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Asymptotic stability and asymptotic solutions of second-order differential equations

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Abstract

We improve, simplify, and extend on quasi-linear case some results on asymptotical stability of ordinary second-order differential equations with complex-valued coefficients obtained in our previous paper [G.R. Hovhannisyan, Asymptotic stability for second-order differential equations with complex coefficients, Electron. J. Differential Equations 2004 (85) (2004) 1–20]. To prove asymptotic stability of second-order differential equations, we establish stability estimates using integral representations of solutions via asymptotic solutions and error estimates. Several examples are discussed.

Keywords: Asymptotic stability; Asymptotic solution; WKB solution; Integral representation; Error estimates

1. Main results

Consider the second-order differential equation

$$\frac{d^2x(t)}{dt^2} + 2\frac{d}{dt}\big(f(t)x(t)\big) + g\big(t, x(t), x'(t)\big)x(t) = 0, \quad t > T > 0, \tag{1.1}$$

where the coefficients $2f(t)$ and $g(t, x(t), x'(t))$ are complex-valued continuous functions of time t.

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The rest state $x(t) = x'(t) = 0$ of (1.1) is called stable if for any $\varepsilon > 0$ there exists $\delta(t_0, \varepsilon) > 0$ such that if $|x(t_0)| + |x'(t_0)| < \delta(t_0, \varepsilon)$ then $|x(t, x(t_0), t_0)| + |x'(t, x(t_0), t_0)| < \varepsilon$ for all $t \geqslant t_0$. The rest state $x(t) = x'(t) = 0$ of (1.1) is called asymptotically stable if it is stable and attractive:

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = 0 \tag{1.2}$$

for every solution of (1.1).

The asymptotic stability for the classical equation (1.1) has been widely studied in [1,2,4,5,7, 8,10–12] by using energy functions and Lyapunov stability and instability theorems. In this paper we use different approach based on integral representations of solutions via different asymptotic solutions (for example WKB solutions) and error estimates technique developed in [3,9,6].

Denote by $L^1(T, \infty)$ the class of Lebesgue integrable in (T, ∞) functions and by $C^p(T, \infty)$ the class of p times differentiable functions on (T, ∞).

From a given function $\mu(t) \in C^2[T, \infty)$ we construct the auxiliary functions

$$\mu_1(t) = \mu(t) - \frac{\mu'(t)}{2\mu(t)} - f(t), \qquad \mu_2(t) = -\mu(t) - \frac{\mu'(t)}{2\mu(t)} - f(t), \tag{1.3}$$

$$k(t) = \frac{\mu'(t)}{2\mu^2(t)}, \qquad I(t) = g\big(t, x(t), x'(t)\big) + f'(t) - f^2(t), \tag{1.4}$$

$$\mathrm{Hov}(t) = \mu(t)k'(t) + \mu^2(t)k^2(t) - \mu^2(t) - I(t). \tag{1.5}$$

Note that $\mu = (\mu_1 - \mu_2)/2$. Also, μ_1 and μ_2 can be used to form the approximate solutions $\varphi_j = \exp\{\int_T^t \mu_j(s)\,ds\}$, $j = 1, 2$, of Eq. (1.1).

Theorem 1.1. *Assume there exists a function $\mu \in C^2(T, \infty)$ which is not identically equal to* 0 *and such that*

$$\int_T^\infty \left|\frac{\mathrm{Hov}(s)}{\mu(s)}\right| e^{\pm 2\int_T^s \Re[\mu]\,dz}\,ds < \infty. \tag{1.6}$$

Then the rest state of (1.1) *is asymptotically stable if and only if*

$$\lim_{t\to\infty} \int_T^t \Re\big[\mu_j(s)\big]\,ds = -\infty, \qquad j = 1, 2, \tag{1.7}$$

$$\lim_{t\to\infty} \int_T^t \Re\left[\mu_j(s) + \frac{\mu_j'(s)}{\mu_j(s)}\right] ds = -\infty, \qquad j = 1, 2. \tag{1.8}$$

Remark 1.1. If condition (1.6) is satisfied for some choice of $\mu(t)$, it means that this choice is good enough to make the error of approximation small enough.

Example 1.1. Consider the linear Euler equation

$$x''(t) + \frac{2ax'(t)}{t} + \frac{bx(t)}{t^2} = 0.$$

Choosing

$$\mu = \frac{\sqrt{(1/2 - a)^2 - b}}{t}$$

we have

$$\mathrm{Hov}(t) \equiv 0, \qquad \mu_{1,2} = \frac{1/2 - a \pm \sqrt{(a-1/2)^2 - b}}{t}.$$

From conditions (1.7) and (1.8) of Theorem 1.1 we get trivial necessary and sufficient conditions of asymptotic stability of Euler equation

$$\Re\big(a - 1/2 \mp \sqrt{(a-1/2)^2 - b}\big) > 0.$$

Example 1.2. Consider the linear equation

$$x''(t) + \frac{2ax'(t)}{t} + \frac{bx(t)}{t^{2\gamma}} = 0, \quad b, \gamma \in R,\ a \in \mathbb{C}.$$

From Theorem 1.1 it follows that if

$$\Re[a] > 0, \quad b > 0, \quad \gamma < \min\big\{1, 2\big|\Re[a]\big|\big\}$$

then this equation is asymptotically stable.

Indeed, choosing

$$\mu = \frac{i\sqrt{b}}{t^\gamma}$$

by direct calculations we get

$$f = \frac{a}{t}, \qquad g = \frac{b}{t^{2\gamma}} + \frac{2a}{t^2}, \qquad k(t) = \frac{i\gamma t^{\gamma-1}}{2\sqrt{b}}, \qquad I(t) = \frac{b}{t^{2\gamma}} + \frac{a-a^2}{t^2},$$

$$\Re[\mu_1] = \frac{\gamma - 2\Re[a]}{2t}, \qquad \Re\left[\mu_1 + \frac{\mu_1'}{\mu_1}\right] = \frac{-\gamma - 2\Re[a]}{2t} + O\big(t^{\gamma-2}\big), \quad t \to \infty,$$

$$\mathrm{Hov} = k'\mu + k^2\mu^2 - \mu^2 - I(t) = O\big(t^{-2}\big), \quad t \to \infty,$$

$$\int_T^\infty \frac{|\mathrm{Hov}(s)| e^{\pm \int_T^s \Re(\mu)\, dz}\, ds}{|\mu(s)|} \leqslant C \int_T^\infty s^{\gamma-2}\, ds < \infty.$$

So conditions (1.6)–(1.8) of Theorem 1.1 are satisfied.

In the cases when the quantity $\Re[\mu(t)]$ is unbounded one of (1.6) is very restrictive. Under additional monotonicity condition

$$\Re\left[\pm\mu - \frac{\mu'}{2\mu} - f(t)\right] \leqslant 0, \quad t > T, \tag{1.9}$$

one can prove the attractivity of the rest state under the conditions less restrictive than (1.6):

Theorem 1.2. *Assume there exists a function $\mu \in C^2(T, \infty)$ which is not identically equal to* 0 *and such that* (1.7), (1.9) *and*

$$\int_T^\infty \frac{|\mathrm{Hov}(s)|\, ds}{|\mu(s)|} < \infty \tag{1.10}$$

are satisfied.

Then every solution of (1.1) *approaches zero as $t \to \infty$.*

Remark 1.2. If in Theorem 1.2 instead of (1.9) the strictly monotonicity conditions

$$\Re\left[\pm\mu - \frac{\mu'}{2\mu} - f(t)\right] \leqslant -\chi, \quad \chi = \text{const} > 0,\ t > T, \tag{1.11}$$

are satisfied then condition (1.7) may be removed, since it follows from (1.11).

Theorem 1.3. *Assume there exists a function $\mu \in C^2(T,\infty)$ which is not identically equal to* 0 *and a positive number C such that* (1.7)–(1.9) *and*

$$\left|\mu_j^{k-1}(t)\right| \int_T^t \frac{|\mathrm{Hov}(s)|\,ds}{|\mu(s)|} \leqslant C, \quad t > T,\ k, j = 1, 2, \tag{1.12}$$

are satisfied. Then the rest state of (1.1) *is asymptotically stable.*

Remark 1.3. Conditions of Theorems 1.1–1.3 are hard to check because of two reasons. First, since there is no construction of the function $\mu(t)$ in these theorems, we should construct $\mu(t)$ (we have the similar situation in the Lyapunov's method: to apply the Lyapunov's method we need to construct the energy function). Second, because the function $\mathrm{Hov}(s)$ depends on solutions, to check (1.6) we should obtain uniform for all solutions estimate for $\mathrm{Hov}(s)$.

Anyway Theorem 1.1 is useful because by using specific asymptotic solutions φ_j of (1.1) we can construct the function $\mu(t) = \frac{1}{2}\frac{d}{dt}\ln(\frac{\varphi_1}{\varphi_2})$, and check condition (1.6) by estimating uniformly the function $\mathrm{Hov}(t)$. For a better approximation the function $\mathrm{Hov}(t)$ becomes smaller and condition (1.6) becomes weaker. If we can construct the ideal function $\mu(t)$ such that $\mathrm{Hov}(t) \equiv 0$, like in Example 1.1, then condition (1.6) disappears.

In Theorems 1.4, 1.5, 1.8, 1.9 we deduce from Theorems 1.1–1.3 some asymptotic stability or attractivity theorems by using specific functions $\mu(t)$.

Choosing

$$\mu(t) = \sqrt{f^2 - g - f'}, \tag{1.13}$$

which means that we are choosing as approximate solutions well-known WKB solutions of (1.1) (see [3]), from Theorems 1.1, 1.2 we obtain the following two theorems.

Theorem 1.4. *Assume $f \in C^3(T,\infty)$, the function $g(t)$ does not depend on solutions, $g \in C^2(T,\infty)$, and*

$$\int_T^t \left|k' + k^2\mu\right| e^{\pm\int_T^s \Re\sqrt{f^2-g-f'}\,dz}\,ds < \infty, \quad k = \frac{\mu'}{2\mu^2}. \tag{1.14}$$

Then the rest state of (1.1) *is asymptotically stable if and only if*

$$\lim_{t\to\infty}\int_T^t \Re[\mu_j]\,dt = -\infty, \quad \lim_{t\to\infty}\int_T^t \Re\left[\mu_j + \frac{\mu_j'}{\mu_j}\right]dt = -\infty, \quad j = 1, 2, \tag{1.15}$$

where μ_j are defined by (1.3), (1.13).

Theorem 1.5. *Assume $f \in C^3(T,\infty)$, $g \in C^2(T,\infty)$, the function $g(t)$ does not depend on solutions,* (1.7), (1.9) *are satisfied, where μ_j are defined by* (1.3), (1.13), *and*

$$\int_T^\infty \left|k'(s)+k^2(s)\mu(s)\right| ds < \infty, \quad k = \frac{\mu'}{2\mu^2}. \tag{1.16}$$

Then every solution of (1.1) *approaches zero as $t \to \infty$.*

Remark 1.4. Condition (1.16) is close to the main assumption of asymptotic stability theorems in Pucci and Serrin [10,11], that $k(t)$ is the function of bounded variation ($\int_T^\infty |k'(t)|\,dt < \infty$).

Example 1.3. Set $f = t^\alpha + it^\beta$, $g(t) = 1 - 2f'(t)$. From Theorem 1.4 it follows that (1.1) is asymptotically stable if $-1 < \alpha < -\beta - 1$, $\alpha < 0$ (see [6]).

Example 1.4. Set $f = t^\alpha + it^\beta$, $g(t) = 1 - 2f'(t)$. From Theorem 1.5 it follows that (1.1) is asymptotically stable if $-1 \leqslant \alpha < 1$, $\beta \leqslant (\alpha+1)/2$ (see [6]).

Using instead of the function $\mu(t)$ another function $F(t)$ related with $\mu(t)$ via transformation

$$\mu(t) = \frac{e^{\int_T^t 2F(z)\,dz}}{2(C + \int_t^T e^{\int_T^s 2F(z)\,dz}\,ds)}, \quad C = \text{const}, \tag{1.17}$$

from Theorems 1.1, 1.2 we deduce the next two theorems.

Theorem 1.6. *Assume there exists a function $F(t) \in C^1(T,\infty)$ such that*

$$\int_T^\infty \frac{|F'(s) - F^2(s) - I(s)| e^{\pm 2\int_T^s \Re[\mu(z)]\,dz}\,ds}{|\mu(s)|} < \infty, \tag{1.18}$$

where the function $I(t)$ is defined in (1.4).

Then the rest state of (1.1) *is asymptotically stable if and only if*

$$\lim_{t\to\infty}\int_T^t \Re[f+F](s)\,ds = \infty, \qquad \lim_{t\to\infty}\int_T^t \Re[f+F+2\mu](s)\,ds = \infty,$$

$$\lim_{t\to\infty}\int_T^t \Re\left[f+F-\frac{(f'+F')}{f+F}\right](s)\,ds = \infty,$$

$$\lim_{t\to\infty}\int_T^t \Re\left[f+F+2\mu-\frac{(f'+F'+2\mu')}{f+F+2\mu}\right](s)\,ds = \infty. \tag{1.19}$$

Remark 1.5. Condition (1.18) is automatically satisfied if one can find the function $F(t)$ which solves the Riccati equation

$$F'(t) - F^2(t) - I(t) = 0.$$

It is well known that Eq. (1.1) is equivalent to this Riccati equation.

Theorem 1.7. *Assume there exists a function $F(t) \in C^1(T,\infty)$ such that*

$$\int_T^\infty \frac{|(F'(s) - F^2(s) - I(s))(C + \int_s^T e^{\int_T^y 2F(z)\,dz}\,dy)|\,ds}{e^{\int_T^s 2Re[F(z)]\,dz}} < \infty, \tag{1.20}$$

$$\Re\big[f(t) + F(t)\big] \geqslant 0, \qquad \Re\left[f(t) + F(t) + \frac{e^{\int_T^t 2F(z)\,dz}}{C + \int_t^T e^{\int_T^s 2F(z)\,dz}\,ds}\right] \geqslant 0, \tag{1.21}$$

$$\lim_{t\to\infty} \int_T^t \Re\big[f(s) + F(s)\big]\,ds = \infty,$$

$$\lim_{t\to\infty} \int_T^t \Re\left[f(s) + F(s) + \frac{e^{\int_T^s 2F(z)\,dz}}{1 + \int_s^T e^{\int_T^y 2F(z)\,dz}\,dy}\right] ds = \infty. \tag{1.22}$$

Then every solution of (1.1) *approaches zero as $t \to \infty$.*

Choosing $F(t) = f(t)$ (which gives different from (1.13) choice of $\mu(t)$, see (1.17)) from Theorem 1.7 it follows

Theorem 1.8. *Assume $f, g \in C(T,\infty)$ and*

$$\int_T^\infty \frac{|g(s,x(s),x'(s))(C + \int_s^T e^{\int_T^y 2f(z)\,dz}\,dy)|\,ds}{e^{\int_T^s 2Re[f(z)]\,dz}} < \infty, \tag{1.23}$$

$$\Re\big[f(t)\big] \geqslant 0, \qquad \Re\left[2f(t) + \frac{e^{\int_T^t 2f\,dz}}{C - \int_T^t e^{\int_T^s 2f(z)\,dz}\,ds}\right] \geqslant 0, \tag{1.24}$$

$$\lim_{t\to\infty} \int_T^t \Re\big[f(s)\big]\,ds = \lim_{t\to\infty} \int_T^t \Re\left[2f(s) + \frac{e^{\int_T^s 2f(z)\,dz}}{C - \int_T^s e^{\int_T^y 2f(z)\,dz}\,dy}\right] ds = \infty. \tag{1.25}$$

Then every solution of (1.1) *approaches zero as $t \to \infty$.*

Remark 1.6. In Theorem 1.8 the function $g(t) = g(t,x(t),x'(t))$ may depend on solutions. Notice that in Theorem 1.8 the classical assumption that the function $g(t,x(t),x'(t)) + 2f'(t)$ is positive (see e.g. Pucci and Serrin [10]) is not required, but to check (1.23) we should estimate $g(t,x(t),x'(t))$ uniformly for all solutions.

Example 1.5. Consider the quasi-linear equation

$$\frac{d^2x(t)}{dt^2} + \frac{d}{dt}\left(\left(\eta t^\alpha + \frac{\alpha}{t}\right)x(t)\right) + x(t)g\big(t,x(t),x'(t)\big) = 0, \tag{1.26}$$

where α, η are positive numbers. From Theorem 1.8 it follows that if

$$\int_T^\infty \frac{\big|g(t,x(t),x'(t))\big(C\eta T^\alpha + 1 - \exp\big(\frac{\eta t^{\alpha+1} - \eta T^{\alpha+1}}{\alpha+1}\big)\big)\big|}{t^\alpha \exp\big(\frac{\eta t^{\alpha+1} - \eta T^{\alpha+1}}{\alpha+1}\big)}\,dt < \infty \tag{1.27}$$

then solutions of (1.26) approach zero as $t \to \infty$.

Indeed, we have

$$2f(t) \equiv \eta t^{\alpha} + \frac{\alpha}{t}$$

and conditions of Theorem 1.8 can be checked by direct calculations. Conditions (1.24), (1.25) are satisfied since

$$2f(t) + \frac{e^{\int_T^t 2f\,dz}}{C - \int_T^t e^{\int_T^s 2f\,dz}\,ds} = \frac{\alpha}{t} + \frac{\eta t^{\alpha}(1 + C\eta T^{\alpha})}{C\eta T^{\alpha} + 1 - \exp\left(\frac{\eta t^{\alpha+1} - \eta T^{\alpha+1}}{\alpha+1}\right)} \geqslant \frac{\alpha}{2t} \geqslant 0.$$

Condition (1.23) turns to (1.27).

If, for example, $g(t,x,x') = \frac{\pm t^{\beta}}{1+|x|^2+|x'(t)|^2}$, $\alpha > \beta + 1$, $\alpha > 0$, then $|g(t,x,x')| \leqslant t^{\beta}$, condition (1.27) is satisfied, and in this case all solutions of (1.26) approach zero as $t \to \infty$.

If $\alpha = 0$ then $2f = \eta > 0$ and condition (1.24) is not satisfied because

$$2f(t) + \frac{e^{\int_1^t 2f\,dz}}{C - \int_1^t e^{\int_1^s 2f\,dz}\,ds} = \frac{\eta(C\eta + 1)}{C\eta + 1 - \exp(\eta t - \eta)} \to -0, \quad \text{as } t \to \infty.$$

In the case $\alpha = 0$, Eq. (1.26) obviously is not asymptotically stable for $g \equiv 0$ although (1.27) is satisfied. Indeed, $x \equiv 1$ is a solution of (1.26) which does not approach zero.

Denote

$$S_{n+1}(t) = \int_T^t \left(g + 2f' + S_n^2\right)(s) e^{2\int_t^s f(y)\,dy}\,ds, \quad S_0(t) \equiv 0, \quad n = 0, 1, 2, \ldots. \tag{1.28}$$

Theorem 1.9. *Let $f \in C^1(T,\infty)$, $g \in C(T,\infty)$ be real functions, g does not depend on solutions,*

$$g(t) + 2f'(t) \geqslant 0, \tag{1.29}$$

and for some non-negative integer n,

$$\int_T^{\infty} \frac{|S_{n+1}^2(t) - S_n^2(t)|}{|\mu(t)|}\,dt < \infty, \qquad \mu(t) \equiv \frac{-1}{2\int_t^{\infty} e^{\int_t^s 2(S_{n+1}(z) - f(z))\,dz}\,ds}, \tag{1.30}$$

$$\int_T^{\infty} S_{n+1}(t)\,dt = \infty. \tag{1.31}$$

Then every solution of (1.1) *approaches zero as $t \to \infty$.*

Example 1.6. Set $f(t) = t^{\alpha}$, $g(t) + 2f'(t) \equiv 1$, $1/(2n+3) < \alpha < 1$ for some $n \geqslant 0$. From Theorem 1.9 it follows that every solution of (1.1) approaches zero as $t \to \infty$.

It is well known that for the case $g(t) + 2f'(t) = \text{const} > 0$ and large damping $f \geqslant \text{const} > 0$ Wintner–Smith's condition (1.31) with $n = 0$ (see [12]) is the necessary and sufficient condition of asymptotic stability of (1.1).

It would be interesting to deduce Smith's result from our approach, or get rid of condition (1.30) of Theorem 1.9, but we do not know if it is possible.

2. Auxiliary theorems

Theorem 2.1 *(Gronwall's inequality). Let the functions $y(t)$, $f(t)$, $K(t)$ be continuous on $[T,b]$, non-negative and*

$$y(t) \leqslant f(t) + \int_t^b K(s)y(s)\,ds, \quad T \leqslant t \leqslant b. \tag{2.1}$$

Then

$$y(t) \leqslant f(t) + \int_t^b K(s)f(s)e^{\int_t^s K\,dz}\,ds, \quad T \leqslant t \leqslant b. \tag{2.2}$$

Proof. Denote

$$M(t) = \int_t^b K(s)y(s)\,ds,$$

then

$$y(t) \leqslant f(t) + M(t),$$
$$M'(t) = -K(t)y(t) \geqslant -K(t)\big(f(t) + M(t)\big).$$

Multiplying the last inequality by $-e^{-\int_t^b K(z)\,dz}$ we get

$$-\big(M(t)e^{-\int_t^b K(z)\,dz}\big)' \leqslant K(t)f(t)e^{-\int_t^b K(z)\,dz}.$$

By integration over (t,b) we have

$$M(t)e^{-\int_t^b K(z)\,dz} \leqslant \int_t^b K(s)f(s)e^{-\int_s^b K(z)\,dz}\,ds,$$

$$y(t) \leqslant f(t) + M(t) \leqslant f(t) + \int_t^b K(s)f(s)e^{\int_t^s K(z)\,dz}\,ds. \qquad \square$$

Consider the system of ordinary differential equations

$$a'(t) = A(t)a(t), \quad t > T, \tag{2.3}$$

where $a(t)$ is an n-vector function and $A(t) \in C(T,\infty)$ is an $(n \times n)$-matrix-function. Suppose we can find the exact solutions of the system

$$\psi'(t) = A_1(t)\psi(t), \quad t > T, \tag{2.4}$$

with the matrix function A_1 close to the matrix-function A.

Let $\Psi(t)$ be the $n \times n$ fundamental matrix of the auxiliary system (2.4). Then the solutions of (2.3) can be represented in the form

$$a(t) = \Psi(t)\big(C + \varepsilon(t)\big), \tag{2.5}$$

where $\varepsilon(t)$ is called an error function and $a(t)$, C, $\varepsilon(t)$ are the n-vector columns $\varepsilon(t) = \text{column}(\varepsilon_1(t), \ldots, \varepsilon_n(t))$, $C = \text{column}(C_1, \ldots, C_n)$, C_k are some constants.

Denote

$$H(t) = \Psi^{-1}(t)\big(A(t)\Psi(t) - \Psi'(t)\big). \tag{2.6}$$

Theorem 2.2. [6] *Assume there exists an invertible matrix-function $\Psi(t) \in C^1[T, \infty)$ such that*

$$\int_T^\infty \big\|H(s)\big\|\, ds < \infty. \tag{2.7}$$

Then every solution of (2.3) *can be represented in the form* (2.5) *and the error vector-function $\varepsilon(t)$ can be estimated as*

$$\big\|\varepsilon(t)\big\| \leqslant \|C\|\big(e^{\int_t^\infty \|H(y)\|\, dy} - 1\big) < \infty, \tag{2.8}$$

where $\|\cdot\|$ is the Euclidean vector (or matrix) norm $\|C\| = \sqrt{C_1^2 + \cdots + C_n^2}$.

Remark 2.1. From estimates (2.8) it follows that the error function $\varepsilon(t)$ is small whenever the expression $\int_t^\infty (\|\Psi^{-1}(s)(A - A_1)(s)\Psi(s)\|)\, ds$ is small.

Theorem 2.3. [6] *Let $\varphi_{1,2}(t) \in C^2(T, \infty)$ be complex-valued functions such that*

$$\int_T^\infty \big|B_{kj}(t)\big|\, dt < \infty, \quad k, j = 1, 2, \tag{2.9}$$

where

$$B_{kj}(t) \equiv \frac{\varphi_k(t) L\varphi_j(t)}{W(\varphi_1, \varphi_2)},$$
$$L \equiv \frac{d^2}{dt^2} + 2f(t)\frac{d}{dt} + 2f'(t) + g\big(t, x(t), x'(t)\big), \quad k, j = 1, 2. \tag{2.10}$$

Then for arbitrary constants C_1, C_2, and some T every solution of the equation $Lx(t) = 0$, $t > T$, can be represented in the form

$$x(t) = \big[C_1 + \varepsilon_1(t)\big]\varphi_1(t) + \big[C_2 + \varepsilon_2(t)\big]\varphi_2(t), \tag{2.11}$$
$$x'(t) = \big[C_1 + \varepsilon_1(t)\big]\varphi_1'(t) + \big[C_2 + \varepsilon_2(t)\big]\varphi_2'(t), \tag{2.12}$$

where the error vector-function $\varepsilon(t) = \binom{\varepsilon_1(t)}{\varepsilon_2(t)}$ is estimated as

$$\big\|\varepsilon(t)\big\| \leqslant \|C\|\left(-1 + \exp\int_t^\infty \big\|B(s)\big\|\, ds\right), \tag{2.13}$$

and the matrix B has entries B_{kj} with Euclidean norm $\|B\| = \sqrt{\sum_{kj} |B_{kj}|^2}$.

Proof of Theorem 1.1. Let us seek the approximate solutions of (1.1) in the form

$$\varphi_j(t) = e^{\int_T^t \mu_j(z)\, dz}, \tag{2.14}$$

where the functions $\mu_j(t)$ are defined by (1.3). From (2.14) we get

$$B_{11} = \frac{\mathrm{Hov}(t)}{2\mu} e^{2\int_T^t \mu\, dz}, \qquad B_{22} = \frac{\mathrm{Hov}(t)}{2\mu} e^{-2\int_T^t \mu\, dz}, \qquad B_{12} = B_{21} = \frac{\mathrm{Hov}(t)}{2\mu}.$$

It follows from (1.6) that condition (2.9) of Theorem 2.3 is satisfied. Using Theorem 2.3 we obtain from representations (2.11), (2.12) the stability estimates

$$|x(t)| + |x'(t)| \leqslant (\|C\| + \|\varepsilon\|)(|\varphi_1(t)| + |\varphi_2(t)| + |\varphi_1'(t)| + |\varphi_2'(t)|).$$

From (1.6) and (2.13) we have $|\varepsilon_j(t)| \leqslant \|\varepsilon(t)\| \leqslant C_3\|C\|$, $j = 1, 2$. From (1.7), (1.8) we get $\varphi_j^{(k-1)}(t) \to 0$, $t \to \infty$, $k, j = 1, 2$. So the stability and the attractivity of the rest solution follows from the stability estimates.

Further we prove that if one of (1.7), (1.8) is not satisfied then there exists asymptotically unstable solution $x(t)$.

Assume for contradiction that (1.2) is satisfied and, for example, first condition of (1.7) is not satisfied. Then there exists the sequence $t_n \to \infty$ such that

$$\lim_{t_n\to\infty} |\varphi_1(t_n)| = \lambda_1 > 0.$$

Further there exists the subsequence $t_{n_j} \equiv t_m$ of the sequence t_n such that

$$\lim_{t_m\to\infty} |\varphi_2(t_m)| = \lambda_2 \geqslant 0.$$

From Theorem 2.3 for any constants C_1, C_2 the solutions $x(t)$ of (1.1) can be represented in the form (2.11), or

$$x(t_m) = [C_1 + \varepsilon_1(t_m)]\varphi_1(t_m) + [C_2 + \varepsilon_2(t_m)]\varphi_2(t_m),$$

where from (2.13) we have

$$|\varepsilon_j(t)| \leqslant \|\varepsilon(t)\| \leqslant \|C\|\left(e^{\int_t^\infty \|B\|\, ds} - 1\right) \to 0$$

as $t = t_m \to \infty$.

From the representation above it follows that λ_1, λ_2 must be finite numbers, otherwise left side of the representation vanishes and right side approaches infinity when t_m approaches infinity. Choosing $C_1 = 1$, $C_2 = 0$ as $t_k \to \infty$ we obtain from the above representation

$$0 = \lambda_1 + \lambda_1 \lim_{t_k\to\infty} \varepsilon_1(t_k) + \lambda_2 \lim_{t_k\to\infty} \varepsilon_2(t_k) = \lambda_1$$

which contradicts the assumption $\lambda_1 > 0$. □

To prove Theorem 1.2, we need the following

Lemma 2.4. *Let the functions* $\varphi_{1,2} \in C^2[0, 1]$ *be given in the form* (2.14), (1.3). *Then they are solutions of the equation*

$$L_1\varphi(t) = 0, \quad t \in [T, \infty), \qquad L_1 \equiv L + \mathrm{Hov}(t), \tag{2.15}$$

and

$$\frac{L\varphi_1}{\varphi_1} = \frac{L\varphi_2}{\varphi_2} = -\mathrm{Hov}(t), \tag{2.16}$$

where the function $\mathrm{Hov}(t)$ *and the operator* L *are defined in* (1.5), (2.10).

The functions φ_1, φ_2 *are solutions of* (1.1) *if and only if* $\mathrm{Hov} \equiv 0$.

In addition, if μ *is not identically equal zero, then* φ_1, φ_2 *are linearly independent functions.*

Proof. Equation (2.15) can be checked by direct substitution (2.14) into (2.15). From (1.3) we get

$$\mu_1+\mu_2+\frac{\mu'}{\mu}+2f\equiv 0. \tag{2.17}$$

We derive the first formula (2.16) from (2.17),

$$\frac{L\varphi_1}{\varphi_1}-\frac{L\varphi_2}{\varphi_2}=\mu_1^2+\mu_1'-\mu_2^2-\mu_2'+2f(\mu_1-\mu_2)\equiv 0.$$

The second formula (2.16) can be checked by direct calculations.

From

$$\frac{W(\varphi_1,\varphi_2)}{\varphi_1\varphi_2}=\mu_2-\mu_1=-2\mu, \qquad W(\varphi_1,\varphi_2)\equiv\varphi_1(t)\varphi_2'(t)-\varphi_1'(t)\varphi_2(t) \tag{2.18}$$

it follows that if $\mu\not\equiv 0$ then the functions φ_1,φ_2 are linearly independent. □

Proof of Theorem 1.2. The solutions of (1.1) or

$$x''(t)+2f(t)x'(t)+\big(2f'(t)+g(t,x,x')+\mathrm{Hov}(t)\big)x(t)=G(x),$$
$$G(x)=\mathrm{Hov}(t)x(t)$$

may be represented in the form

$$x(t)=\varphi_1(t)C_1+\varphi_2(t)C_2+\varphi_2(t)\int_b^t\frac{\varphi_1G(s)\,ds}{W[\varphi_1(s),\varphi_2(s)]}-\varphi_1(t)\int_b^t\frac{\varphi_2G(s)\,ds}{W[\varphi_1(s),\varphi_2(s)]}$$

or

$$x(t)=\varphi_1(t)C_1+\varphi_2(t)C_2+\int_b^t\left(\frac{\varphi_1(t)}{\varphi_1(s)}-\frac{\varphi_2(t)}{\varphi_2(s)}\right)\frac{\varphi_1(s)L\varphi_2(s)}{W[\varphi_1(s),\varphi_2(s)]}x(s)\,ds,$$
$$T\leqslant b\leqslant t, \tag{2.19}$$

where φ_1,φ_2 are exact solutions of the homogeneous equation (2.15).

In view of (2.16), (2.18) we have

$$x(t)=\varphi_1(t)C_1+\varphi_2(t)C_2+\int_b^t\left(\frac{\varphi_1(t)}{\varphi_1(s)}-\frac{\varphi_2(t)}{\varphi_2(s)}\right)\frac{\mathrm{Hov}(s)x(s)\,ds}{2\mu(s)}.$$

From condition (1.9) it follows that $|\varphi_j(t)|=|\varphi_j(T)|\exp\{\int_T^t\Re[\frac{\varphi_j'(s)\,ds}{\varphi_j(s)}]\}$ are non-increasing, bounded functions, so we get the estimates

$$\big|x(t)\big|\leqslant\big|\varphi_1(t)C_1+\varphi_2(t)C_2\big|+\int_b^t\frac{|\mathrm{Hov}(s)x(s)|\,ds}{|\mu(s)|},\quad T\leqslant b\leqslant t. \tag{2.20}$$

From (1.7) we get $\varphi_1(t)\to 0$, $\varphi_2(t)\to 0$ as $t\to\infty$. So for any $\varepsilon>0$ there exists t_0 such that

$$\big|C_1\varphi_1(t)+C_2\varphi_2(t)\big|\leqslant\frac{\varepsilon}{e^{\int_b^\infty|\mathrm{Hov}(s)|\,ds/|\mu(s)|}},\quad t>t_0.$$

Further from (2.20) we have the inequality

$$\left|x(t)\right| \leqslant \frac{\varepsilon}{e^{\int_b^\infty |\mathrm{Hov}|/|\mu|}} + \int_b^t \frac{|\mathrm{Hov}(s)x(s)|}{|\mu|}\,ds. \tag{2.21}$$

Applying Gronwall's inequality we get

$$\left|x(t)\right| \leqslant \frac{\varepsilon e^{\int_b^t |\mathrm{Hov}|/|\mu|}}{e^{\int_b^\infty |\mathrm{Hov}|/|\mu|}} \leqslant \varepsilon \tag{2.22}$$

from which it follows that every solution of (1.1) approaches zero as $t \to \infty$. □

Proof of Theorem 1.3. In view of Theorem 1.2 it is enough to prove the stability and that for every solution $\lim_{t\to\infty} x'(t) = 0$.

From (2.19),

$$x'(t) = \varphi_1'(t)C_1 + \varphi_2'(t)C_2 + \int_b^t \left(\frac{\varphi_1'(t)}{\varphi_1(s)} - \frac{\varphi_2'(t)}{\varphi_2(s)}\right)\frac{\varphi_1(s)L\varphi_2(s)}{W[\varphi_1(s),\varphi_2(s)]}x(s)\,ds,$$

$$\begin{aligned} x'(t) &= \mu_1(t)\varphi_1(t)C_1 + \mu_2(t)\varphi_2(t)C_2 \\ &\quad + \int_b^t \left(\frac{\mu_1(t)\varphi_1(t)}{\varphi_1(s)} - \frac{\mu_2(t)\varphi_2(t)}{\varphi_2(s)}\right)\frac{\mathrm{Hov}(s)x(s)\,ds}{\mu(s)} \end{aligned} \tag{2.23}$$

in view of (1.9) we get

$$\left|x'(t)\right| \leqslant \left|\varphi_1'(t)C_1 + \varphi_2'(t)C_2\right| + \left(\left|\mu_1(t)\right| + \left|\mu_2(t)\right|\right)\int_b^t \frac{|\mathrm{Hov}(s)x(s)|}{|\mu(s)|}\,ds. \tag{2.24}$$

From (1.8), $\varphi_1'(t)C_1 + \varphi_2'(t)C_2 \to 0$, $t \to \infty$, so for any positive number ε we can find t_0 such that for all $t > t_0$,

$$\left|x'(t)\right| \leqslant e^{-2C}\varepsilon + \left(\left|\mu_1(t)\right| + \left|\mu_2(t)\right|\right)\int_b^t \frac{|\mathrm{Hov}(s)x(s)|}{|\mu(s)|}\,ds. \tag{2.25}$$

By using inequalities (2.22), (1.12) we get

$$\left|x'(t)\right| \leqslant C_1\varepsilon,$$

and the attractivity (1.2) is proved.

Let us prove the stability estimates. From (2.20) and Gronwall's inequality we get

$$\left|x(t)\right| \leqslant \left|\varphi_1(t)C_1 + \varphi_2(t)C_2\right| + \int_b^t \frac{|\mathrm{Hov}(s)|}{|\mu(s)|}e^{\int_b^s \frac{|\mathrm{Hov}(y)|\,dy}{|\mu(y)|}}\left|\varphi_1(s)C_1 + \varphi_2(s)C_2\right|ds, \tag{2.26}$$

or in view of (1.12),

$$\left|x(t)\right| \leqslant C\Big(|C_1|\sup_{b\leqslant s\leqslant t}\left|\varphi_1(s)\right| + |C_2|\sup_{b\leqslant s\leqslant t}\left|\varphi_2(s)\right|\Big), \quad t \geqslant b. \tag{2.27}$$

From (2.23) and (2.27) we have for all $t \geqslant b$,

$$\left|x'(t)\right| \leqslant \left|C_1\mu_1\varphi_1(t) + C_2\mu_2\varphi_2(t)\right| + C\left(\left|\mu_1(t)\right| + \left|\mu_2(t)\right|\right)$$
$$\times \int_b^t \frac{|\mathrm{Hov}(s)|}{|\mu(s)|}\left(|C_1| \sup_{b\leqslant s\leqslant t}\left|\varphi_1(s)\right| + |C_2| \sup_{b\leqslant s\leqslant t}\left|\varphi_2(s)\right|\right) ds. \tag{2.28}$$

From (1.7), (1.8) it follows that for some $b > T$ we have

$$\left|\varphi_j(t)\right| = e^{\int_T^t \Re[\mu_j]\,dy} \leqslant 1, \quad t > b,\ j = 1, 2,$$
$$\left|\mu_j\varphi_j(t)\right| = e^{\int_T^t \Re[\mu_j + \frac{\mu_j'}{\mu_j}]\,dy} \leqslant 1, \quad t > b,\ j = 1, 2.$$

Choosing $|C_j| < \delta/2$ from (2.20), (2.24) we get

$$\left|x(b)\right| = \left|C_1\varphi_1(b) + C_1\varphi_2(b)\right| \leqslant |C_1| + |C_2| < \delta,$$
$$\left|x'(b)\right| = \left|C_1\mu_1\varphi_1(b) + C_1\mu_2\varphi_2(b)\right| \leqslant |C_1| + |C_2| < \delta.$$

The stability follows from condition (1.12) and stability estimates (2.27), (2.28). □

Theorems 1.4, 1.5 are deduced directly from Theorems 1.1, 1.2.

Proof of Theorem 1.6. From (1.17) we have

$$\frac{\mu'(t)}{2\mu(t)} = F(t) + \mu(t) = k\mu, \quad k = \frac{\mu'}{2\mu^2} = 1 + \frac{F}{\mu},$$
$$\mathrm{Hov} = f^2 - g - f' - \mu^2 + \mu\left(k' + \mu k^2\right) = f^2 - g - f' - F^2 + F',$$
$$\mu_1 = -f - F, \qquad \mu_2 = -f - F - 2\mu.$$

Theorem 1.6 follows from these calculations and Theorem 1.1. □

Theorem 1.7 follows from Theorem 1.2.
Theorem 1.8 follows from Theorem 1.7 by choosing $F(t) = f(t)$.

Proof of Theorem 1.9. Set in (1.17)

$$F(t) = S_{n+1}(t) - f(t),$$

where the sequence S_k is defined from the first-order linear differential equations

$$S_{k+1}'(t) + 2f(t)S_{k+1}(t) = S_k^2(t) + g(t) + 2f'(t), \quad S_0 = 0, \quad k = 0, \ldots, n. \tag{2.29}$$

Solving these equations we get representation (1.28) for $S_{n+1}(t)$.

Let us check that all conditions of Theorem 1.2 are satisfied. By appropriate choice of C in $\mu(t)$ from (1.17) we have

$$\mu(t) = \frac{e^{\int_T^t 2F(z)\,dz}}{2\int_t^\infty e^{\int_T^s 2F(z)\,dz}\,ds} = \frac{1}{2\int_t^\infty e^{\int_t^s 2(S_{n+1}(z) - f(z))\,dz}\,ds}.$$

By direct calculations

$$\mathrm{Hov}(t) = F' - F^2 - I = S_{n+1}' - f' - (S_{n+1} - f)^2 - g - f' + f^2 = S_n^2 - S_{n+1}^2.$$

So condition (1.30) implies (1.10).

Furthermore, since μ, μ_1, μ_2 are real-valued functions we have

$$-\mu_1 = \frac{\mu'}{2\mu} - \mu + f = F(t) + f(t) = S_{n+1}(t) \geqslant 0,$$
$$-\mu_2 = \frac{\mu'}{2\mu} + \mu + f = F(t) + 2\mu + f(t) \geqslant F + f = S_{n+1}(t) \geqslant 0,$$

and condition (1.9) is satisfied. Condition (1.31) implies (1.7),

$$\int_T^\infty \big(-\mu_j(s)\big)\,ds \geqslant \int_T^\infty S_{n+1}(s)\,ds = \infty, \quad j = 1, 2.$$

So Theorem 1.9 follows from Theorem 1.2. □

Proof of Example 1.6. Suppose that $f(t) = t^\alpha$, $0 \leqslant \alpha \leqslant 1$, $t \geqslant 1$.

Denote

$$\hat{f}(t,s) = \frac{1}{t-s}\int_s^t f(y)\,dy, \tag{2.30}$$

$$Q(t) = f^{2n+1}(t)\int_T^t \frac{e^{2\int_t^s f\,dy}\,ds}{f^{2n}(s)}. \tag{2.31}$$

Then we have for some $t_0 > T$,

$$\frac{f(t)}{2} \leqslant \frac{f(t)}{\alpha+1} \leqslant \hat{f}(t,s) \leqslant f(t), \quad 0 \leqslant s \leqslant t, \tag{2.32}$$
$$0 \leqslant Q(t) \leqslant 1, \quad t > t_0. \tag{2.33}$$

To prove (2.32) consider the function

$$g(z) = t^{-\alpha}(\alpha+1)\hat{f}(t,s) = \frac{1-(1-z)^{\alpha+1}}{z}, \quad z = 1 - \frac{s}{t}.$$

From

$$g'(z) = \frac{(1-z)^\alpha(1+z\alpha)-1}{z^2} \leqslant \frac{(1-z)^\alpha(1+z)^\alpha - 1}{z^2} \leqslant 0, \quad 0 \leqslant z \leqslant 1,$$

we get that $g(z)$ is decreasing on [0, 1]. So $1 = g(1) \leqslant g(z) \leqslant g(0) = \alpha + 1$, from which we get (2.32).

Further by using l'Hospital's rule we have

$$\lim_{t\to\infty} 2Q(t) = \lim_{t\to\infty}\left(\frac{2\int_1^t e^{\int_1^s 2f(y)\,dy} f^{-2n}(s)\,ds}{f^{-2n-1}(t)e^{\int_1^t 2f(y)\,dy}}\right)$$
$$= \lim_{t\to\infty}\left(\frac{2f(t)}{2f(t) - (n+1)f'(t)/f(t)}\right) = 1,$$

from which we obtain (2.33).

Using (2.32) we have

$$S_1(t)=\int_T^t e^{2\int_t^s f\,dy}\,ds=\int_T^t e^{2(s-t)\hat f(t,s)}\,ds\leqslant\int_T^t e^{(s-t)f(t)}\,ds\leqslant\frac{1}{f(t)},$$

$$S_1(t)-S_0(t)=S_1(t)\leqslant\frac{1}{f(t)}.$$

Denote the sequence A_n by recurrent formulas

$$A_{n+1}=1+A_n^2,\quad A_0=0,\quad n=0,1,2,\ldots.$$

Inequalities

$$S_n\leqslant\frac{A_n}{f(t)}\leqslant A_n,\quad n=1,2,\ldots,\tag{2.34}$$

$$S_n-S_{n-1}\leqslant\frac{A_n-A_{n-1}}{f^{2n-1}(t)},\quad n=1,2,\ldots,\tag{2.35}$$

can be proved by induction

$$S_{n+1}=\int_T^t\big(1+S_n^2\big)e^{2\int_t^s f\,dy}\,ds\leqslant\big(1+A_n^2\big)\int_T^t e^{2\int_t^s f\,dy}\,ds\leqslant\frac{A_{n+1}}{f(t)},$$

$$\begin{aligned}S_{n+1}-S_n&=\int_T^t(S_n-S_{n-1})(S_n+S_{n-1})e^{2\int_t^s f\,dy}\,ds\\&\leqslant(A_n-A_{n-1})(A_n+A_{n-1})\int_T^t\frac{e^{2\int_t^s f\,dy}\,ds}{f^{2n}(s)}\\&=(A_{n+1}-A_n)\int_T^t\frac{e^{2\int_t^s f\,dy}\,ds}{f^{2n}(s)}\end{aligned}$$

from which using (2.33) we get

$$S_{n+1}-S_n\leqslant\frac{A_{n+1}-A_n}{f^{2n+1}(t)}.$$

To finish the proof of Example 1.6 let us check that conditions (1.30), (1.31) of Theorem 1.9 are satisfied for Example 1.6. From the estimate

$$\begin{aligned}\frac{1}{2|\mu|}&=\int_t^\infty e^{\int_t^s 2(S_{n+1}(z)-f(z))\,dz}\,ds\leqslant\int_t^\infty e^{-2\int_t^s f(z)\,dz}\,ds\\&=\int_t^\infty e^{2(t-s)\hat f(t,s)}\,ds\leqslant\int_t^\infty e^{(t-s)f(t)}\,ds\leqslant\frac{1}{f(t)}\end{aligned}$$

we have

$$\big|\mu(t)\big|\geqslant\frac{f(t)}{2},$$

and from (2.34), (2.35), and $\alpha > \frac{1}{2n+3}$ it follows that condition (1.30) is satisfied

$$\int_T^\infty \frac{|\mathrm{Hov}(t)|\,dt}{|\mu(t)|} = \int_T^\infty \frac{S_{n+1}^2(t) - S_n^2(t)}{|\mu(t)|}\,dt \leqslant \int_T^\infty \frac{2(A_{n+2} - A_{n+1})\,dt}{f^{2n+3}(t)}$$
$$= \int_T^\infty \frac{C\,dt}{t^{\alpha(2n+3)}(t)} < \infty.$$

Condition $\alpha < 1$ implies that the Wintner–Smith condition

$$\int_T^\infty S_1(t)\,dt = \infty \tag{2.36}$$

is satisfied.

In view of $S_n \leqslant S_{n+1}$ condition (1.31) for any n follows from the Wintner–Smith condition.

So all conditions of Theorem 1.9 are satisfied and the statement of Example 1.6 is true. □

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