ADIABATIC INVARIANT FOR A LINEAR OSCILLATOR IN EXTERNAL FIELD

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SUMMARY

A mathematical definition of an adiabatic invariant is proposed. We find out an adiabatic invariant for a linear oscillator in an external variable field using this definition. Estimates of changes of this invariant are proved. This formula contains a denominator which decreases when the differences of frequencies tend to zero (resonance).

§ 1. MAIN RESULTS

An adiabatic invariant of the physical system is called the physical quantity that changes small, when the parameters of the system vary slowly. In other words the adiabatic invariant is an approximate first integral of the system. In [1] a connection between approximate integrals of energy and Wronskians of approximate solutions has been shown. Using this connection we find out an adiabatic invariant for a linear oscillator in external variable field and estimate its changes.

It is known (see, for example [2]) that for linear harmonic oscillator

$$\frac{d^2x}{dt^2} + \omega^2(\varepsilon t)x = 0, \quad t \in R,$$

an adiabatic invariant is the quantity

$$J(t, \varepsilon) = \frac{\dot{x}^2 + \omega^2 x^2}{2\omega} = \frac{E}{\omega},$$

where x is a solution of the equation with Cauchy data, independent of ε. The full change of $J(t, \varepsilon)$,

$$J(\varepsilon) = J(\infty, \varepsilon) - J(-\infty, \varepsilon)$$

may be estimated as

$$J(\varepsilon) = O(\varepsilon^m), \quad \varepsilon \to 0.$$

Here m is a natural number which depends on ω. Note that $m = \infty$ if ω is a holomorphic function in some neighbourhood of the real axis.

We consider the following differential equation of linear oscillator in external variable field:

$$\ddot{x} + \omega_1^2(\varepsilon t)x = \cos \int_0^t \omega_2(\varepsilon s)ds, \quad t \in R, \tag{1.1}$$

where $\ddot{x} = \frac{d^2x}{dt^2}$ and $\varepsilon > 0$ is a small parameter.
Definition. *We call adiabatic invariant of (1.1) the quantity*

$$J(t,\varepsilon) = I(t, x - x', \varepsilon) = \mid W(x - x', \varphi) \mid^2,$$

where x and x' are exact solutions of (1.1) with bounded by ε Cauchy data and φ is an asymptotic solution of the corresponding homogeneous equation.

Let frequencies ω_1, ω_2 satisfy the following conditions:
1° $\omega_1(\tau) \in C^4(R)$, $\omega_1(\tau) > 0$, and there exist limits $\omega_1(\pm\infty) > 0$.
2° $\omega_2(\tau) \in C^2(R)$, $\omega_2^2(\tau) \neq \omega_1^2(\tau)$, $\tau \in R$, there exist limits $\omega_2(\pm\infty)$, and $\omega_2^2(\pm\infty) \neq \omega_1^2(\pm\infty)$.
3°

$$\int\limits_{-\infty}^{\infty} \omega_1^{(k)}(\tau) d\tau < \infty, \quad k = 1, \ldots, 4 \quad \int\limits_{-\infty}^{\infty} \mid \omega_2^{(k)}(\tau) \mid d\tau < \infty, \quad k = 1, 2.$$

In this paper we find out an adiabatic invariant

$$J(t,\varepsilon) = \frac{E}{\omega_1 \left(1 - \frac{\omega_2^2}{\omega_1^2}\right)}, \tag{1.2}$$

where

$$E = \left(\left(1 - \frac{\omega_2^2}{\omega_1^2}\right) \dot{x} + \frac{\omega_2}{\omega_1^2} \sin \int\limits_0^t \omega_2(\varepsilon s) ds \right)^2 + \omega_1^2 \left(\left(1 - \frac{\omega_2^2}{\omega_1^2}\right) x - \frac{1}{\omega_1^2} \cos \int\limits_0^t \omega_2(\varepsilon s) ds \right)^2 \tag{1.3}$$

$x = x(t,\varepsilon)$ is a solution of (1.1) with bounded by ε Cauchy data. The changes of this adiabatic invariant satisfy the following estimates:
1) there exist C, ε', such that

$$\mid J(t_1,\varepsilon) - J(t_2,\varepsilon) \mid \leq C\varepsilon \tag{1.4}$$

for every $t_1, t_2 \in (-\infty; \infty)$, $0 < \varepsilon < \varepsilon'$,
2)

$$J(\varepsilon) = J(\infty,\varepsilon) - J(-\infty,\varepsilon) = O(\varepsilon), \quad \text{when} \quad \varepsilon \to 0. \tag{1.5}$$

Evidently the quantity E is the first integral of (1.1) when $\omega_m = \text{const}$. We also show that the full change of $J(t,\varepsilon)$, $J(\varepsilon) = J(\infty,\varepsilon) - J(-\infty,\varepsilon)$ is exponentially small if the following additional conditions are satisfied:
4° The function $\omega_1(\tau)$ is holomorphic and $\omega_1(\tau) \neq 0$ in the one-connected domain D of the complex plane τ, which contains the real axis. By the function

$$S(0,\tau) = \int\limits_0^\tau \omega_1(s) ds$$

the domain D is one-to-one mapped onto the band $\mathrm{H}_a : \mid \mathrm{Im} S \mid < \mathrm{a}$, 5°

$$\int\limits_{l} (|\, \dot{\omega}_1(t)\, |^2 + |\, \ddot{\omega}_1(t)\, |)\, |\, dt\, |< \infty,$$

where the integrals are taken over lines $\mathrm{Im}S(0,t) = \mathrm{c}$, $\mathrm{c} \in (-\mathrm{a}, \mathrm{a})$.

§ 2. ASYMPTOTIC SOLUTIONS OF THE EQUATION (1.1)

We can rewrite (1.1) in variable $\tau = \varepsilon t$

$$\ddot{v} + \frac{\omega_1^2(\tau)}{\varepsilon^2} v = \frac{1}{\varepsilon^2} \cos\left(\frac{1}{\varepsilon} \int\limits_0^\tau \omega_2(s) ds \right), \quad \tau \in R, \tag{2.1}$$

where $v = v(\tau, \varepsilon) = x(t, \varepsilon)$ and $\ddot{v} = \frac{d^2 v}{d\tau^2}$. First we will find the solutions of the auxiliary equation

$$\ddot{z} + \frac{\omega_1^2(\tau)}{\varepsilon^2} z = \frac{1}{\varepsilon^2} \exp\left(\frac{i}{\varepsilon} \int\limits_0^\tau \omega_2(s) ds \right), \quad \tau \in R. \tag{2.2}$$

The linearly independent solutions of the corresponding homogeneous equation

$$Pu = \ddot{u} + \frac{\omega_1^2(\tau)}{\varepsilon^2} u = 0, \quad \tau \in R,$$

with their derivatives can be represented in the following form (see, for example [3]):

$$u_j^{\pm,(k-1)} = \widetilde{u}_j^{(k-1)} (1 + \varepsilon^2 \rho_{j,k}^{\pm}), \quad j, k = 1, 2,$$

$$\text{where} \quad \widetilde{u}_j = \omega_1^{-1/2} \exp\left(\int\limits_0^\tau \left(\pm i \frac{\omega_1}{\varepsilon} + \varepsilon \frac{3\dot{\omega}_1^2 - 2\omega_1 \ddot{\omega}_1}{8\omega_1^3} \right) ds \right),$$

$\rho_{jk}^{\pm}(\tau, \varepsilon)$ are bounded for $\tau \in (-\infty; \infty)$, $\varepsilon > 0$ and $\rho_{jk}^{\pm} \to 0$ when $\tau \to \pm\infty$ for every fixed ε. The functions

$$z_0^{\pm} = \int\limits_0^\tau \frac{u_1^{\pm}(s,\varepsilon) u_2^{\pm}(\tau,\varepsilon) - u_1^{\pm}(\tau,\varepsilon) u_2^{\pm}(s,\varepsilon)}{\varepsilon^2 W(s, u_1^{\pm}, u_2^{\pm})} \exp\left(\frac{i}{\varepsilon} \int\limits_0^s \omega_2(l) dl \right) ds$$

are the solutions of (2.2). Here $W(s, u_1^{\pm}, u_2^{\pm}) = u_1^{\pm} \dot{u}_2^{\pm} - u_2^{\pm} \dot{u}_1^{\pm}$ are the Wronskians of the solutions $u_1^{\pm}(s, \varepsilon)$ and $u_2^{\pm}(s, \varepsilon)$. Using $1° - 3°$ and the expressions (see [3])

$$W(s, u_1^{\pm}, u_2^{\pm}) = -\frac{2i}{\varepsilon},$$

$$c \mid \varepsilon^2 \rho_{jk}^{\pm}(\tau, \varepsilon) \mid \leq \left\{ \exp \int\limits_{\tau}^{\infty} \sum_{q,r=1}^{2} \mid \widetilde{u}_r P \widetilde{u}_q W^{-1}(s, \widetilde{u}_1, \widetilde{u}_2) \mid ds \right\} - 1,$$

$$P\widetilde{u}_q = \left[\pm i\varepsilon \frac{d}{ds} \left(\frac{3\dot{\omega}_1^2 - 2\omega_1\ddot{\omega}_1}{8\omega_1^3} \right) \pm \varepsilon \frac{\dot{\omega}_1}{i\omega_1} \frac{3\dot{\omega}_1^2 - 2\omega_1\ddot{\omega}_1}{8\omega_1^3} - \varepsilon^2 \left(\frac{3\dot{\omega}_1^2 - 2\omega_1\ddot{\omega}_1}{8\omega_1^3} \right)^2 \right] \widetilde{u}_q,$$

$$\int\limits_{0}^{\tau} \exp(\theta_j) ds = \left. \frac{\exp\theta_j}{\dot{\theta}_j} \right|_0^{\tau} + \left. \frac{\ddot{\theta}_j \exp\theta_j}{\dot{\theta}_j^3} \right|_0^{\tau} - \int\limits_{0}^{\tau} \frac{d}{ds} \left(\frac{\ddot{\theta}_j}{\dot{\theta}_j^3} \right) \exp(\theta_j) ds,$$

where $c > 0$, and

$$\theta_j = \frac{i}{\varepsilon} \int\limits_{0}^{s} \omega_2 dl - \frac{1}{2} \ln \omega_1 \pm i \int\limits_{0}^{s} \left(\frac{\omega_1}{\varepsilon} + \varepsilon \frac{3\dot{\omega}_1^2 - 2\omega_1\ddot{\omega}_1}{8\omega_1^3} \right) dl,$$

we obtain

$$z_0^{\pm} = \frac{i}{2\varepsilon} u_2^{\pm}(\tau, \varepsilon) \left[\int_0^{\tau} \exp\theta_1 ds + \varepsilon^2 \left(\int\limits_{0}^{\infty} - \int\limits_{\tau}^{\infty} \right) \widetilde{u}_1(s, \varepsilon) \rho_{11}^{\pm}(s, \varepsilon) ds \right] -$$

$$- \frac{i}{2\varepsilon} u_1^{\pm}(\tau, \varepsilon) \left[\int_0^{\tau} \exp\theta_2 ds + \varepsilon^2 \left(\int\limits_{0}^{\infty} - \int\limits_{\tau}^{\infty} \right) \widetilde{u}_2(s, \varepsilon) \rho_{21}^{\pm}(s, \varepsilon) ds \right],$$

or

$$z_0^{\pm} = a_1^{\pm} u_1^{\pm} + a_2^{\pm} u_2^{\pm} + \frac{1}{\omega_1^2 - \omega_2^2} \exp \left(\frac{i}{\varepsilon} \int\limits_{0}^{\tau} \omega_2 ds \right) + \varepsilon q_1^{\pm},$$

$$\varepsilon \dot{z}_0^{\pm} = \varepsilon a_1^{\pm} \dot{u}_1^{\pm} + \varepsilon a_2^{\pm} \dot{u}_2^{\pm} + \frac{i\omega_2}{\omega_1^2 - \omega_2^2} \exp \left(\frac{i}{\varepsilon} \int\limits_{0}^{\tau} \omega_2 ds \right) + \varepsilon q_2^{\pm},$$

where $a_i^{\pm}$ depend on ε, q_j^-, q_j^+ are bounded for $\tau \in (-\infty, \infty)$, $\varepsilon > 0$ and $\lim\limits_{\tau \to \pm\infty} q_j^{\pm}(\tau, \varepsilon) = 0$. Consequently there exist solutions of (2.1) which have the form

$$v_0^{\pm} = \frac{1}{\omega_1^2 - \omega_2^2} \cos \left(\frac{1}{\varepsilon} \int\limits_{0}^{\tau} \omega_2 ds \right) + \varepsilon Re q_1^{\pm},$$

$$\varepsilon \dot{v}_0^{\pm} = - \frac{\omega_2}{\omega_1^2 - \omega_2^2} \sin \left(\frac{1}{\varepsilon} \int\limits_{0}^{\tau} \omega_2 ds \right) + \varepsilon Re q_2^{\pm}.$$

§ 3. ESTIMATES OF CHANGES OF $J(t,\varepsilon)$

Solutions v of (2.1) can be written in the forms

$$v = a^- u_1^- + b^- u_2^- + v_0^- = a^+ u_1^+ + b^+ u_2^+ + v_0^+ \tag{3.1}$$

where the coefficients $a^\pm, b^\pm$ depend on ε. Differentiating (3.1) and solving the obtained systems with respect to a^+, b^+ and a^-, b^-, correspondingly, it is easy to check that these coefficients are bounded for a solution of (1.1) with bounded by ε Cauchy data.

Theorem. *If the conditions* $1^\circ - 3^\circ$ *are fulfilled, then the quantity (1.2) with bounded Cauchy data* $x_0 = x(0,\varepsilon), x_1 = \dot{x}(0,\varepsilon)$ *is an adiabatic invariant of (1.1) and its changes satisfy the estimates* (1.4), (1.5). *In addition if* $4^\circ - 5^\circ$ *are satisfied, then*

$$J(\varepsilon) = J(\infty,\varepsilon) - J(-\infty,\varepsilon) = O(\exp(-b\varepsilon^{-1})),$$

where b *is an arbitrary number, such that* $0 < b < a$.

Proof. From (3.1) we have

$$\frac{i\varepsilon}{2} W(v - v_0^\pm, u_2^\pm) =$$

$$= \frac{\widetilde{u}_2^\pm}{2}\left[\omega_1\left(v - \frac{1}{\omega_1^2-\omega_2^2}\cos\frac{1}{\varepsilon}\int\limits_0^\tau \omega_2 ds\right) - i\left(\varepsilon\dot{v} + \frac{\omega_2}{\omega_1^2-\omega_2^2}\sin\frac{1}{\varepsilon}\int\limits_0^\tau \omega_2 ds\right)\right] + \varepsilon h_2^\pm = a^\pm$$

$$\frac{i\varepsilon}{2} W(u_1^\pm, v - v_0^\pm) =$$

$$= \frac{\widetilde{u}_1^\pm}{2}\left[\omega_1\left(v - \frac{1}{\omega_1^2-\omega_2^2}\cos\frac{1}{\varepsilon}\int\limits_0^\tau \omega_2 ds\right) + i\left(\varepsilon\dot{v} + \frac{\omega_2}{\omega_1^2-\omega_2^2}\sin\frac{1}{\varepsilon}\int\limits_0^\tau \omega_2 ds\right)\right] + \varepsilon h_1^\pm = b^\pm,$$

where $h_j^-(\tau,\varepsilon), h_j^+(\tau,\varepsilon)$ are bounded for $\tau \in (-\infty;\infty)$, $\varepsilon > 0$ and $\lim\limits_{\tau\to\pm\infty} h_j^\pm(\tau,\varepsilon) = 0$. Thus we have

$$J(t,\varepsilon) = a^-b^- + \varepsilon h^- = a^+b^+ + \varepsilon h^+, \tag{3.2}$$

where $h^\pm(t,\varepsilon)$ are bounded and $\lim\limits_{t\to\pm\infty} h_j^\pm(t,\varepsilon) = 0$. The estimate (1.4) immediately follows from (3.2). As it was mentioned above the coefficients $a^\pm, b^\pm$ are bounded. From results of [4] it follows that

$$a^+b^+ - a^-b^- = O(\varepsilon),$$

if $1^\circ - 3^\circ$, are fulfilled and

$$a^+b^+ - a^-b^- = O(\exp(-b\varepsilon^{-1})),$$

if $1^\circ - 5^\circ$ are fulfilled. The theorem is proved.

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