ESTIMATES FOR ERROR FUNCTIONS OF ASYMPTOTIC SOLUTIONS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS

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1. Introduction

Many problems of mathematical physics lead to differential equations which cannot be solved exactly, but permit asymptotic solutions. For example, no exact nontrivial solution is known for general $q(t)$ for the equation

$$u''(t) + q(t)\,u(t) = 0, \qquad t \in [0, T]. \tag{0.1}$$

However it is not difficult to get asymptotic solutions. Indeed, taking in (0.1) $q = 0$ we obtain an auxiliary equation $u''(t) = 0$ whose general solution is

$$u = C_1\varphi_1 + C_2\varphi_2,$$

where

$$\varphi_1 = 1, \quad \varphi_2 = t \tag{0.2}$$

are the fundamental solutions. It is well known that if

$$q \in L_1([0, T]), \tag{0.3}$$

then the general solution of (0.1) can be represented in the form

$$u(t) = (C_1 + \varepsilon_1(t))\,\varphi_1(t) + (C_2 + \varepsilon_2(t))\,\varphi_2(t), \qquad \lim_{t\to 0} \varepsilon_j(t) = 0, \quad j = 1, 2. \tag{0.4}$$

In other words, (0.2) are asymptotic solutions of (0.1). The functions ε_j are called error functions.
If the function $q(t)$ has singularity at $t = 0$, then for the equation (0.1) the Cauchy data $u(0)$, $u'(0)$ are meaningless. For example, if $q(t) = -2t^{-2}$, then $u = C_1\psi_1 + C_2\psi_2$ with fundamental solutions $\psi_1 = t^{-1}$, $\psi_2 = t^2$. In our earlier paper [5] we have in such cases proposed to use weighted initial data in Wronskian form. Let us illustrate this on the example of (0.1).
We take a general solution of (0.1) written in terms of fundamental solutions ψ_1, ψ_2 and arbitrary constants C_1, C_2:

$$u = C_2\psi_1 - C_1\psi_2, \qquad u' = C_2\psi_1' - C_1\psi_2'.$$

Solving this system we get

$$C_j = W(u, \psi_j)/W(\psi_1, \psi_2), \qquad j = 1, 2. \tag{0.5}$$

Here $W(\psi_1, \psi_2) = \psi_1\psi_2' - \psi_2\psi_1'$ is the Wronskian. If $\varphi_1(t)$, $\varphi_2(t)$ are asymptotic solutions of (0.1) such that

$$\partial_t^{k-1}\psi_j(t) = [1 + \varepsilon_j(t)]\,\partial_t^{k-1}\varphi_j(t), \qquad \lim_{t\to 0} \varepsilon_j(t) = 0, \quad k, j = 1, 2 \tag{0.6}$$

then necessarily

$$\frac{W(u,\psi_j)}{W(\psi_1,\psi_2)} = \lim_{t\to 0} \frac{W(u,\varphi_j)}{W(\varphi_1,\varphi_2)}, \quad j = 1,2$$

and we get from (0.5)

$$\lim_{t\to 0} \frac{W(u,\varphi_j)}{W(\varphi_1,\varphi_2)} = C_j, \quad j = 1,2. \tag{0.7}$$

These relations can replace the initial data. The ratios in (0.7) we call *Wronskian data*. Since we can always choose φ_1, φ_2, ψ_1, ψ_2 to have $W(\psi_1,\psi_2) = W(\varphi_1,\varphi_2) = 1$, (0.7) reduces to

$$\lim_{t\to 0} W(u,\varphi_j) = C_j, \quad j = 1,2.$$

If (0.3) is satisfied, then Wronskian data for the equation (0.1) reduce to Cauchy data. So, in the absence of singularities Wronskian data generalize usual Cauchy data.

Construction of an asymptotic solution usually consists of the following steps:

1) choice of appropriate probe functions,

2) proof that the difference of an exact solution and the probe function is small in some norm,

3) proof that asymptotic representation is differentiable (compare with (0.6)).

In this paper we apply this analysis to the equation

$$Lu(t,\xi) = \sum_{k=0}^{m} q_{m-k}(t,\xi)\, \partial_t^k\, u(t,\xi) = 0, \quad q_0 = 1, \quad t \in]0,T[. \tag{0.8}$$

The main tool is a generalization of Levinson asymptotic theorem (Theorem 2). This theorem estimates error functions for higher order ordinary linear differential equations with parameter. We apply Theorem 2 to singular initial value problems with Wronskian data.

2. The main results

In this section we formulate six theorems. Theorem 1 is a uniqueness theorem for (0.8) with Wronskian data. Theorem 2 provides a sufficient criterion for asymptotic solutions. Theorem 3 is a modification of Theorem 2 for first order systems. Theorem 4 is a particular case of Theorem 2 for second order differential equations. Theorem 5 (Petrovsky) is a well-known necessary and sufficient criterion of correctness of the Cauchy problem. Theorem 6 is an asymptotic version of Theorem 5.

Consider the ordinary differential equation

$$Lu(t,\xi) = 0, \qquad t \in]0,T[, \tag{1.1}$$

where L is in (0.8); as regards the coefficients, we will assume that

$$q_{m-k}(.,\xi) \in C(]0,T]), \qquad k = 0,\ldots,m. \tag{1.2}$$

For any real s we denote by $H^s = H^s(R^n)$ the usual Sobolev space with the norm

$$||v||_s = \{(2\pi)^{-n} \int (1+|\xi|^2)^s\, |\widehat{v}(\xi)|^2\, d\xi\}^{1/2} = ||\widehat{v(\xi)}||_s, \tag{1.3}$$

where $\widehat{v}$ is the x-Fourier transformation of $v(x)$:

$$\widehat{v}(\xi) = \int v(x^1, \ldots, x^n) \exp(-ix\,\xi)\, dx^1 \ldots dx^n, \qquad \xi \in R^n. \tag{1.4}$$

Denote by $C^k(]0,T], H^p)$ the space of k times continuously differentiable mappings of $]0,T]$ to H^p, $H^\infty = \cap H^s$.
Given the functions

$$\varphi_j(.,\xi) \in C^m(]0,T[), \qquad j = 1, \ldots m, \tag{1.5}$$

we consider a $m \times m$ matrix

$$\Phi = (\partial_t^{k-1} \varphi_j)_{k,j=1}^m \tag{1.6}$$

and Wronskian

$$W(\varphi_1, \ldots, \varphi_m) = W(\varphi) = \det \Phi. \tag{1.7}$$

By Φ^{mj} we denote the algebraic minors, obtained from Φ by deleting mth line and jth column.
Assume that the functions φ_j have been chosen to satisfy

$$\int_0^T |W^{-1}(\varphi)\, \Phi^{mj}\, L\varphi_k|\, dt \le c_1 + c_2 \ln < \xi >, \qquad k,j = 1, \ldots, m, \quad \xi \in R^n \tag{1.8}$$

where c_i are some nonnegative constants, $< \xi >= (9 + |\xi|^2)^{1/2}$, $\quad |\xi|^2 = \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2$. We have chosen $< \xi >$ to have $\ln < \xi > \ge 1$. Consider initial value problem (1.1) with Wronskian data (1.9)

$$\lim_{t \to 0} W(\varphi_1, \ldots, \varphi_{j-1}, u, \varphi_{j+1}, \ldots, \varphi_m)/W(\varphi) = g_j(\xi), \quad j = 1, \ldots, m, \tag{1.9}$$

Then the following theorem is true.

Theorem 1 If the conditions (1.5), (1.8) are satisfied then in $C^m(]0,T])$ the problem (1.1), (1.9) with $g_j \equiv 0$, $j = 1, ..., m$ has only trivial solution.

Theorem 2 Let the conditions (1.2), (1.5), (1.8) are satisfied. Then for any choice of $g_j \in L_2(R^n)$ there exist a solution $u(t,\xi)$ of the equation (1.1), representable in the form

$$\partial_t^{k-1} u(t,\xi) = \sum_{j=1}^m g_j(\xi)\,[1 + \varepsilon_j(t,\xi)]\, \partial_t^{k-1}\, \varphi_j(t,\xi), \qquad k = 1, \ldots, m, \tag{1.10}$$

$$\lim_{t \to 0} ||\varepsilon_j(t,.)||_p = 0, \qquad j = 1, \ldots, m, \quad \text{for some} \quad p \in R, \tag{1.10$'$}$$

with the estimates

$$|\varepsilon_j(t,\xi)| \le \sum_{p=1}^m |g_p^(\xi)| \{-1 + \exp \int_0^t \sum_{k,j=1}^m |W^{-1}(\varphi)\Phi^{ms} L\varphi_k|\, dt\}, \qquad j = 1, \ldots, m \tag{1.11}$$

or, in a weaker form

$$||\varepsilon_j(t,\xi)||_p \le c \sum_{j=1}^m ||g_j(\xi)||_{p+c_2 m^2}, \qquad j = 1, \ldots, m, \quad \text{for some} \quad p \in R. \tag{1.11$'$}$$

Remark 1. If in (1.8) $c_2 = 0$, then the relations (1.10) hold, but instead of (1.10′) we have

$$\lim_{t\to 0} \varepsilon_j(t,\xi) = 0, \qquad k, j = 1,\ldots,m, \tag{1.10''}$$

uniformly for $\xi \in R^n$.
Theorem 2 remains valid if one replaces the semi-interval $]0,T]$ in (1.1), (1.2), (1.5), (1.8) and (1.11) by $[T,\infty[$ and in (1.10′), (1.10″) replaces $t \to 0$ by $t \to \infty$.
For the first order $m \times m$ system of ordinary differential equations with parameter ξ

$$v_t = A(t,\xi)\, v(t,\xi), \quad t \in]0,T[\tag{1.12}$$

we have the following version of Theorem 2.

Theorem 3 Let there exist nonnegative constants c_1, c_2 and a matrix function $\Psi(.,\xi) \in C^1(]0,T])$ such that

$$(c_1 + c_2 \ln < \xi >)^{-1}\, ||\Psi^{-1}\,(A\Psi - \Psi_t)||_M \in L_1([0,T]), \quad \text{for all} \quad \xi \in R^n, \tag{1.13}$$

where $||.||_M$ is the matrix norm. If

$$A(.,\xi) \in C(]0,T]), \tag{1.14}$$

then for any $C(\xi) \in L_2(R^n)$ there exists a solution $v(.,\xi)$ of the system (1.12) representable in the form

$$v(t) = \Psi(t,\xi)(C(\xi) + \varepsilon(t,\xi)), \tag{1.15}$$

where $C(\xi)$ and ε are m-vectors, and for some p

$$\lim_{t\to 0} ||\varepsilon(t,\xi)||_p = 0, \tag{1.16}$$

Remark 2. If Ψ is the Cauchy matrix of the system $w_t = Bw$ where B does not depend on ξ i. e. Ψ satisfies the equation

$$\Psi_t = B\Psi,$$

then the condition (1.13) becomes

$$||\Psi^{-1}(A - B)\,\Psi||_M \in L_1([0,T]). \tag{1.17}$$

Remark 3. If the diagonal part of A is chosen for B i. e.

$$B = \mathrm{diag}(a_{11}, a_{22}, \ldots, a_{mm}), \qquad A = \|a_{ij}\|, \tag{1.18}$$

then the classical theorem of N. Levinson (see Theorem 3.1 in the §3 or [2],[4]) follows from Theorem 3 (with $c_2 = 0$ in (1.13)).
For the second order ordinary differential equation

$$L_2 u = [\partial_t^2 + q_1(t,\xi)\,\partial_t + q_2(t,\xi)]\, u(t,\xi) = 0, \quad t \in]0,T[, \quad q_i(\cdot,\xi) \in C^2(]0,T[), \tag{1.19}$$

we obtain the following corollary of Theorem 2.

Theorem 4 Let non-negative constants c_1, c_2 and functions

$$\varphi_1(.,\xi),\ \varphi_2(.,\xi) \in C^2(]0,T[) \tag{1.20}$$

exist such that for $k, j = 1, 2,\ \xi \in R^n$

$$\int_0^T \|W^{-1}(\varphi_1,\varphi_2)\varphi_{3-j}\, L_2\varphi_k\|\, dt = \int_0^T |\varphi_j^{-1}(\varphi_{1t}\varphi_1^{-1} - \varphi_{2t}\varphi_2^{-1})^{-1} L_2\varphi_k| dt \le c_1 + c_2 \ln < \xi > . \tag{1.21}$$

Then for any $g_j(\xi) \in L_2(R^n)$, $j = 1, 2$, there exists a solution $u(t, \xi)$ of the equation (1.19) representable in the form (1.10), (1.10'), with $m = 2$.
Remark 4. If we take

$$\varphi_j(t, \xi) = \exp \int_T^t \sigma_j(t, \xi)\, dt, \quad j = 1, 2, \tag{1.22}$$

then the condition (1.21) becomes

$$\int_0^T |\varphi_j^{-1}(\sigma_1 - \sigma_2)^{-1}\varphi_k(\sigma_{jt} + \sigma_j^2 + q_1\sigma_j + q_2)|\, dt \le c_1 + c_2 \ln < \xi > \tag{1.23}$$

for all $\xi \in R^n, \quad k, j = 1, 2.$
If instead of (1.22) we take

$$\varphi_1 = \exp \int_T^t \sigma(t, \xi) dt, \quad \varphi_2 = \varphi_1(t, \xi) \int_T^t \varphi_{2,1}(t, \xi) dt, \tag{1.24}$$

where $\varphi_{2,1}$ is the solution of the initial value problem

$$(\partial_t + 2\varphi_{1t}\varphi_1^{-1} + q_1)\varphi_{2,1}(t, .) = 0, \quad t \in]0, T[, \quad \varphi_{2,1}\mid_{t=T} = 1, \tag{1.25}$$

then

$$\varphi_1^{-1} L\varphi_1 = \varphi_2^{-1} L\varphi_2, \quad W(\varphi_1, \varphi_2) = \varphi_1^2\varphi_{21}. \tag{1.25'}$$

Remark 5. Let there exist a function

$$\sigma(., \xi) \in C^1(]0, T[) \tag{1.26}$$

such that for some nonnegative constants c_1, c_2

$$\int_0^T |\varphi_1^2(t)(\int_T^t (\varphi_{21}(\tau) d\tau)^s (\sigma_t + \sigma^2 + q_1\sigma + q_2)| dt \le c_1 + c_2 \ln < \xi >; \quad s = 0, 1, 2, \quad x \in R^n. \tag{1.27}$$

By Theorem 4 there exists a solution $u(t, \xi)$ of the equation (1.19), representable in the form (1.10), (1.10′) with $m = 2$, where the functions φ_1, φ_2 are defined by (1.24).
Remark 6. If in the equation (1.19) $q_1 = 0$, $q_2 = q > 0$ and in (1.22) $\sigma_1 = iq^{1/2} - q_t/(4q)$, $\sigma_2 = -iq^{1/2} - q_t/(4q)$, then by Remark 5 the well known (see [4]) JWKB- estimates (1.10),(1.10″) with $m = 2$ follow. Indeed, for the equation

$$u_{tt} + q(t, \xi)u(t, \xi) = 0, \tag{1.28}$$

where the functions φ_1, φ_2 are defined by

$$\varphi_1 = q^{-1/4} \exp \left\{ i \int_T^t q^{1/2}\, dt \right\}, \quad \varphi_2 = q^{-1/4} \exp \left\{ -i \int_T^t q^{1/2}\, dt \right\}, \tag{1.29}$$

the conditions (1.21) (with $c_2 = 0$) take the form

$$q^{-1/4}(q^{-1/4})_{tt} \in L_1([0, T]). \tag{1.30}$$

Remark 7. If in (1.24)

$$\sigma = i(q - q_t^2 q^{-2}/16)^{1/2} q_t q^{-1}/4, \quad S(t) = (-q^{-1/2}/2)_t = q_t/(4q^{3/2}), \tag{1.31}$$

then by Remark 5 the representations (1.10), (1.10′′) hold, where the conditions (1.27) become

$$\int_0^T \mid \varphi_1^2(t)\left(\int_T^t (\varphi_1^{-2}(y)\,dy\right)^k ((S^2-1)^{1/2}-S)_t \mid dt \le c_1+c_2 \ln <\xi>, \quad k=0,1,2. \tag{1.32}$$

This follows from $\sigma_t + \sigma^2 + q(t) = q^{1/2}((S^2-1)^{1/2} - S)_t$.

Remark 8. If we define φ_1, φ_2 by (1.24), (1.31), then for the equation (1.28) the conditions (1.27) become

$$\int_0^T |\varphi_j\varphi_k(\sigma_t+\sigma^2+q|\,dt \le c_1+c_2\ln <\xi>, \quad j,k=1,2. \tag{1.33}$$

If in (1.24)

$$\sigma(t) = \alpha_1(t) + \alpha_2(t) + \ldots + \alpha_n(t), \tag{1.34}$$

where α_j are the solutions of the first order equations

$$\alpha_t(t) + q(t) = 0, \quad \alpha_j'(t) + \alpha_{j-1}^2(t) + 2\alpha_j(t)\left[\alpha_1 + \ldots + \alpha_{j-1}\right] = 0, \quad j = 2,\ldots,n, \tag{1.35}$$

then we obtain

$$\sigma_t + \sigma^2 + q(t) = \alpha_n^2. \tag{1.36}$$

Remark 9. Let for some $n \ge 1$

$$\varphi_j\varphi_k\alpha_n^2 \in L_1([0,T]), \quad k,j=1,\ldots,m, \tag{1.37}$$

where the functions φ_j are defined by (1.24), (1.34), (1.35). Then there exists a solution $u(.,\xi) \in C^2(]0,T])$ of the equation (1.28), representable in the form (1.10), (1.10′′) with $m = 2$.
Consider the m-th order ($m \ge 2$) partial differential equation

$$Lu = \partial_t^m u + \sum_{k=0}^{m-1}\sum_{|\alpha|\le m-k} a_{k\alpha}(t)\partial_t^k\partial_x^\alpha u(t,x) = 0, \quad t \in]0,T[, \quad x \in R^n, \tag{1.38}$$

where α is a multi-index and

$$\partial_t = \partial/\partial_t, \quad \partial_x = (\partial/\partial x^1, \ldots, \partial/\partial x^n). \tag{1.39}$$

.
Using Fourier x-transformation (1.4), from (1.38) we get the ordinary differential equation (1.1) with

$$q_0 = 1, \quad q_{m-k}(t,\xi) = \sum_{|\alpha|\le m-k} a_{k\alpha}(t) i^{|\alpha|}\xi^\alpha. \tag{1.40}$$

The Cauchy problem

$$\partial_t^k u(0,x) = g_k(x), \quad x \in R^n, \quad k = 0,\ldots,m-1 \tag{1.41}$$

for the equation (1.38) is called well posed or correct if for any $g_k(x) \in H^\infty$ there exist unique solution $u \in C^m([0,T]), H^\infty)$ of (1.38), (1.41) and for any $l, \varepsilon \ge 0$ there exist $p, \delta \ge 0$ such that

$$\sum_{k=0}^{m-1} ||g_k(x)||_p \le \delta \quad \text{implies} \quad \sum_{j=0}^{m-1} \max_{[0,T]} ||\partial_t^j u(t,x)||_l \le \varepsilon.$$

Let $\{\psi_j(t,\xi)\}_{j=1}^m$ be a fundamental system of solutions of (1.1), (1.40).
The following theorem of Petrovsky is well known.

Theorem 5 The Cauchy problem (1.38), (1.41) is well posed if and only if there exist $c > 0$, $p \in R$ such that

$$|\psi_j(t,\xi)| \leq c\,(1 + |\xi|^p), \quad j = 1, \dots m, \quad \xi \in R^n, \quad t \in [0,T]. \tag{1.42}$$

From Theorems 2 and 5 follows the following statement

Theorem 6 Let there exist functions $\{\varphi_j(t,\xi)\}_{j=1}^m$ satisfying conditions (1.2), (1.5) and (1.8). The conditions

$$|\varphi_j(t,\xi)| \leq c\,(1 + |\xi|^p), \quad j = 1, \dots m, \quad \xi \in R^n, t \in [0,T] \tag{1.43}$$

are necessary and sufficient for correctness of the Cauchy problem (1.38), (1.41).

3. New formula for energy. Proof of theorem 1

Let $\{\psi_j(t,\xi)\}_{j=1}^m$ be the fundamental system of solutions of the equation (1.1). Then the general solution of the (1.1) is

$$u(t,\xi) = C_1(\xi)\psi_1(t,\xi) + \ldots + C_m(\xi)\psi_m(t,\xi), \tag{2.1}$$

where $C_j(\xi) \in L_2(R^n)$ are arbitrary functions.
Differentiating (2.1) k times by t $(k = 0, 1, \dots, m-1)$ and resolving by $C_j(\xi)$ we get

$$C_j(\xi) = (1/W(\psi))\, W(\psi_1, \dots, \psi_{j-1}, u, \psi_{j+1}, \dots, \psi_m), \tag{2.2}$$

The nonnegative expression

$$\tilde{E}(t,\xi) = \sum_{j=1}^{m} |C_j(\xi)|^2 \tag{2.3}$$

we call energy density.
Then energy conservation law $\partial_t \tilde{E} = 0$ follows from independence of $C_j(\xi)$ on time.
For arbitrary $\varphi_j(.,\xi) \in C^m(]0,T])$, $j = 1, \dots, m$, with $W(\varphi) \neq 0$ we define auxiliary functions $u^j(t,\xi)$, $j = 1, \dots, m$ putting

$$\partial_t^k u = u^j(t,\xi)\, \partial_t^k \varphi_j(t,\xi), \quad k = 0, \dots, m-1, \tag{2.4}$$

where summations over $j = 1, \dots, m$ are done. Bellow for φ_j we will choose approximate solutions of the equation (1.1).
From (2.4) using (1.9) we get

$$u^j(t,\xi) = W(\varphi_1, \dots, \varphi_{j-1}, u, \varphi_{j+1}, \dots, \varphi_m)/W(\varphi)\},$$

$$u^j(t,\xi) = g_j(\xi) + o(1), \quad t \to 0, \quad j = 1, \dots, m \tag{2.5}$$

while (2.4) and (1.1) imply

$$u_t^j(t,\xi)\partial_t^k \varphi_j(t,\xi) = 0, \quad k = 0, 1, \dots, m-2, \quad u_t^j(t,\xi)\, \partial_t^{m-1} \varphi_j(t,\xi) = -u^j(t,\xi) L\varphi_j(t,\xi), \quad j = 1, ..., m \tag{2.6}$$

and

$$\partial_t u^j(t,\xi) = -u^k(t,\xi)\, \phi^{mj} L\varphi_k / W(\varphi), \quad j = 1, \dots, m, \tag{2.7}$$

where Φ^{mj} are algebraic minors of the matrix Φ.

Multiplying (2.7) by $\bar{u^j}$ and adding the complex conjugate relation, we get

$$\partial_t E(t,\xi) \le cK(t,\xi)E(t,\xi), \tag{2.8}$$

where

$$K(t,\xi) = \sum_{k,j=1}^{m} |W^{-1}(\varphi)\Phi^{mj} L\varphi_k|, \quad E(t,\xi) = \sum_{j=1}^{m} |u^j(t,\xi)|^2. \tag{2.9}$$

Here E is the approximate energy density considered in [6]. Using Gronwall inequality, we obtain from (2.8) the main energy estimate

$$E(t,\xi) \le E(0,\xi) \exp\{c \int_0^t K(\tau,\xi)\, d\tau\}. \tag{2.10}$$

In view of (1.8), (2.5) and $E(0,\xi) = \sum\limits_{j=1}^{m} |g_j|^2$ this implies Theorem 1.

4. Proof of theorems 2 and 3

We remind Levinson's asymptotic theorem. Consider the system of ordinary differential equations

$$y'(t) = [R(t) + B(t)]y(t), \quad t \in]0,T[, \tag{3.1}$$

where $R(t) = \mathrm{diag}\{r_1(t), \dots, r_m(t)\}$ is a diagonal matrix. We assume that $R(t)$ and $B(t)$ are $m \times m$ matrices and

$$R(t), \quad B(t) \in C(]0,T]). \tag{3.2}$$

If $B(t) = 0$, then (3.1) has the system of solutions

$$y_j(t) = \exp \int_T^t r_j(s)\, ds, \quad j = 1, \dots, m. \tag{3.3}$$

Theorem 3.1 (Levinson) The conditions

1) $\int_0^T |||B(s)||\, ds < \infty$ and

2) $Re\{r_j(s) - r_k(s)\}$ do not change the sign for any fixed j and $k = 1, \dots, m.$ $s \in]0,T]$

imply that the system (3.1) has solutions $y_j(t), \quad j = 1, \dots, m$ such that

$$y_{jk}(t) = (\delta_{jk}(t) + \epsilon_{jk}(t)) \exp \int_T^t r_j(s)\, ds, \tag{3.4}$$

$$\lim_{t \to 0} \epsilon_{kj}(t) = 0, \quad k,j = 1, \dots, m. \tag{3.4'}$$

Proof of Theorem 2 Transform (1.1) to the form

$$v_t + Av = 0, \quad t \in]0,T[, \tag{3.5}$$

where

$$v = \mathrm{colon}(u, u_t, \dots, \partial_t^{m-1} u), \quad A = \begin{pmatrix} 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_m & \dots & \dots & q_2 & q_1 \end{pmatrix}$$

Letting

$$v = \Phi w \tag{3.6}$$

from (3.5) we obtain the system

$$w_t + Gw = 0, \quad t \in]0, T[, \tag{3.7}$$

where $G = \Phi^{-1}(A\Phi + \Phi_t)$. By direct computation we get

$$G_{kj} = \Phi^{mj} L\varphi_k/(W(\varphi), \quad k, j = 1, \ldots, m. \tag{3.8}$$

Applying to Theorem 3.1 (3.7) with $R = 0$, $B = -G$, in view of (1.8) ($c_2 = 0$), we get

$$w_{kj} = \delta_{kj} + \epsilon_{kj}(t, \xi), \quad k, j = 1, \ldots, m, \tag{3.9}$$

where δ_{kj} is the Kroneker symbol, and

$$\epsilon_{kj} \to 0, \quad \text{when} \quad t \to 0. \tag{3.10}$$

Returning to the previous variables (3.6) we get (1.10), (1.10$''$).
Let us prove the estimates (1.11). Integrating (2.7) over $[0, t]$ yields

$$u^j(t, \xi) = g_j(\xi) - \int_0^t W^{-1}(\varphi) u^k(\tau, \xi) \phi^{mj} L\varphi_k(\tau, \xi)\, d\tau, \quad j = 1, \ldots, m \tag{3.11}$$

or

$$|u^j(t, \xi)| \le |g_j(\xi)| + \int_0^t |W^{-1}(\varphi) u^k(\tau, \xi)\, \phi^{mj}\, L\varphi_k(\tau, \xi)|\, d\tau, \quad j = 1, \ldots, m.$$

Adding up these estimates we obtain

$$U(t, \xi) \le G(\xi) + \int_0^t K(\tau, \xi)\, U(\tau, \xi)\, d\tau, \tag{3.12}$$

where

$$U(t, \xi) = \sum_{j=1}^{m} |u^j(t, \xi)|, \quad G(\xi) = \sum_{j=1}^{m} |g_j(\xi)|,$$

$$K(\tau, \xi) = \sum_{k,j=1}^{m} |W^{-1}(\varphi) \phi^{mj}\, L\varphi_k(\tau, \xi)|.$$

By Gronwall inequality, from (3.12) we get

$$U(t, \xi) \le G(\xi) \exp \int_0^t K(\tau, \xi)\, d\tau. \tag{3.13}$$

For $\varepsilon_j(t, \xi) = u^j(t, \xi) - g_j(\xi), \quad j = 1, \ldots, m$, in view of (3.11), (3.13) we obtain

$$|\varepsilon_j(t, \xi)| \le \int_0^t |W^{-1}(\varphi)\, u^k(\tau, \xi) \Phi^{mj}\, L\varphi_k(\tau, \xi)\,|\, d\tau \le \int_0^t K(\tau, \xi)) U(\tau, \xi)\, d\tau \le$$

$$\le G(\xi) \int_0^t K(\tau, \xi) \left\{ \exp \int_0^\tau K(y, \xi)\, dy \right\} d\tau \le G(\xi) \left[-1 + \exp \int_0^t K(\tau, \xi)\, d\tau \right].$$

The proof of (1.11) is complete. Expressions (1.10), (1.10$'$) may be obtained in similar way, after we note that using Lebesgue theorem from (1.8) $c_2 = 0$, (1.11) we can get (1.10$'$).
From (1.8) we get $\exp \int_0^t K(\tau, \xi)\, d\tau \le c < \xi >^{m^2 c_2}$, and from (1.11) we obtain the estimates (1.11$'$).

Proof of Theorem 3 Letting

$$v = \Psi w \tag{3.14}$$

from (1.12) we get the system of equations

$$w_t = \Psi^{-1}(A\Psi - \Psi_t)w. \tag{3.15}$$

Applying Theorem 3.1 to (3.15) we obtain Theorem 3.

5. Examples

Example 1. Consider mth order ordinary differential equation

$$\partial_t^m y(t) + \sum_{j=1}^{m} q_j(t)\, \partial_t^{m-j}\, y(t) = 0. \tag{4.1}$$

Let p_j $(j = 1, \dots, m)$ be the roots of the characteristic equation

$$l(t,p) = p^m + \sum_{j=1}^{m} q_j(t) p^j = 0, \tag{4.2}$$

and the functions φ_j be defined as (see [4]):

$$\varphi_j = \exp \int_T^t \sigma_j(s)\, ds, \quad \sigma_j(s) = p_j(s) - \sum_{j=1}^{m} \frac{p'(s)}{p_j(s) - p_k(s)}, \quad j = 1, \dots, m. \tag{4.3}$$

Under assumptions (1.8) of Theorem 2, solutions of the equation (4.1) exist that can be represented in the form (1.10), (1.10$''$).

Example 2. Let in the equation (1.28)

$$q(t,\xi) = \beta t^{\varepsilon - 2}. \tag{4.4}$$

Case 1: $\varepsilon < 0$, $\beta > 0$. In this case the condition (1.30) is fulfilled and one can get asymptotic solutions using JWKB-estimates (see Remark 6).
Case 2: $\varepsilon = 0$. (Euler equation). In this case exact solutions have the form t^λ.
Case 3: $\varepsilon > 0$. If we choose in (1.24), (1.34)

$$\alpha_1(t) = \beta(1-\varepsilon)^{-1}\, t^{\varepsilon - 1}, \quad \varepsilon \neq 1,$$

$$\alpha_1(t) = -\beta \ln t, \quad \varepsilon = 1, \tag{4.5}$$

then from (1.35) we get the expressions

$$\alpha_{k+1}(t) \sim t^{2^k \varepsilon - 1}, \quad k = 1, \dots, n. \tag{4.6}$$

If $\varepsilon \geq 2^{-n}$, then the conditions (1.37) are fulfilled and Remark 9 can be applied.

Example 3. Consider the equation (1.19) with $q = |\xi|^2 + \alpha(t)$, that is

$$[\partial_t^2 + |\xi|^2 + \alpha(t)]\widehat{u}(t,\xi) = 0, \quad t \in]0,T[. \tag{4.7}$$

Letting

$$\sigma_{1,2} = \pm i \mid \xi \mid + \beta(t), \tag{4.8}$$

from the condition (1.23) ($c_2 = 0$) we obtain

$$\varphi_j^{-1}\varphi_k \left\{ \beta + \frac{\beta_t + \alpha + \beta^2}{2i|\xi|} \right\} \in L_1([0,T]), \quad k,j = 1,2, \quad \xi \in R^n, \tag{4.9}$$

or

$$\beta, \quad \beta_t + \beta^2 + \alpha \in L_1([0,T]). \tag{4.10}$$

Let

$$\beta(t) = \alpha_1(t) + \ldots + \alpha_n(t), \tag{4.11}$$

where α_n are defined by (1.35).
In the case

$$\alpha = t^{-\gamma}, \quad 0 < \gamma < 2 \tag{4.12}$$

from (1.35) we get

$$\alpha_1 = (\gamma - 1)^{-1}\, t^{1-\gamma}, \quad \alpha_r \sim t\alpha_{r-1}^2 \sim t^w, \quad w = 2^r - 1 - 2^{r-1}\gamma. \tag{4.13}$$

From (4.13) follow the conditions (1.37) of Remark 9:

$$\alpha_n^2 \in L_1([0,T]), \quad \text{for} \quad 0 < \gamma < 2. \tag{4.14}$$

From Remark 9 the following proposition follows.

Proposition 1 Let the condition (4.14) be satisfied. Then for the solutions u of the equation (4.7) the representations (1.10), (1.10$''$) (with $m = 2$) hold, where the functions φ_j are defined by (1.22), (4.8), (4.11), (1.35).

Example 4. Consider the radial Shrödinger equation

$$L\psi = [\partial_r^2 + k^2 - V(r)]\psi = 0 \tag{4.15}$$

with generalized Coulomb potential

$$V = \rho_0 r^{-2} + \rho_1 r^{-5/3} + \rho_2 r^{-4/3} + \rho_3 r^{-1} + \rho_4 r^{-2/3} + \rho_5 r^{-1/3},$$
$$\rho_0 = l(l+1), \quad l \geq 0.$$

Case 1: $r \to 0$. We look for approximate solutions of (4.15) in the form

$$\varphi_1 = r^{\alpha_1}\mu_1(r)\exp(ikr), \quad \varphi_2 = r^{\alpha_2}\mu_2(r)\exp(-\mathrm{ikr}), \tag{4.16}$$
$$\mu_1 = 1 + \beta_1 r^{1/3} + \beta_2 r^{2/3} + \beta_3 r + \ldots + \beta_s r^{s/3},$$
$$\mu_2 = 1 + \gamma_1 r^{1/3} + \gamma_2 r^{2/3} + \gamma_3 r + \gamma_4 r^{4/3} + \ldots + \gamma_s r^{s/3}.$$

The coefficients σ_j and m_j we define from the expressions:

$$\mu_1^{-1}\mu_1' = \sigma_0 r^{-2/3} + \sigma_1 r^{-1/3} + \sigma_2 + \sigma\dot{3} r^{1/3} + \ldots, \quad \sigma_0 = \beta_1/3,$$
$$\mu_1^{-1}\mu_1'' = m_0 r^{-5/3} + m_1 r^{-4/3} + m_2 r^{-1} + \ldots, \quad m_0 = -2\beta_1/9. \tag{4.17}$$

By direct computation

$$\varphi_1^{-1}\, L\varphi_1 = (\alpha^2 - \alpha - \rho_0)r^{-2} + (2\alpha\sigma_0 + m_0 - \rho_1)r^{-5/3} + (2\alpha\sigma_1 + m_1 - \rho_2)r^{-4/3} + \ldots + br^{(s-6)/3} + 0(r^{(s-5)/3}). \tag{4.18}$$

We choose $\alpha, \beta_j,\ j = 1, \ldots, s$ to reduce the coefficients by the powers of r to zeros. This yields

$$\alpha_1 = l + 1, \quad \alpha_2 = -l, \quad \beta_1 = \frac{9\rho_1}{2(3\alpha - 1)} \tag{4.19}$$

We also have

$$W(\varphi_2, \varphi_1) = \varphi_1\varphi_2[(\alpha_1 - \alpha_2)/r + 2ik + (\ln(\mu_1/\mu_2))'] = 1 + 0(r^{1/3}), \quad r \to 0.$$
$$W^{-1}(\varphi_1, \varphi_2)\varphi_k L\varphi_j \leq c r^{-2l + (s-5)/3} \in L_1([0,1]), \quad k, j = 1, 2, \tag{4.20}$$

if $s > 6l + 2$.

Case 2: $k > 0$, $r \to \infty$. We choose approximate solutions of (4.15) in the form

$$\varphi_{1,2} = \exp\left\{\pm i \int \{k^2 - V_1\}^{1/2}\, dr\right\}, \quad V_1(r) = \rho_3/r + \rho_4/r^{2/3} + \rho_5/r^{1/3}. \tag{4.21}$$

The condition $k > 0$ implies existence of R such that $k^2 - V_1 > 0$ for $r > R$.
We have

$$W(\varphi_2, \varphi_1) = 2i\{k^2 - V_1\}^{1/2} = 0(1); \quad \varphi_1, \varphi_2 = O(1), \quad r \to \infty,$$

$$W^{-1}(\varphi_1, \varphi_2)\varphi_k L\varphi_j = O(r^{-4/3}) \in L_1[1, \infty), \quad k, j = 1, 2.$$

Case 3: $k = 0$, $\rho_5 < 0$, $\quad r \to \infty$.
We choose approximate solutions of (4.15) in the form

$$\varphi_{1,2} = V_2^{-1/4} \exp \pm i \left\{ \int V_2^{1/2}(r)\, dr \right\}, \quad V_2 = -\rho_3/r - \rho_4/r^{2/3} - \rho_5/r^{1/3}, \tag{4.22}$$

$$W^{-1}(\varphi_1, \varphi_2)\varphi_k L\varphi_j = O(r^{-7/6}) \in L_1[1, \infty), \quad k, j = 1, 2.$$

For the general solution ψ of the equation (4.15) we will have

$$\partial_t^k \psi = [C_1 + o(1)]\partial_r^k \varphi_1 + [C_2 + o(1)]\partial_r^k \varphi_2, \quad r \to 0 \quad \text{or} \quad r \to \infty, \quad k = 0, 1, \tag{4.23}$$

where φ_1, φ_2 are defined by (4.16),(4.21) and (4.22).
Note that in case $k = 0$, $\rho_5 > 0$, $r \to \infty$ the conditions of Theorem 2 are satisfied and representations (4.23) with asymptotic solutions (4.22) can be proved only for $k = 0$ ([3]).

Example 5. Consider the Chaplygin equation ([1])

$$F_{\theta\theta} - \frac{1 - ev^2c^{-2}}{c^{-2} - v^{-2}} F_{vv} + vF_v = 0, \quad e = \frac{\gamma + 1}{\gamma - 1} \tag{4.24}$$

or

$$F_{vv} - \lambda^2 F_{\theta\theta} - v\lambda^2 F_v = 0,$$

where

$$\lambda = \sigma(v - c)^{1/2}c^{-3/2}, \quad \sigma = v^{-1}c^{3/2}(c + v)^{1/2}(c^2 - ev^2)^{-1/2}.$$

We want to find asymptotic solutions of the equation (4.24) near singular hyperplane $v = c$. Substituting

$$F = a(v)u(v, \theta) \tag{4.25}$$

we obtain the equation

$$u_{vv} - \lambda^2 u_{\theta\theta} + \left(\frac{2a_v}{a} - v\lambda^2\right) u_v + \left(\frac{a_{vv}}{a} - v\lambda^2 \frac{a_v}{a}\right) u = 0.$$

Choosing

$$a = \exp\left[\frac{1}{2}\int_c^v v\lambda^2(v)\, dv\right] \tag{4.26}$$

and applying Fourier transformation

$$\hat{u}(v, \xi) = \int \exp(-i\theta\xi) u(v, \theta) d\theta,$$

we get the equation

$$\hat{u}_{vv} + (\xi^2\lambda^2 + 2^{-1}(v\lambda^2)_v - 2^{-2}v^2\lambda^4)\hat{u} = 0. \tag{4.27}$$

The transformation

$$\hat{u} = z(\tau)\{|\xi|\lambda(v)\}^{1/2}, \quad \tau = |\xi| \int_c^v \lambda(v) dv \tag{4.28}$$

yields

$$P_1 z = z_{\tau\tau} + (1 + \beta\tau^{-2})z(\tau) = 0, \tag{4.29}$$

where

$$\beta = [1/2 + v\lambda_v/\lambda - v^2\lambda^2/4 + 4^{-1}\lambda^{-4}(3\lambda_v^2 - 2\lambda\lambda_{vv})]\left[\int_c^v \lambda(v)\,dv\right]^2. \qquad (4.30)$$

We look for asymptotic solutions of (4.29) near singular points $\tau = 0$, $\tau = \infty$. Denoting $x = \frac{v}{c} - 1$, we get

$$\lambda = x^{1/2}\sigma c^{-1}, \quad \sigma = (x+1)^{-1}(x+2)^{1/2}[1 - e(x+1)^2]^{-1/2}.$$

Near $x = 0$ we have

$$\sigma = \sigma_0 + \sigma_1 x + \sigma_2 x^2 + \sigma_3 x^3 + \ldots,$$

where

$$\sigma_0 = 2^{1/2}(1-e)^{-1/2} = (\gamma+1)^{1/2}, \quad \frac{\sigma_1}{\sigma_0} = \frac{e}{1-e} - \frac{3}{4} = \frac{\gamma-1}{2} - \frac{3}{4},$$

and so on. We determine ρ_0, $\rho_1, \ldots$ from the relations

$$\frac{\lambda_x}{\lambda} = \frac{1}{2x} + \frac{\sigma_x}{\sigma} = \frac{1}{2x} + \rho_0 + \rho_1 x + \rho_2 x^2 + O(x^3),$$

where

$$\rho_0 = \frac{\sigma_1}{\sigma} = (\gamma-1)/2 - 3/4, \quad \rho_1 = \frac{2\sigma_2}{\sigma_0} - (\sigma_1/\sigma_0)^2 = \frac{5}{4}(\gamma-1)^2 + \frac{\gamma-1}{4} + \frac{37}{16},$$

and so on. From (4.28) we get

$$\tau = 2x^{3/2}|\xi|\{\sigma_0/3 + x\sigma_1/5 + \ldots\} = 2|\xi|\left(\frac{v}{c} - 1\right)^{3/2}[(\gamma+1)/3]^{1/2}$$

and

$$x = \frac{v}{c} - 1 = [3\tau|\xi|^{-1}(\gamma+1)^{-1/2}]^{2/3}2^{-2/3}.$$

We define k_0, $k_1, \ldots$ by the relation

$$\left[\int_c^v \lambda(v)\,dv\right]^2 \lambda^{-2} = 4c^2x^2(k_0 + k_1 x + O(x^2).$$

This yields

$$k_0 = 1/9, \quad k_1 = \frac{1}{15} - \frac{\sigma_2}{9\sigma_0} = -\frac{(\gamma-1)^3}{12} + \frac{11}{72}(\gamma-1)^2 - \frac{97}{720}(\gamma-1) + \frac{67}{480},$$

$$\frac{3\lambda_v^2}{4\lambda^2} - \frac{\lambda_{vv}}{2\lambda}\left(\int_c^v \lambda(s)ds\right)^2\lambda^{-2} = \frac{5}{36} + \frac{5}{48}\left[\frac{7}{8} - \frac{13}{12}(\gamma-1) + \frac{11}{6}(\gamma-1)^2 - (\gamma-1)^3\right]x + O(x^2).$$

Representing (4.30) in the form

$$\beta = \beta_0 + \beta_1\tau^{1/3} + \beta_2\tau^{2/3} + +O(\tau),$$

we find

$$\beta_0 = 5/36, \quad \beta_1 = 0, \quad \beta_2 = \frac{5}{48}\left[\frac{7}{8} - \frac{13}{12}(\gamma-1) + \frac{11}{6}(\gamma-1)^2 - (\gamma-1)^3\right]\left[3|\xi|^{-1}(\gamma+1)^{-1/2}2^{-1}\right]^{2/3}.$$

We are looking for solution z of the equation (4.29) in the form

$$z_1 = \tau^{\gamma_1} + m_1\tau^{\alpha_1} + m_2\tau^{\alpha_2} + 0(\tau^{\alpha_3}), \quad z_2 = \tau^{\gamma_2}. \qquad (4.31)$$

For small τ, substitution of (4.31) in (4.29) and the choice

$$\gamma(\gamma-1) + \beta_0 = 0, \quad \alpha_1 = \gamma_1 + 1/3, \quad \alpha_2 = \gamma_1 + 2/3, \quad m_j = \frac{\beta_j}{\alpha_j(1-\alpha_j)}, \quad j = 1, 2.$$

or

$$\gamma_1 = 1/6, \quad \gamma_2 = 5/6, \quad \alpha_1 = 1/2, \quad \alpha_2 = 5/6, \quad m_1 = 0, \quad m_2 = \frac{\beta_2}{\alpha_2(1-\alpha_2)} = \frac{36}{5}\beta_2.$$

yields

$$P_1 z_1 = [\gamma_1(\gamma_1 - 1 + \beta_0)\tau^{\gamma_1 - 2} + \tau^{\alpha_1 - 2}[m_1\alpha_1(\alpha_1 - 1) + \beta_1] + \tau^{\alpha_2 - 2}[m_2\alpha_2(\alpha_2 - 1) + \beta_2] + \beta_3\tau^{\gamma_1 - 1} + \ldots = O(\tau^{\gamma_1 - 1})$$

or

$$z_1 P_1 z_1 = O(\tau^{2\gamma_1 - 1}), \quad z_2 P_1 z_2 = O(\tau^{2\gamma_2 - 1}).$$

In view of $\gamma_1 > 0, \quad 2\gamma_2 > 1$, we obtain estimations for error functions

$$\int_0^1 |z_j P_1 z_k| \, d\tau < \infty, \quad k, j = 1, 2.$$

Applying Theorem 2 we get

$$z = \begin{cases} [C_1 \exp(i\tau) + C_2 \exp(-i\tau)](1 + o(1)), & \tau \to \infty, \quad \tau \quad \text{is real}, \\ [C_3(\tau^{1/6} + m_2\tau^{\alpha_2}) + C_4\tau^{5/6}](1 + o(1)), & \tau \to 0, \quad \tau \quad \text{is real}, \\ C_5(1 + o(1)) \exp(-i\tau), & \tau \to \infty, \quad Re(i\tau) > 0, \end{cases}$$

or in the previous variables

$$\widehat{u} = \begin{cases} (C_1\psi_1 + C_2\psi_2)(1 + o(1)), & \tau \geq N \to \infty, \quad v > c, \\ (C_3\psi_3 + C_4\psi_4)(1 + o(1)), & \tau \leq N \to 0, \quad v > c, \\ C_5(1 + o(1))\psi_5, \quad \tau \geq N \to \infty, & c > v, \end{cases}$$

where

$$\psi_{1,2} = \{|\xi|\lambda(v)\}^{-1/2} \exp\left\{\pm i|\xi| \int_c^v \lambda(s)\, ds\right\}, \quad \psi_4 = |\xi|^{1/3}\lambda^{-1/2}\left(\int_c^v \lambda(s)\, ds\right)^{5/6},$$

$$\psi_3 = \{|\xi|\lambda(v)\}^{-1/2}\left(|\xi|\int_c^v \lambda(s)\, ds\right)^{1/6}\left[1 + \nu\left(\int_c^v \lambda(s)\, ds\right)^{2/3}\right],$$

$$\nu = (\gamma + 1)^{-1/3} 3^{5/3} 2^{-8/3}\left[\frac{7}{8} - \frac{13}{12}(\gamma - 1) + \frac{11}{6}(\gamma - 1)^2 - (\gamma - 1)^3\right].$$

If $\gamma = 1$, then $\nu = 3^{5/3}2^{-6}7$ and

$$\psi_5 = (|\xi|\mu)^{-1/2} \exp\left\{-|\xi| \int_v^c \mu(s) ds\right\}, \quad \mu = \frac{1}{v}(c^2 - v^2)^{1/2}(c^2 - ev^2)^{-1/2}.$$

Note that if $\mathrm{Re}\mu > 0$ then $\mu = i\lambda$.
Using the asymptotic behavior of λ we get

$$\lambda = (\gamma + 1)^{1/2} c^{-3/2}(v - c)^{1/2}, \quad \int_c^v \lambda(s)\, ds = 23^{-1}(\gamma + 1)^{1/2} c^{-3/2}(v - c)^{3/2},$$

and obtain an asymptotic representation for solutions of Chaplygin equation:

$$\widehat{u} = \begin{cases} [1 + o(1)]|\xi|^{-1/2}(v - c)^{-1/4}\{C_1 \exp[\frac{2i|\xi|}{3}(\gamma + 1)^{1/2}(v - c)^{3/2}c^{-3/2}\} + \\ C_2 \exp[\frac{-2i|\xi|}{3}(\gamma + 1)^{1/2}(v - c)^{3/2}c^{-3/2}\}, \quad \text{if} \quad v > c, \quad |\xi|(v - c)^{3/2} \to \infty, \\ (1 + o(1))\{C_3|\xi|^{-1/3}[1 + \nu_1(v - c)^{3/2}] + C_4(v - c)|\xi|^{1/3}\}, \quad v > c, \quad |\xi|(v - c)^{3/2} \to 0, \\ (1 + o(1))C_5|\xi|^{-1/2}(c - v)^{-1/4} \exp[\frac{-2i|\xi|}{3}(\gamma + 1)^{1/2}(c - v)^{3/2}c^{-3/2}\}, \\ \text{if} \quad c > v \quad |\xi|(c - v)^{3/2} \to \infty \end{cases}$$

$$\nu_1 = (\gamma+1)^{1/6} 3^{2/3} 2^{-5/3} c^{-3/2} \left[\frac{7}{8} - \frac{13}{2}(\gamma-1) + \frac{11}{6}(\gamma-1)^2 - (\gamma-1)^3 \right].$$

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