lim_{p\searrow\nu(\tau)} \frac{Log|1-p\theta_j(\tau)|}{-p}\nabla\tau\right), \quad j=1,2$$

are non increasing.

If $\nu \equiv 0$ then the functions $|\widehat{e}_{\theta_j}(t,t_0)|$ are non-increasing in view of

$$\frac{|\widehat{e}_{\theta_j}(t,t_0)|^\nabla}{|\widehat{e}_{\theta_j}(t,t_0)|} = \frac{|\widehat{e}_{\theta_j}(t,t_0)|'}{|\widehat{e}_{\theta_j}(t,t_0)|} = \Re[\theta_j] \leq 0. \tag{2.37}$$

Because the functions $|\widehat{e}_{\theta_j}(t,t_0)|, \quad j=1,2$, are non-increasing we get

$$|\widehat{e}_{\theta_j}(t,t_0)| \leq |\widehat{e}_{\theta_j}(t_0,t_0)| = 1, \quad \left|\frac{\widehat{e}_{\theta_j}(t,t_0)}{\widehat{e}_{\theta_j}(s,t_0)}\right| \leq 1, \quad t \geq s \geq t_0. \tag{2.38}$$

From condition (2.31) it follows that $|U_j| \le C$ and inequality (2.33) is true. Inequality (2.34) follows from the formula:

$$\Psi(t)\Psi^{-1}(s) = \frac{1}{(U_2 - U_1)(s)} \times$$

$$\begin{bmatrix} \frac{\widehat{e}_{\theta_1}(t,t_0)}{\widehat{e}_{\theta_1}(s,t_0)}U_2(s) - \frac{\widehat{e}_{\theta_2}(t,t_0)}{\widehat{e}_{\theta_2}(s,t_0)}U_1(s) & \frac{\widehat{e}_{\theta_2}(t,t_0)}{\widehat{e}_{\theta_2}(s,t_0)} - \frac{\widehat{e}_{\theta_1}(t,t_0)}{\widehat{e}_{\theta_1}(s,t_0)} \\ \frac{\widehat{e}_{\theta_1}(t,t_0)}{\widehat{e}_{\theta_1}(s,t_0)}U_1(t)U_2(s) - \frac{\widehat{e}_{\theta_2}(t,t_0)}{\widehat{e}_{\theta_2}(s,t_0)}U_1(s)U_2(t) & \frac{\widehat{e}_{\theta_2}(t,t_0)}{\widehat{e}_{\theta_2}(s,t_0)}U_2(t) - \frac{\widehat{e}_{\theta_1}(t,t_0)}{\widehat{e}_{\theta_1}(s,t_0)}U_1(t) \end{bmatrix}. \quad (2.39)$$

Further using (2.20),(2.22) we get estimate (2.35):

$$\|A - \Psi^{\nabla}\Psi^{-1}\| \le |Q_1| + |Q_0| = \left|\frac{Hov_1}{a_{12}} - U_1Q_0\right| + |Q_0| \le \left|\frac{Hov_1}{a_{12}}\right| + \sigma|Q_0|. \quad (2.40)$$

□

Lemma 2.4. *If conditions of Lemma 2.3 and (1.21) are satisfied then*

$$\|\Psi(t)H(s)\Psi^{-1}(s)\| \le K(s), \quad s \in \mathbb{T}_\infty \bigcap [t_0, t], \quad (2.41)$$

where $K(s)$ is defined in (1.16).

Proof. Denote

$$\Omega(s) \equiv \Psi(1 - \nu\Psi^{-1}\Psi^{\nabla})\Psi^{-1}(s) = 1 - \nu\Psi^{\nabla}\Psi^{-1} = 1 - \nu A - \nu \begin{pmatrix} 0 & 0 \\ Q_1 & Q_0 \end{pmatrix}. \quad (2.42)$$

Then

$$\|\Omega\| \le 1 + \nu(\|A\| + |Q_1| + |Q_0|)$$
$$\|\Omega^{co}\| \le 1 + \nu(\|A\| + |Q_1| + |Q_0|),$$

where Ω^{co} is the adjoint of the matrix Ω.

Using (2.22) we have

$$a_{11}Q_0 - a_{12}Q_1 = a_{11}Q_0 - a_{12}\left(\frac{Hov_1}{a_{12}} - U_1Q_0\right) = \theta_1 Q_0 - Hov_1. \quad (2.43)$$

From (2.20) we get

$$|det(\Omega)| = \left|det[1 - \nu\Psi^{\nabla}\Psi^{-1}]\right| = \left|det\begin{bmatrix} 1 - \nu a_{11} & -\nu a_{12} \\ -\nu(Q_1 + a_{21}) & 1 - \nu(Q_0 + a_{22}) \end{bmatrix}\right| =$$
$$|1 - \nu(Q_0 + TrA) + \nu^2(|A| + a_{11}Q_0 - a_{12}Q_1)|.$$

In view of (2.43)

$$|det(\Omega)| = |1 - \nu(Q_0 + TrA) + \nu^2(|A| + \theta_1 Q_0 - Hov_1)|,$$

and from assumption (1.21) we have

$$|det(\Omega)| \ge \beta > 0. \quad (2.44)$$

Further

$$\|\Omega^{-1}\| = \frac{\|\Omega^{co}\|}{|det(\Omega)|} \le \frac{1 + \nu(\|A\| + |Q_1| + |Q_0|)}{|det(\Omega)|} \le \frac{1 + \nu(\|A\| + |Q_0| + |Q_1|)}{\beta}$$

and using (2.34), (2.35) we get

$$\|\Psi(t)H(s)\Psi^{-1}(s)\| \le \|\Psi(t)\Psi^{-1}(s)\| \cdot \|\Psi(1 - \nu\Psi^{-1}\Psi^{\nabla})^{-1}\Psi^{-1}(A - \Psi^{\nabla}\Psi^{-1})\|(s) \le$$

$$\le C\left|\frac{a_{12}(s)}{\theta(s)}\right| \|\Omega^{-1}(A - \Psi^{\nabla}\Psi^{-1})\|(s) \le C\left|\frac{a_{12}}{\theta}\right| (|Q_1| + |Q_0|)(1 + \nu(\|A\| + |Q_1| + |Q_0|))$$

$$\le c\left|\frac{a_{12}(s)}{\theta(s)}\right|\left(\sigma|Q_0|+\frac{|Hov_1|}{|a_{12}|}\right)\left(1+\nu(\|A\|+\sigma|Q_0|+\frac{|Hov_1|}{|a_{12}|}\right)=K(s).$$

□

Lemma 2.5. [5, 12] *Assume* $y, f \in C_{ld}(\mathbb{T}), \quad f, y \ge 0, \quad K \in \mathbb{R}_\nu^+$. *Then*

$$y(t) \le f(t) + \int_{t_0}^{t} K(s)y(s)\nabla s \quad \textit{for all} \quad t \in \mathbb{T}_\infty \tag{2.45}$$

implies

$$y(t) \le f(t) + \int_{t_0}^{t} \widehat{e}_K(t,\rho(s))K(s)f(s)\nabla s, \quad \textit{for all} \quad t \in \mathbb{T}_\infty. \tag{2.46}$$

Proof of Theorem 1.2. By integration from the system (2.2) we get

$$v(t) = C + \int_{t_0}^{t} H(s)v(s)\nabla s. \tag{2.47}$$

Multiplying this representation by $\Psi(t)$ we have

$$\Psi(t)v(t) = \Psi(t)C + \int_{t_0}^{t} \Psi(t)H(s)v(s)\nabla s, \tag{2.48}$$

or using notation $u(t) = \Psi(t)v(t)$ we get

$$u(t) = \Psi(t)C + \int_{t_0}^{t} \Psi(t)H(s)\Psi^{-1}(s)u(s)\nabla s. \tag{2.49}$$

In view of (2.41) we get

$$\|u(t)\| \le \|\Psi(t)C\| + \int_{t_0}^{t} K(s)\|u(s)\|\nabla s. \tag{2.50}$$

Using Gronwall's inequality (2.46) to (2.50) we get the stability estimate

$$\|u(t)\| \le \|\Psi(t)C\| + \int_{t_0}^{t} \widehat{e}_K(t,\rho(s))K(s)\|\Psi(t)C\|\nabla s, \quad t \in \mathbb{T}_\infty. \tag{2.51}$$

From (1.22), (1.23) it follows that

$$\lim_{t\to\infty} \|\Psi(t)C\| = 0, \tag{2.52}$$

so for any $\varepsilon > 0$ there exists t_0 such that for $t \in \mathbb{T}_\infty$ we have

$$\|\Psi(t)C\| \le \varepsilon. \tag{2.53}$$

Hence it follows from (2.51)

$$\|u(t)\| = \varepsilon\left(1 + \int_{t_0}^{t} \widehat{e}_K(t,\rho(s))K(s)\nabla s\right), \quad t \in \mathbb{T}_\infty. \tag{2.54}$$

Further

$$\int_{t_0}^{t} \widehat{e}_K(t,\rho(s))K(s)\nabla s = \widehat{e}_K(t,t_0) - \widehat{e}_K(t,t), \tag{2.55}$$

and so

$$\|u(t)\| \le \varepsilon\widehat{e}_K(t,t_0) \le C\varepsilon, \tag{2.56}$$

from which we get asymptotic stability of dynamic system (1.1).

□

Example 2.1. *Consider system (1.1) with*

$$A = \begin{bmatrix} 0 & 1 \\ -\bar{q} & -\bar{p} \end{bmatrix}, \quad \bar{q} = \frac{b}{t\rho}, \quad \bar{p} = \frac{a}{\rho}, \quad t_0 > 0, \tag{2.57}$$

$$TrA = -\bar{p} = -\frac{a}{\rho}, \quad |A| = \bar{q} = \frac{b}{t\rho}.$$

From (1.15) it follows that if exist two different phase functions such that generalized characteristic equation (see (1.13))

$$Hov(t) = \theta^2 - \theta Tr(A) + |A| - a_{12}(1-\nu\theta)\left(\frac{a_{11}-\theta}{a_{12}}\right)^{\nabla} = 0,$$

is satisfied, then $M_j \equiv 0, j = 1,2$ and condition (1.17) of Theorem 1.1 disappears.

For the Euler system (1.1) with the matrix $A(t)$ given by (2.57) this equation turns to

$$Hov(t) = \theta^2(t) + \frac{a\theta(t)}{\rho(t)} + \frac{b}{t\rho(t)} + (1-\nu(t)\theta(t))\theta^{\nabla}(t) = 0. \tag{2.58}$$

Solution of this non-linear Riccati equation we seek in the form

$$\theta(t) = \frac{\lambda}{t}.$$

In view of

$$\frac{\theta^{\nabla}(t)}{\theta(t)} = -\frac{1}{\rho(t)}, \quad \nu(t) = t - \rho(t), \quad \theta_j^2 - \nu\theta_j^{\nabla}\theta_j = \frac{t\theta_j^2}{\rho}$$

characteristic equation (2.58) turns to

$$Hov(t) = \frac{\lambda^2 - (1-a)\lambda + b}{t\rho(t)} = 0$$

or

$$\lambda^2 - (1-a)\lambda + b = 0,$$

which is the usual characteristic quadratic equation with solutions

$$\lambda_{1,2} = \frac{1-a}{2} \pm \lambda, \qquad \lambda = \sqrt{\frac{(1-a)^2}{4} - b}. \tag{2.59}$$

Choosing the phase functions

$$\theta_j(t) = \frac{\lambda_j}{t}, \quad j = 1,2, \tag{2.60}$$

we have

$$Hov_j(t) = M_j(t) \equiv 0.$$

Well known exact solutions of the Euler system can be constructed by using the phase functions (2.60).

So condition (1.17) is satisfied and from Theorem 1.1 it follows the trivial result that system (1.1) with matrix $A(t)$ defined by (2.57) is asymptotically stable if and only if the condition

$$\lim_{t\to\infty} \theta_j^{k-1}\widehat{e}_{\theta_j}(t,t_0) = 0, \quad k,j = 1,2 \tag{2.61}$$

is satisfied.

Example 2.2. *Consider system (1.1) with*

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{tb}{\rho(t)(t^2+1)} & -\frac{a}{\rho(t)} \end{bmatrix}, \quad t_0 > 0, \tag{2.62}$$

$$TrA = -\frac{a}{\rho}, \quad |A| = \frac{tb}{\rho(t)(1+t^2)}.$$

For this system we can't solve generalized characteristic equation

$$Hov(t) = \theta^2(t) + \frac{a\theta(t)}{\rho(t)} + \frac{tb}{\rho(t)(t^2+1)} + (1-\nu(t)\theta(t))\theta^{\nabla}(t) = 0. \tag{2.63}$$

Anyway choosing phase functions by formula (2.60), the same way as in Example 2.1, from (1.13) we get

$$Hov_j(t) = \frac{\lambda_j^2 + (a-1)\lambda_j}{t\rho} + |A| = -\frac{b}{t\rho(t)} + \frac{tb}{\rho(t)(1+t^2)} = -\frac{b}{t\rho(t)(1+t^2)},$$

$$Q_0 = \frac{Hov_1 - Hov_2}{2\theta} \equiv 0, \quad K(t) \le \left|\frac{CHov_1}{\theta}\right| \le \frac{C}{t\rho(t)(1+t^2)}. \tag{2.64}$$

Condition (1.21) turns to the condition:

$$\left|1 + \frac{at\nu(t) + b\nu^2(t)}{t\rho(t)}\right| \ge \beta > 0, \quad \textit{for all} \quad t \in \mathbb{T}_\infty. \tag{2.65}$$

Condition (1.22) is satisfied and conditions (1.30), (1.31) turn to

$$2\Re[\lambda_j] < \frac{\nu(t)}{t}|\lambda_j|^2, \quad \int_{t_0}^{\infty} \frac{\nabla s}{\nu(s)} = \infty, \quad j = 1, 2. \tag{2.66}$$

Using estimate (2.64) one can simplify condition (1.19):

$$\lim_{t\to\infty} \widehat{e}_{K_0}(t, t_0) < \infty, \quad K_0 = \frac{C}{t^3\rho(t)}. \tag{2.67}$$

So from conditions (2.65)-(2.67) in view of Theorem 1.2 it follows the asymptotic stability of system (1.1) with matrix $A(t)$ as in (2.62).

Note that of $\nu(t) \equiv 0$ conditions (2.65), (2.67) are satisfied and (2.66) turns to $\Re[\lambda_j] < 0$.

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