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Research Article

Asymptotic Solutions of *n*th Order Dynamic Equation and Oscillations

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We establish a new asymptotic theorem for the nth order nonautonomous dynamic equation by its transformation to the almost diagonal system and applying Levinson's asymptotic theorem. Our transformation is given in the terms of unknown phase functions and is chosen in such a way that the entries of the perturbation matrix are the weighted characteristic functions. The characteristic function is defined in the terms of the phase functions and their choice is exible. Further applying this asymptotic theorem we prove the new oscillation and nonoscillation theorems for the solutions of the nth order linear nonautonomous differential equation with complex-valued coefficients. We show that the existence of the oscillatory solutions is connected with the existence of the special pairs of phase functions.

1. Introduction

Consider an ordinary nonautonomous differential equation of the nth order

$$L(u(t)) = \sum_{j=0}^{n} a_j(t)\, u^{(j)}(t) = 0, \quad t > t_0,\ a_n = 1, \tag{1}$$

with complex-valued continuous variable coefficients $a_j(t)$, $j = 0, \dots n-1$.

A solution of (1) is said to be *oscillatory* if it has an infinite sequence of zeros in (t_0, ∞) and *nonoscillatory* otherwise. Equation (1) is said to be *nonoscillatory* if all nontrivial solutions are nonoscillatory.

Oscillation theorems for ordinary differential equation of the nth order in the case of real variable coefficients have been studied in many papers (see [1, 2] and references therein). To the best of the author's knowledge the oscillations of the solutions of nonautonomous nth order equations with complex coefficients have not been studied yet (except [3]).

Let $C^k(t_0, \infty)$ be the set of k times differentiable functions on (t_0, ∞) and $L_1(t_0, \infty)$ the set of Lebesgue absolutely integrable functions on (t_0, ∞).

To consider the case of complex coefficients we are using asymptotic solutions of (1) in Euler form $u(t) = e^{\int_{t_0}^{t} f(s)ds}$. Define the characteristic function of (1) depending on a phase function $f_j(t) \in C^{n-1}(t_0, \infty)$

$$Ch_j(t) = e^{-\int_{t_0}^{t} f_j(s)ds} L\left(e^{\int_{t_0}^{t} f_j(s)ds}\right), \quad j = 1, 2, \dots, n. \tag{2}$$

By direct calculations for $j = 1, 2, \dots, n$ we get

$$Ch_j(t) = \sum_{k=1}^{n+1} f_{kj}(t)\, a_{k-1}(t) = f_{n+1,j}(t) + f_{nj} a_{n-1}(t) + \dots + f_{2j} a_1(t) + f_{1j} a_0(t), \tag{3}$$

where

$$f_{1j}(t) = 1, \qquad f_{k+1,j}(t) = \left(\frac{d}{dt} + f_j(t)\right) f_{kj}(t), \quad k = 1, 2, 3, \dots, n. \tag{4}$$

For example,

$$f_{2j}(t) = f_j(t), \qquad f_{3j}(t) = f_j^2(t) + f_j'(t),$$
$$f_{4j}(t) = f_j^3(t) + 3f_j' f_j(t) + f_j''(t). \tag{5}$$

Define the auxiliary square matrix $F(t)$ with the entries $f_{kj}(t)$ (see (4))

$$F(t) = \left\| f_{kj}(t) \right\|_{k,j=1}^{n}, \tag{6}$$

and denote by $F_j(t)$, $j = 1, \ldots, n$ the (n, j) minors of the matrix $F(t)$.

2. Main Theorems

The basic method of this paper is a new version of Levinson's asymptotic theorem (see [4, 5]).

Theorem 1. *Assume there exist complex-valued functions $f_j(t) \in C^{n-1}(t_0, \infty)$, $j = 1, 2, \ldots, n$ such that for all $t > t_0$ expressions $\Re[f_j(t) - f_k(t)]$, $1 \le k < j \le n$ do not change a sign; that is,*

$$\Re\left[f_j(t) - f_k(t)\right] \le 0 \quad \text{or} \quad \Re\left[f_j(t) - f_k(t)\right] \ge 0, \qquad 1 \le k < j \le n, \; t > t_0, \tag{7}$$

$$\int_{t_0}^{\infty} \left| \frac{F_j(t)\, Ch_k(t)}{F_0(t)} \right| dt < \infty, \qquad j = 1, 2, \ldots, n-1, \qquad k = 1, 2, \ldots, n, \tag{8}$$

$$\frac{F_0'(t)}{F_0(t)} + \sum_{j=1}^{n} f_j(t) + a_{n-1}(t) = 0, \quad t > t_0, \tag{9}$$

where characteristic functions $Ch_k(t)$, $k = 1, 2, \ldots, n$ are defined in (2), (3), *and*

$$F_0(t) \equiv \det(F(t)). \tag{10}$$

Then solutions of (1) *may be represented in the form*

$$u(t) = \sum_{j=1}^{n} \varphi_j(t)\left(1 + \varepsilon_j(t)\right) C_j, \qquad \lim_{t\to\infty} \varepsilon_j(t) = 0, \qquad j = 1, 2, \ldots, n, \tag{11}$$

where

$$\varphi_j(t) = e^{\int_{t_0}^{t} f_j(s)ds}, \quad j = 1, 2, \ldots, n. \tag{12}$$

Note that condition (9) is the well-known Abel's identity that is satisfied if (12) are exact solutions of (1).

Theorem 1 means that the error functions $\varepsilon_j(t) \to 0$ as $t \to \infty$ if the weighted characteristic functions $F_j(t)Ch_k(t)/F_0(t)$ are absolutely integrable in (t_0, ∞).

In Theorem 1 the weighted characteristic functions are the entries of the perturbation matrix from Levinson's Theorem (see Remark 18 in Section 3).

Note that the error functions $\varepsilon_j(t)$ in representation (11) may be estimated via the characteristic function (see, e.g., Theorem 2.2 in [6]).

Theorem 1 may be used also for the applications mentioned in [5], the stability theory, and the Dirac equation with complex coefficients (see, e.g., [6–8]).

We will say that (1) has the asymptotic solutions $e^{\int_{t_0}^{t} f_j(s)ds}$ corresponding to the phase functions $f_j(t) \in C^{n-1}(t_0, \infty)$, $j = 1, 2, \ldots, n$ if (7)–(9) are satisfied.

Theorem 2. *The asymptotic solution of* (1) *(corresponding to the phase function $f_k(t)$) generates the oscillatory solution of* (1) *if there exists another asymptotic solution with the phase $f_j(t)$ such that*

$$\Re\left[f_k(t) - f_j(t)\right] \equiv 0, \tag{13}$$

and one of the following conditions is satisfied:

$$\Im\left[f_k(t) + f_j(t)\right] \equiv 0, \qquad \int_{t_0}^{\infty} \Im\left[f_k(t)\right] dt = \infty \quad \text{or} \quad \int_{t_0}^{\infty} \Im\left[f_k(t)\right] dt = -\infty, \tag{14}$$

$$\int_{t_0}^{\infty} \Im\left[f_k(t) - f_j(t)\right] dt = \infty \quad \text{or} \quad \int_{t_0}^{\infty} \Im\left[f_k(t) - f_j(t)\right] dt = -\infty. \tag{15}$$

The interesting question is to obtain the converse of Theorem 2.

Conditions (13) and (14) mean that the phase functions $f_k(t), f_j(t)$ are complex congugate. For the equations with real coefficients the complex phase functions appear in complex conjugate pairs. But for the equations with complex coefficients this is not true.

Since every solution of (1) with the constant complex coefficients (without repeated characteristic roots) is the linear combination of the exponents, in view of Theorem 2, the following questions arise.

Question 1. Describe the complex numbers $z_1, z_2, \ldots, z_n$ such that

$$e^{z_1} + e^{z_2} + \cdots + e^{z_n} = 0. \tag{16}$$

Question 2. For which complex constant coefficients a_j, $j = 0, 1, \ldots, n$ a polynomial $\sum_{j=0}^{n} a_j z^j$, $n \ge 2$ has at least one pair of complex conjugate zeros with nonzero imaginary parts?

Question 1 is complicated in the multidimensional case (see [9]), but by the substitution $e^{y_{n-1}} = e^{z_{n-1}} + e^{z_n}$ one can reduce n dimensional equation to the two equations: $(n-1)$-dimensional $e^{z_1} + e^{z_2} + \cdots + e^{z_{n-2}} + e^{y_{n-1}} = 0$ and 3-dimensional $e^{y_{n-1}} = e^{z_{n-1}} + e^{z_n}$.

For the two dimensional case there is a simple answer to Question 1:

$$e^{z_1} + e^{z_2} = 0 \tag{17}$$

if and only if

$$\begin{gathered}(A)\,\Re\left[z_1 - z_1\right] = 0,\\ \Im\left[z_1 - z_2\right] = (1+2k)\pi,\\ k = 0, \pm 1, \pm 2, \ldots.\end{gathered} \tag{18}$$

One can answer Question 2 for quadratic, cubic, and quartic equations with the complex coefficients since the simple formulas for solutions are available for these cases (see Remark 14, Theorem 15 below).

Theorem 2 shows that the oscillations of the solutions could be produced not only by the complex conjugate phase functions, which one can see from the following example.

Example 3. The equation

$$u''''(t) + \frac{3+4i}{2t^2}u''(t) - \frac{3+4i}{t^3}u'(t) + \frac{45+40i}{16t^4}u(t) = 0 \tag{19}$$

is oscillatory, but there is no complex conjugate pair among its phase functions with nonzero imaginary parts:

$$\begin{gathered}f_1(t) = \frac{1+2i}{2t}, \qquad f_2(t) = \frac{5-2i}{2t},\\ f_3(t) = \frac{1}{2t}, \qquad f_4(t) = \frac{5}{2t}.\end{gathered} \tag{20}$$

Note that $u_j(t) = e^{\int_{t_0}^{t} f_j(s)ds}$, $j = 1, 2, 3, 4$ are exact solutions of (19), $u_1(t) - u_3(t)$ is an oscillatory solution of (19), and (13), (15) are satisfied.

Example 4. If $r(t)$, $\varphi(t) \in [0, \pi)$ are a real-valued $C^2(t_0, \infty)$ functions such that

$$\left(r^{-1/2}(t)\,e^{-\varphi(t)/2}\right)'' r^{-1/2}(t)\,e^{-\varphi(t)/2} \in L_1(t_0, \infty), \tag{21}$$

then the equation

$$u''(t) + r^2(t)\,e^{2i\varphi(t)}u(t) = 0 \tag{22}$$

is oscillatory if and only if

$$\varphi(t) \equiv 0, \qquad \int_{t_0}^{\infty} r(s)\,ds = \infty, \qquad t > t_0. \tag{23}$$

The following theorem is inspired by the asymptotic theorems for two-term differential equation from [10, 11]. We deduce it from Theorem 1 by choosing the phase function $f_k(t)$ as an approximate solution of the characteristic equation $Ch_k(t) = 0$ (see [10]):

$$f_k(t) = b_k a(t) - \frac{(n-1)\,a'(t)}{2a(t)}, \qquad k = 1, 2, \ldots, n. \tag{24}$$

Note that Theorem 1 may be used to obtain other oscillation theorems by taking different approximate solutions of the characteristic equation (see [10, 12]).

Theorem 5. *Assume there exists a complex-valued function $a(t) \in C^{n-1}(t_0, \infty)$, $a(t) \neq 0$, and complex numbers b_k, $k = 1, 2, \ldots, n$ such that*

$$b_1 + b_2 + \cdots + b_n = 0 \tag{25}$$

$$\Re\left[\left(b_k - b_j\right)a(t)\right] \text{ do not change a sign on } (t_0, \infty), \quad 1 \le k < j \le n, \tag{26}$$

$$\begin{gathered}\int_{t_0}^{\infty}\left|a^{(1-n)/2}(t)\,e^{-b_k\int_{t_0}^{t}a(s)ds} \times L\left(a^{(1-n)/2}(t)\,e^{b_k\int_{t_0}^{t}a(s)ds}\right)\right| dt < \infty,\\ k = 1, \ldots, n,\end{gathered} \tag{27}$$

the set $\{b_k a(t)\}_{k=1}^{n}$ contains a set $S = \{b_k a(t)\}_{k=1}^{2r}$ of r pairs of complex conjugate functions, and

$$\int_{t_0}^{\infty} \Im\left[b_k a(t)\right] dt = \pm\infty, \quad k = 1, \ldots, 2m, \quad m \le r, \tag{28}$$

$$\left|\int_{t_0}^{\infty} \Im\left[b_k a(t)\right] dt\right| < \infty, \quad k = 2m+1, m+2, \ldots, 2r. \tag{29}$$

Then (1) with $a_{n-1}(t) \equiv 0$ has $2m$ oscillatory and $(n-2m)$ nonoscillatory linearly independent solutions.

By taking $a(t) = 1/t$ from Theorem 5 we get the following corollary.

Corollary 6. *Assume that for some complex numbers $b_1, b_2, \ldots, b_n$, the set $\{b_j\}_{j=1}^{n}$ contains a set $S = \{b_j\}_{k=1}^{2r}$ of r pairs of complex conjugate numbers, and conditions (25), (26),*

$$\begin{gathered}\Im\left[b_j\right] \neq 0, \quad j = 1, \ldots, 2m,\\ \Im\left[b_j\right] \equiv 0, \quad j = 2m+1, \ldots, 2r,\end{gathered} \tag{30}$$

$$a_0(t)\,t^{n-1} + \sum_{j=1}^{n} a_j(t)\,t^{n-1-j}\prod_{p=1}^{j}\left(b_k + \frac{n+1-2p}{2}\right) \in L_1(t_0, \infty) \tag{31}$$

are true for all $k = 1, 2, \ldots, n$. Then (1) with $a_{n-1}(t) \equiv 0$ has $2m$ oscillatory and $(n-2m)$ nonoscillatory linearly independent solutions.

From Corollary 6 follows the well-known result.

Corollary 7 (see [1]). *Assume that conditions*

$$\int_{t_0}^{\infty} t^{n-1-j}\left|a_j(t)\right| dt < \infty, \quad j = 0, 1, 2, \ldots, n-2 \tag{32}$$

are satisfied. Then (1) with $a_{n-1}(t) \equiv 0$ is nonoscillatory.

For the fourth order equation we deduce the following result.

Corollary 8. *Assume that for some real number $\varepsilon > 0$ and for the fixed number j from the set $\{0, 1, 2\}$*

$$t^{3-j}\left|a_j(t) + \frac{9j^2 - 15j + 18 - 2^{j+1}\varepsilon^2}{32t^{4-j}}\right| \in L_1(t_0, \infty), \qquad t^{3-k}a_k \in L_1(t_0, \infty), \quad k = 0, 1, 2, \ k \neq j. \tag{33}$$

Then equation

$$u''''(t) + a_2(t)u''(t) + a_1(t)u'(t) + a_0(t)u(t) = 0 \tag{34}$$

is nonoscillatory.

Remark 9. If $j = 0$ conditions (33) turn to

$$\int_{t_0}^{\infty} t^3\left|a_0(t) + \frac{9-\varepsilon^2}{16t^4}\right| dt < \infty, \qquad \int_{t_0}^{\infty}\left|t^{3-k}a_k(t)\right| dt < \infty, \quad k = 1, 2. \tag{35}$$

Remark 10. To compare condition (35) with the well-known real coefficient case, note that (34) with $a_1(t) = a_2(t) \equiv 0$ and real valued $a_0(t)$ is nonoscillatory if (see [1])

$$\int_t^{\infty} s^2\left(a_0(s) + \frac{9}{16s^4}\right) ds = \int_t^{\infty} s^2 a_0(s)\, ds + \frac{9}{16t} > 0. \tag{36}$$

By taking $\varepsilon = i\xi$, $j = 0$ one can get another result.

Corollary 11. *Assume that for some real number ξ conditions*

$$\int_{t_0}^{\infty} t^{3-k}\left|a_k(t)\right| dt < \infty, \qquad \int_{t_0}^{\infty} t^3\left|a_0(t) + \frac{9+\xi^2}{16t^4}\right| dt < \infty, \quad k = 1, 2, \tag{37}$$

are satisfied. Then (34) has 2 oscillatory and 2 nonoscillatory (linearly independent) solutions.

For two terms equation:

$$u^{(n)}(t) + a_0(t)u(t) = 0, \qquad a_0(t) = \left|a_0(t)\right| e^{i\varphi}, \qquad n \geq 2, \quad \varphi = 0 \quad \text{or} \quad \varphi = \pi, \tag{38}$$

with real valued coefficient $a_0(t)$ by choosing

$$a(t) = \left|a_0\right|^{1/n}, \qquad b_j = e^{i\varphi/n}e^{i\pi(2j+1)/n}, \quad j = 1, \ldots, n, \tag{39}$$

from Theorem 5 one can deduce the following theorem.

Theorem 12. *Assume $a(t) = |a_0|^{1/n} \in C^{n-1}(t_0, \infty)$, and for all $k = 1, \ldots, n$*

$$\int_{t_0}^{\infty}\left|a^{(1-n)/2}(t)\, e^{-b_k\int_{t_0}^t a(s)ds}\left(\frac{d^n}{dt^n} + a_0(t)\right) \times \left(a^{(1-n)/2}(t)\, e^{b_k\int_{t_0}^t a(s)ds}\right)\right| dt < \infty, \tag{40}$$

$$\int_{t_0}^{\infty}\left|a_0(t)\right|^{1/n} dt = \infty. \tag{41}$$

Then (38) has $2m = 2[(n+\varphi/\pi)/2] - (2\varphi/\pi)$ oscillatory and $n - 2m$ nonoscillatory linearly independent solutions. Here $[\cdot]$ is the integral part of a real number.

Remark 13. If $a_0(t) = t^\gamma$, then conditions (40), (41) are simplified to

$$\gamma > -\frac{2}{n-1}. \tag{42}$$

Remark 14. For the quadratic equations with complex coefficients $z^2 + a_1 z + a_0 = 0$, it is easy to see that it has one pair of complex conjugate zeros with nonzero imaginary parts if and only if the coefficients a_0, a_1 are real, and $a_1^2 - 4a_0 < 0$.

Theorem 15. *The cubic equation with complex constant coefficients*

$$P(z) = z^3 + a_2 z^2 + a_1 z + a_0 = 0 \tag{43}$$

has a pair of complex conjugate solutions with nonzero imaginary parts if and only if one of the following conditions is true:

$$\overline{a_0} = a_0, \qquad \overline{a_1} = a_1, \qquad \overline{a_2} = a_2, \qquad D_3(t) > 0 \tag{44}$$

or

$$\overline{a_2} \neq a_2, \qquad D_2(t) < 0, \qquad P(z_1) = 0, \tag{45}$$

where

$$D_3 = -108\left(\frac{a_1}{3} - \frac{a_2^2}{9}\right)^3 - 108\left(\frac{a_0}{2} - \frac{a_1a_2}{6} + \frac{a_2^3}{27}\right)^2 = a_1^2a_2^2 - 4a_1^3 - 4a_2^3a_0 - 27a_0^2 + 18a_0a_1a_2, \tag{46}$$

$$D_2 = \left(\frac{a_1 - \overline{a_1}}{2(a_2 - \overline{a_2})}\right)^2 - \frac{\overline{a_0} - a_0}{\overline{a_2} - a_2}, \tag{47}$$

are the discriminants of (43) and $P(z_1) - \overline{P}(z_1) = 0$ correspondingly,

$$z_1 = \sqrt{D_2} - \frac{a_1 - \overline{a_1}}{2(a_2 - \overline{a_2})}. \tag{48}$$

Example 16. For the equation

$$z^3 - (5+4i)z^2 + (11+8i)z - 15 - 20i = 0 \tag{49}$$

we have

$$a_0 - \overline{a_0} = -40i, \qquad a_1 - \overline{a_1} = 16i, \qquad a_2 - \overline{a_2} = -8i \neq 0,$$
$$D_2 = \left(\frac{a_1 - \overline{a_1}}{2(a_2 - \overline{a_2})}\right)^2 - \frac{\overline{a_0} - a_0}{\overline{a_2} - a_2} = 1 - 5 < 0,$$
$$z_1 = \sqrt{1-5} + 1 = 2i + 1, \qquad P(z_1) = P(\overline{z_1}) = 0 \tag{50}$$

and condition (45) is satisfied.

3. Proofs

We are going to use Levinson's asymptotic theorem as it appears in [5].

Theorem 17 (see [5]). *Let $\Lambda(t) = \operatorname{diag}(\lambda_1(t), \dots, \lambda_n(t))$ be an $n \times n$ diagonal matrix-function which satisfies dichotomy condition.*

For each pair of integers i and j in $[1, n]$ $(i \neq j)$ exist constants K_1, K_2 such that for all x and t, $t_0 \leq t \leq x < \infty$

$$\int_t^x \Re\left[\lambda_i(s) - \lambda_j(s)\right] ds \leq K_1,$$
$$\text{or} \quad \int_t^x \Re\left[\lambda_i(s) - \lambda_j(s)\right] ds \geq K_2. \tag{51}$$

Let the $n \times n$ matrix $P(t)$ satisfy $P(t) \in L_1(t_0, \infty)$ or

$$\int_t^x |P(t)|\, ds < \infty, \tag{52}$$

by which one means that each entry in $P(t)$ has an absolutely convergent infinite integral. Then the system

$$Y'(t) = (\Lambda(t) + P(t))\, Y(t) \tag{53}$$

has a vector solution $Y(t)$ with the asymptotic form

$$Y(t) = (E + \varepsilon(t))\, e^{\int_{t_0}^t \Lambda(s) ds} C, \qquad \lim_{t \to \infty} \varepsilon(t) = 0, \tag{54}$$

where E is the identity matrix, $\varepsilon(t)$ is the $n \times n$ error matrix-function, and $C = (C_1, \dots C_n)^{tr}$ is a constant column-vector.

Proof of Theorem 1. Rewrite (1) as a system

$$y'(t) = A(t)\, y(t), \tag{55}$$

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & 0 & \cdots & 1 \\ -a_0(t) & -a_1(t) & -a_2(t) & -a_3 & \cdots & -a_{n-1}(t) \end{pmatrix},$$
$$y(t) = \begin{pmatrix} u(t) \\ u'(t) \\ \vdots \\ u^{(n-1)} \end{pmatrix}. \tag{56}$$

By transformation

$$y(t) = \Phi(t)\, z(t), \tag{57}$$

where matrix-function $\Phi(t)$ is defined via phase functions $f_j(t)$:

$$\Phi(t) = F(t)\, e^{\int_{t_0}^t f_1(s) ds}, \tag{58}$$

and the entries of the matrix $F(t)$ are defined in (4), (6), we get

$$z'(t) = \Phi^{-1}(t) \left(A(t)\, \Phi(t) - \Phi'(t)\right) z(t), \tag{59}$$

or, in view of identity,

$$A(t)\, \Phi(t) - \Phi'(t) = \Phi(t) \left(\Lambda(t) + \frac{B(t)}{F_0(t)}\right), \tag{60}$$

we have

$$z'(t) = \left(\Lambda(t) + \frac{B(t)}{F_0(t)}\right) z(t), \tag{61}$$

where

$$\Lambda(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & f_2(t) - f_1(t) & 0 & 0 \\ 0 & 0 & f_3(t) - f_1(t) & 0 \\ 0 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & f_n(t) - f_1(t) \end{pmatrix}, \tag{62}$$

$$B(t) = \begin{pmatrix} -F_1 Ch_1 & -F_1 Ch_2 & \cdots & -F_1 Ch_n \\ F_2 Ch_1 & F_2 Ch_2 & \cdots & F_2 Ch_n \\ -F_3 Ch_1 & -F_3 Ch_2 & \cdots & -F_3 Ch_n \\ \cdots & \cdots & \cdots & \cdots \\ (-1)^n F_n Ch_1 & (-1)^n F_n Ch_2 & \cdots & (-1)^n F_n Ch_n \end{pmatrix}. \tag{63}$$

Here and further we often suppress the time variable t for the simplicity. □

Remark 18. Remarkable formulas (61)–(63) show that any nth order (1) may be transformed to the first order system $z'(t) = A_1(t) z(t)$, where the matrix $A_1(t)$ is a sum of diagonal matrix $\Lambda(t)$ and perturbation matrix $P(t)$; that is, $A_1(t) = \Lambda(t) + P(t)$ (Levinson form). Moreover, if the diagonal matrix is chosen in terms of phase functions as in (62), then the perturbation matrix $P(t) = B(t)/F_0(t)$ is the weighted characteristic function (i.e., the entries of matrix B are proportional to the characteristic functions $Ch_j(t)$). Our conjecture is that it is true not only for (1), but also for any first order system. For planar systems it was proved in [7].

To prove basic formula (60) note that by using Laplace expansion by the minors of the determinants:

$$F_0 = \det\begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{pmatrix},$$
$$0 = \det\begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ f_{k1} & f_{k2} & \cdots & f_{kn} \end{pmatrix}, \quad k = 1, \dots, n-1, \tag{64}$$

we get identities:

$$F_1 f_{n1} - F_2 f_{n2} + \cdots + (-1)^{n-1} F_n f_{nn} = F_0(t),$$
$$F_1 f_{k1} - F_2 f_{k2} + \cdots + (-1)^{n-1} F_n f_{kn} = 0, \quad k = 1, \dots, n-1, \tag{65}$$

or $(f_{1j} \equiv 1,\ f_{2j} = f_j)$

$$F_1 - F_2 + F_3 + \cdots + (-1)^{n-1} F_n = 0,$$
$$f_1 F_1 - f_2 F_2 + \cdots + (-1)^{n-1} f_n F_n = 0,$$
$$\left(f_1^2 + f_1'\right) F_1 - \left(f_2^2 + f_2'\right) F_2 + \left(f_3^2 + f_3'\right) F_3 + (-1)^{n-1}\left(f_n^2 + f_n'\right) F_n = 0. \tag{66}$$

By direct calculations we get

$$\left(A\Phi - \Phi'\right)_{11} = e^{\int^t f_1 ds}\left(f_{21} - f_1\right) = 0,$$
$$\left(\Phi\left(\Lambda + \frac{B}{F_0}\right)\right)_{11} = \frac{\left(-F_1 + F_2 - F_3 + \cdots + (-1)^n F_n\right) Ch_1}{F_0} = 0,$$
$$\left(A\Phi - \Phi'\right)_{21} = \left(f_2 - f_1\right) e^{\int^t f_1 ds}$$
$$\left(\Phi\left(\Lambda + \frac{B}{F_0}\right)\right)_{21} = \left(f_2 - f_1\right) e^{\int^t f_1 ds} + \frac{e^{\int^t f_1 ds}\left(-F_1 + F_2 + \cdots + (-1)^n F_n\right) Ch_2}{F_0} = \left(f_2 - f_1\right) e^{\int^t f_1 ds},$$
$$\left(A\Phi - \Phi'\right)_{n1} = \left(-a_0 - a_1 f_1 - a_2 f_{31} - \cdots - a_{n-1} f_{n1} - f_{n1} - f_{n1}' f_1\right) e^{\int^t f_1 ds} = -e^{\int^t f_1 ds} Ch_1,$$
$$\left(\Phi\left(\Lambda + \frac{B}{F_0}\right)\right)_{n1} = \frac{e^{\int^t f_1 ds}\left(-f_{n1} F_1 + f_{n1} F_2 + \cdots + (-1)^n f_{nn} F_n\right) Ch_1}{F_0} = -e^{\int^t f_1 ds} Ch_1, \tag{67}$$

and the same way we get

$$\left(A\Phi - \Phi'\right)_{jk} = \left(\Phi\left(\Lambda + \frac{B}{F_0}\right)\right)_{jk}, \quad j, k = 1, 2, \dots, n. \tag{68}$$

To apply Theorem 17 to system (61) note that from (7) follows dichotomy condition (51) of Theorem 17. Condition (52) of Theorem 17 turns to $B(t)/F_0(t) \in L_1(t_0, \infty)$:

$$\frac{F_j Ch_k(t)}{F_0(t)} \in L_1\left(t_0, \infty\right), \quad k, j = 1, 2, 3, \dots, n, \tag{69}$$

or (8) in view of $-F_1 + F_2 - \cdots + (-1)^n F_n = 0$.

From Theorem 17 applied to system (61) we get

$$z(t) = z_0(t)(E + \varepsilon(t)) C, \qquad \lim_{t \to \infty} \varepsilon(t) = 0,$$
$$z_0(t) = e^{\int_{t_0}^t \Lambda(s) ds} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\int_{t_0}^t (f_2 - f_1) ds} & 0 & 0 \\ 0 & 0 & e^{\int_{t_0}^t (f_3 - f_1) ds} & 0 \\ 0 & 0 & 0 & e^{\int_{t_0}^t (f_4 - f_1) ds} \end{pmatrix}. \tag{70}$$

From (57):

$$y(t) = \Phi(t) z(t) = \Phi(t) z_0(t)(E + \varepsilon(t)) C, \tag{71}$$

we get representation (11). If functions $\{f_j(t)\}_{j=1}^n$ are exact solutions of $Ch(f_j) = 0$, $j = 1, 2, \dots n$, then $F(t) e^{\int_{t_0}^t f_1 ds} e^{\int_{t_0}^t \Lambda ds}$ is the fundamental solution of (55). Abel's identity

$$C = \det(F(t)) e^{\int_{t_0}^t f_1 ds} e^{\int_{t_0}^t \Lambda ds} e^{-\int_{t_0}^t \operatorname{Tr}(A(s)) ds} = F_0(t) e^{\int_{t_0}^t (f_1 + f_2 + \cdots + f_n + a_{n-1})(s) ds} \tag{72}$$

may be written in the form (9). We always choose (approximate) phase functions $f_j(t)$, $j = 1, 2, \dots, n$ such that (9) is satisfied.

Proof of Theorem 2. From the assumptions (13), (14) of Theorem 2 one can generate an asymptotic solution as the difference of two asymptotic solutions:

$$u_0(t) = e^{\int_{t_0}^t f_k(s) ds} - e^{\int_{t_0}^t f_j(s) ds} = 2i e^{\int_{t_0}^t \Re[f_k(s)] ds} \sin\left(\int_{t_0}^t \Im\left[f_k(s)\right] ds\right), \tag{73}$$

which is oscillatory. Indeed if $\int_{t_0}^{t} \Im[f_k(s)]ds = \infty$, or $\int_{t_0}^{t} \Im[f_k(s)]ds = -\infty$, then the infinite sequence of zeros t_p of $u_0(t)$ is given by

$$\int_{t_0}^{t_p} \Im\left[f_k(s)\right] ds = p\pi, \qquad p = 1, 2, 3, \ldots \quad \text{or} \quad p = -1, -2, -3, \ldots. \tag{74}$$

From the assumptions (13), (15) for the solution $u_0(t)$ one can generate another infinite sequence of zeros t_m that is given by

$$\int_{t_0}^{t_m} \Im\left[f_k(s) - f_j(s)\right] ds = 2m\pi, \qquad m = 1, 2, 3, \ldots, \quad \text{or} \quad m = -1, -2, -3, \ldots. \tag{75}$$

Indeed we have the following:

$$u_0(t_m) = e^{\int_{t_0}^{t_m} f_j(s)ds}\left(e^{\int_{t_0}^{t} \Re[f_k(s)-f_j(s)]ds} e^{i\int_{t_0}^{t} \Im[f_k(s)-f_j(s)]ds} - 1\right) = 0. \tag{76}$$

Theorem 2 is followed from representation (11). □

Proof of Example 4. Asymptotic solutions of (22) we choose in the form

$$u_j(t) = e^{\int_{t_0}^{t} f_j(s)ds}, \qquad f_j = x_j - \frac{x_j'(t)}{2x_j(t)}, \qquad x_1(t) = ir(t)e^{i\varphi(t)}, \qquad x_2(t) = -x_1(t) \tag{77}$$

or

$$u_j(t) = Cx_j^{-1/2}(t)e^{\int_{t_0}^{t} x_j(s)ds}, \qquad j = 1, 2. \tag{78}$$

From (4) we have the following:

$$\begin{gathered} F(t) = \begin{pmatrix} 1 & 1 \\ f_1(t) & f_2(t) \end{pmatrix}, \\ F_0(t) = \det(F) = f_2(t) - f_1(t) = -2x_1(t), \\ F_1 = F_2 = 1, \\ Ch_j = f_j'(t) + f^2(t) - x_1^2(t) = \left(x_1^{-1/2}\right)''(t)\, x_1^{1/2}(t). \end{gathered} \tag{79}$$

So if (8) is satisfied, that is,

$$\left(x_1^{-1/2}\right)''(t)\, x_1^{1/2}(t) \in L_1(t_0, \infty), \tag{80}$$

then solutions of $u''(t) + r^2(t)e^{2i\varphi(t)}u(t)$ may be written in the form following:

$$u(t) = C_1(1 + \varepsilon_1(t))u_1(t) + C_2(1 + \varepsilon_2(t))u_2(t), \qquad \lim_{t\to\infty} \varepsilon_j(t) = 0, \qquad j = 1, 2. \tag{81}$$

If $u(t)$ has a zero for some complex numbers $C_{1,2}$, $|C_1| + |C_2| \neq 0$, then

$$x_1^{-1/2}(t)\left(C_1 e^{\int_{t_0}^{t} x_1(s)ds} + C_2 e^{\int_{t_0}^{t} x_2(s)ds}\right) = 0, \tag{82}$$

or

$$C_1 e^{iy(t)} + C_2 e^{-iy(t)} = 0, \qquad y(t) = \int_{t_0}^{t} r(s)\left[\cos(\varphi(s)) + i\sin(\varphi(s))\right] ds, \tag{83}$$

or

$$-\frac{C_2}{C_1} = e^{2iy(t)}. \tag{84}$$

By taking the derivative by t of both sides we get

$$0 = 2iy'(t)e^{2iy(t)} \tag{85}$$

and since

$$y'(t) = r(t)\left[\cos(\varphi(t)) + i\sin(\varphi(t))\right] \neq 0, \tag{86}$$

we get

$$0 = e^{2iy(t)} = e^{2i\int_{t_0}^{t} r(s)\cos(\varphi(s))ds} e^{-2\int_{t_0}^{t} r\sin(\varphi(s))ds}, \tag{87}$$

or

$$0 = e^{-2\int_{t_0}^{t} r\sin(\varphi(s))ds}. \tag{88}$$

By taking the derivative by t of the both sides we get

$$0 = -2r(t)\sin(\varphi(t))e^{-2\int_{t_0}^{t} r\sin(\varphi(s))ds} \tag{89}$$

or

$$\varphi(t) \equiv 0. \tag{90}$$

So condition (8) turns to (21). □

Proof of Theorem 5. We deduce Theorem 5 from Theorem 1 by choosing $f_j(t)$ as in (24), (25). By calculations we get

$$\begin{aligned} F_0(t) &= \prod_{1\le k<m\le n} (b_k - b_m)\, a^{1+2+\cdots+n-1} \\ &= \prod_{1\le k<m\le n} (b_k - b_m)\, a^{(n-1)n/2}. \end{aligned} \tag{91}$$

Note that (9) is satisfied in view of $b_1 + \cdots + b_n = 0$:

$$\begin{aligned} f_1 + f_2 + \cdots + f_n + \frac{F_0'}{F_0} &= \left(b_1 + \cdots + b_n - \frac{n(n-1)a'}{2a^2}\right)a(t) \\ &\quad + \frac{n(n-1)a'}{2a} = 0. \end{aligned} \tag{92}$$

Further we have the following:

$$F_j = \prod_{k<m<n;k,m\neq j} (b_k - b_m)\, a^{1+2+\cdots+n-2} = \prod_{k<m<n;k,m\neq j} (b_k - b_m)\, a^{(n-1)(n-2)/2}, \tag{93}$$

and condition (8), in view of Definition (2), turns to (27):

$$\begin{aligned}\frac{F_j(t)\, Ch_k(t)}{F_0(t)} &= C_j Ch_k(t)\, a^{1-n}(t) \\ &= C_j a^{1-n} a^{(n-1)/2} e^{-b_k \int_{t_0}^{t} a(s)ds} \\ &\quad \times L\left(a^{(1-n)/2} e^{b_k \int_{t_0}^{t} a(s)ds}\right), \\ C_j &= \frac{\prod_{k<m<n;k,m\neq j}(b_k - b_m)}{\prod_{1\leq k<m\leq n}(b_k - b_m)}.\end{aligned} \tag{94}$$

Further (7) turns to (26). So under conditions of Theorem 5 conditions (7)–(9) are satisfied and Theorem 1 is applicable. Further from (28) we get (13), (14) is satisfied for $k = 1, 2, \ldots, 2m$, and Theorem 5 is followed from Theorem 1 and Theorem 2. □

Proof of Corollary 6. Corollary 6 is followed from Theorem 5 by taking $a(t) = 1/t$. From (27) we have

$$t^{(n-1)/2-b_k} \sum_{j=0}^{n} a_j(t) \left(t^{b_k+(n-1)/2}\right)^{(j)} \in L_1(t_0, \infty), \qquad k = 1, \ldots, n, \tag{95}$$

or, since $(t^{\beta})^{(j)} = \beta(\beta-1)\cdots(\beta-j+1)t^{\beta-j}$, $j = 1, 2, \ldots, n$, we get

$$\begin{aligned}&\sum_{j=1}^{n} a_j(t)\, t^{n-1-j} \left(b_k + \frac{n-1}{2}\right) \\ &\quad \times \left(b_k + \frac{n-3}{2}\right) \cdots \left(b_k + \frac{n+1-2j}{2}\right) \\ &\quad + a_0(t)\, t^{n-1} \in L_1(t_0, \infty),\end{aligned} \tag{96}$$

or condition (27) turns to (31). Further conditions (28),(29) turn to (30). □

Proof of Corollary 7. Corollary 7 is followed from Corollary 6 by taking $b_k = k - 1 - ((n-1)/2)$, $k = 1, 2, \ldots, n$, $m = 0$. □

Proof of Corollary 8. We deduce Corollary 8 from Corollary 6.

In the case $n = 4$, $a_3 \equiv 0$ condition (29) turns to

$$\begin{aligned}&t^{-1}\left(b_k + \frac{3}{2}\right)\left(b_k + \frac{1}{2}\right)\left(b_k - \frac{1}{2}\right)\left(b_k - \frac{3}{2}\right) \\ &\quad + t a_2 \left(b_k + \frac{3}{2}\right)\left(b_k + \frac{1}{2}\right) \\ &\quad + t^2 a_1 \left(b_k + \frac{3}{2}\right) \\ &\quad + t^3 a_0 \in L_1(t_0, \infty), \quad k = 1, 2, 3, 4.\end{aligned} \tag{97}$$

Considering the case $j = 0$ of Corollary 8 assume $t^{3-k} a_k \in L_1(t_0, \infty)$, $k = 1, 2$. Then condition (97) turns to

$$t^3 \left(a_0 + \frac{(b_k + 3/2)(b_k + 1/2)(b_k - 1/2)(b_k - 3/2)}{t^4}\right) \in L_1(t_0, \infty). \tag{98}$$

For any real ε by choosing

$$\begin{aligned}b_1 &= \frac{1}{2}\sqrt{5 - \sqrt{25 - \varepsilon^2}}, \qquad b_2 = -b_1, \\ b_3 &= \frac{1}{2}\sqrt{5 + \sqrt{25 - \varepsilon^2}}, \qquad b_4 = -b_3,\end{aligned} \tag{99}$$

we get $(b_k^2 - 9/4)(b_k^2 - 1/4) = (9 - \varepsilon^2)/16$ and (97) turns to (33) with $j = 0$.

In the case $j = 1$, assuming $t^{3-k} a_k \in L_1(t_0, \infty)$, $k = 0, 2$ (97) turns to

$$t^2 (b_k + 3/2)\left(a_1 + \frac{\left(b_k^2 - 1/4\right)(b_k - 3/2)}{t^3}\right) \in L_1(t_0, \infty). \tag{100}$$

By choosing $b_4 = -3/2$, and b_1, b_2, and b_3 as real solutions of

$$\left(b_k^2 - \frac{1}{4}\right)\left(b_k - \frac{3}{2}\right) = \frac{3 - \varepsilon^2}{8}, \qquad b_1 + b_2 + b_3 = \frac{3}{2} \tag{101}$$

with the negative discriminant (see (46)), we get conditions (33) with $j = 1$:

$$t^2 \left(a_1 + \frac{3 - \varepsilon^2}{8t^3}\right), \quad t^{3-k} a_k \in L_1(t_0, \infty), \quad k = 0, 2. \tag{102}$$

In the case $j = 2$, assume $t^{3-k} a_k \in L_1(t_0, \infty), k = 0, 1$. Condition (97) turns to

$$t (b_k + 3/2)(b_k + 1/2)\left(a_2 + \frac{(b_k - 3/2)(b_k - 1/2)}{t^2}\right) \in L_1(t_0, \infty). \tag{103}$$

By choosing $b_3 = -3/2$, $b_4 = -1/2$, and $b_{1,2}$ as real solutions of

$$\left(b_k - \frac{3}{2}\right)\left(b_k - \frac{1}{2}\right) = \frac{3 - \varepsilon^2}{4}, \qquad b_1 + b_2 + b_3 + b_4 = 0, \tag{104}$$

or

$$b_1 = 1 - \sqrt{\frac{1-\varepsilon^2}{4}}, \qquad b_2 = 1 + \sqrt{\frac{1-\varepsilon^2}{4}}, \tag{105}$$

we get nonoscillation conditions (33) with $j = 2$:

$$t\left(a_2 + \frac{3-\varepsilon^2}{8t^2}\right), \quad t^{3-k}a_k \in L_1(t_0, \infty), \quad k = 0, 1. \tag{106}$$

□

Proof of Corollary 11. Proof of Corollary 11 is similar to proof of Corollary 8 in case $j = 0$ with the choice $b_1 = (1/2)\sqrt{5 - \sqrt{25 + \xi^2}}$, $b_2 = -b_1$, $b_3 = (1/2)\sqrt{5 + \sqrt{25 + \xi^2}}$, and $b_4 = -b_3$. Note since $\text{Im}[b_1] > 0$, $\text{Im}[b_2] < 0$, and $\text{Im}[b_3] = \text{Im}[b_4] = 0$ we get $2m = 2$. □

Proof of Theorem 12. We deduce Theorem 12 from Theorem 5.

From the choice (39) we get

$$b_j = e^{i\varphi/n}e^{i\pi(2j+1)/n}, \quad (-a_0(t))^{1/n} = b_j a(t), \quad j = 1, \dots, n. \tag{107}$$

It is easy to see that condition (25) of Theorem 5 is satisfied. Condition (27) turns to (40). To check condition (26) consider

$$\begin{aligned}\Re\left[b_k - b_j\right] &= \cos\left(\frac{\varphi+\pi}{n} + \frac{2\pi k}{n}\right) - \cos\left(\frac{\varphi+\pi}{n} + \frac{2\pi j}{n}\right) \\ &= 2\sin\left(\frac{\varphi + \pi(1+j+k)}{n}\right)\sin\left(\frac{\pi(j-k)}{n}\right).\end{aligned} \tag{108}$$

Assuming $1 \le k < j \le n$, we have

$$\sin\left(\frac{\pi(j-k)}{n}\right) > 0, \tag{109}$$

and since $\varphi \equiv 0$ or $\varphi \equiv \pi$, the sign of $\Re[b_k - b_j]$ does not depend on t and (26) is true. Let m be the number of indices $j \in \{1, \dots, n\}$ for which

$$\Im\left[b_j a(t)\right] = |a_0(t)|^{1/n}\sin\left(\frac{\varphi + \pi(2j+1)}{n}\right) \neq 0. \tag{110}$$

Then (28) is true in view of (41).

If for some $j \in \{1, \dots, n\}$

$$\sin\left(\frac{\varphi + \pi(2j+1)}{n}\right) = 0, \tag{111}$$

then a nonoscillatory solution exists. This condition

$$\frac{\alpha + 2j + 1}{n} \in Z, \qquad \alpha = \frac{\varphi}{\pi}, \tag{112}$$

that is, $\alpha = 0, 1$ since $\varphi = 0$ or $\varphi = \pi$.

Consider the case

$$\varphi = 0, \qquad \alpha = 0. \tag{113}$$

If n is odd, then $(2j+1)/n$ is integer if and only if $j = (n-1)/2$ which means that the solution with

$$j = \frac{n-1}{2}, \quad n \ge 2 \tag{114}$$

is nonoscillatory, so other $2m = n - 1 = 2[n/2]$ solutions are oscillatory.

If $\varphi = 0$ and n is even, then and all $2m = n = 2[n/2]$ solutions are oscillatory.

Further in the case

$$\varphi = \pi, \quad \alpha = 1, \tag{115}$$

and n is even we have the solutions with

$$j = \frac{n}{2} - 1, \quad j = n - 1 \tag{116}$$

are nonoscillatory, and the other $n-2$ solutions are oscillatory.

In the case $\varphi = \pi$ and n is odd we have only one nonoscillatory solution with $j = n - 1$ and $2m = n - 1$.

So

$$2m = \begin{cases} 2\left[\dfrac{n}{2}\right], & \varphi \equiv 0, \\ n-2, & \varphi \equiv \pi, \ n \text{ is even}, \\ n-1, & \varphi \equiv \pi, \ n \text{ is odd}, \end{cases} \tag{117}$$

or

$$2m = 2\left[\frac{n + \varphi/\pi}{2}\right] - \frac{2\varphi}{\pi}. \tag{118}$$

□

Proof of Remark 14. If $z^2 + a_1 z + a_0 = 0$ has 1 pair of complex conjugate zeros $z_1, z_2 = \overline{z_1}$, then a_0, a_1 are real numbers since $a_0 = z_1 z_2 = |z_1|^2 \ge 0$ and $a_1 = -z_1 - z_2 = -z_1 - \overline{z_1}$, and quadratic equation $z^2 + a_1 z + a_0 = 0$ has two complex conjugate roots if $a_1^2 - 4a_0 < 0$. Assuming that a_0, a_1 are real numbers and $a_1^2 - 4a_0 < 0$ from quadratic formula the equation $z^2 + a_1 z + a_0 = 0$ has two complex conjugate roots. □

Proof of Theorem 15. Assume there exists a pair of complex conjugate solutions $z_1, \overline{z_1}$ of $P(z) = z^3 + a_2 z^2 + a_1 z + a_0 = 0$; that is,

$$P(z_1) = 0, \qquad P(\overline{z_1}) = 0, \tag{119}$$

then denoting

$$\overline{P}(z) = z^3 + \overline{a_2}z^2 + \overline{a_1}z + \overline{a_0}, \tag{120}$$

we have $\overline{P}(z_1) = 0$ as well, and

$$P(z_1) - \overline{P}(z_1) = (a_2 - \overline{a_2})z_1^2 + (a_1 - \overline{a_1})z_1 + a_0 - \overline{a_0} = 0. \tag{121}$$

Consider 3 cases.

First case: $a_2 = \overline{a_2}$, $a_1 = \overline{a_1}$, $a_0 = \overline{a_0}$, $D_3 > 0$.

In this case $P(z_1) - \overline{P}(z_1) = 0$ is satisfied. The coefficients of $P(z) = 0$ are real and (43) has two complex conjugate solutions since discriminant D_3 of $P(z) = 0$ is positive. We drop the case $a_2 = \overline{a_2}$, $a_1 = \overline{a_1}$, $a_0 = \overline{a_0}$, $D_3 < 0$ since in that case all solutions are real.

Second case: $a_2 \neq \overline{a_2}$.

In this case by solving quadratic equation $P(z_1) - \overline{P}(z_1) = 0$ we get

$$z_1 = -\frac{a_1 - \overline{a_1}}{2(a_2 - \overline{a_2})} + \sqrt{D_2},$$
$$D_2 = \left(\frac{a_1 - \overline{a_1}}{2(a_2 - \overline{a_2})}\right)^2 - \frac{\overline{a_0} - a_0}{\overline{a_2} - a_2}, \quad (122)$$

and $P(z_1) = \overline{P}(z_1) = 0$ implies another solution z_2 can be found by division:

$$P(z) = (z - z_1) Q(z),$$
$$Q(z) = z^2 + (z_1 + a_2) z + a_1 + a_2 z_1 + z_1^2, \quad (123)$$
$$Q(z_2) = 0.$$

Note that if discriminant D_2 of the quadratic equation $P(z_1) - \overline{P}(z_1) = 0$ is negative, then by solving this equation we get two complex conjugate roots with nonzero imaginary parts:

$$z_2 = \overline{z_1} = -\frac{a_1 - \overline{a_1}}{2(a_2 - \overline{a_2})} - \sqrt{D_2}. \quad (124)$$

If

$$D_2 \geq 0, \quad (125)$$

then z_1 is real, and to find another (complex) solution z_3 one should solve

$$Q(z_3) = z_3^2 + (z_1 + a_2) z_3 + a_1 + a_2 z_1 + z_1^2 = 0. \quad (126)$$

Since $\overline{z_3}$ must be a solution as well, we have

$$Q(z_3) - \overline{Q}(z_3) = (a_2 - \overline{a_2})(z_3 + z_1) + a_1 - \overline{a_1} = 0, \quad (127)$$

and z_3 is a real number given by

$$z_3 = -z_1 - \frac{a_1 - \overline{a_1}}{a_2 - \overline{a_2}}. \quad (128)$$

Further since z_1, z_3 are solutions by division

$$Q(z) = (z - z_3) R(z), \qquad R(z) = z + z_1 + z_3 + a_2, \quad (129)$$

and one can find the third solution z_4:

$$z_4 = -z_1 - z_3 - a_2. \quad (130)$$

This case does not work since we have two real solutions z_1, z_3 and one complex solution z_4 of $P(z) = 0$.

Third case: $a_2 = \overline{a_2}$, $a_1 \neq \overline{a_1}$.

In this case assuming z_5, $\overline{z_5}$ are complex conjugate solutions of $P(z) = 0$ from $P(z_5) - \overline{P}(z_5) = (a_1 - \overline{a_1}) z_5 + a_0 - \overline{a_0} = 0$ we get z_5 which is a real number given by

$$z_5 = -\frac{a_0 - \overline{a_0}}{a_1 - \overline{a_1}} \quad (131)$$

and $P(z_5) = 0$.

If z_6 is another solution of $P(z) = 0$, by division we get

$$P(z) = (z - z_5) Q(z),$$
$$Q(z) = z^2 + (z_5 + a_2) z + a_1 + a_2 z_5 + z_5^2, \quad (132)$$
$$Q(z_6) = 0.$$

Since $z_6 = \overline{z_5}$ is also solution of $Q(z) = 0$, we get

$$Q(z_5) - \overline{Q}(z_5) = a_1 - \overline{a_1} = 0, \quad (133)$$

and we come back to the case 1.

To prove that (45) is sufficient condition in view of $P(\overline{z_1}) = P(z_1)$, it is enough to show that

$$\overline{z_1} = -\frac{a_1 - \overline{a_1}}{2(a_2 - \overline{a_2})} - \sqrt{\left(\frac{a_1 - \overline{a_1}}{2(a_2 - \overline{a_2})}\right)^2 - \frac{\overline{a_0} - a_0}{\overline{a_2} - a_2}}, \quad (134)$$

but it is true since $((a_1 - \overline{a_1})/2(a_2 - \overline{a_2}))^2 - (\overline{a_0} - a_0)/(\overline{a_2} - a_2) < 0$.

It is easy to see that (44) is a sufficient condition as well. □

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