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POISSON'S INEQUALITY FOR A DIRICHLET PROBLEM ON A TIME SCALE

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This paper is dedicated to Professor Lynn Erbe

ABSTRACT. By separation of variables we derive Poisson's inequality for a Dirichlet problem in a circle on a time scale.

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1. Introduction. Poisson's inequality

The fundamental Poisson's formula characterizes solutions of the Dirichlet problem by their averages over circles (see [7]). Many generalizations of this formula have been proven (see for example [2, 3, 5, 6]), including extensions to partial differential equations on manifolds. In this paper we prove the version of Poisson's formula on a time scale, introduced by Hilger in [4].

Consider the Dirichlet problem in a circle with radius $r_0 > 0$

$$u^{\Delta\Delta}(r,\varphi) + \frac{u^{\Delta}(r,\varphi)}{\sigma(r)} + \frac{u_{\varphi\varphi}(r,\varphi)}{r\sigma(r)} = 0, \quad 0 < r < r_0, \quad 0 \le \varphi \le 2\pi \tag{1.1}$$

$$u(r_0,\varphi) = f(\varphi), \quad 0 \le \varphi \le 2\pi. \tag{1.2}$$

Here $u^{\Delta}(r,.)$ is the delta derivative (see [4, 1]) with respect to the variable r on a time scale $\mathbb{T}$. Also, $u_{\varphi}(\cdot,\varphi)$ denotes the partial derivative of the function $u(r,\varphi) : \mathbb{T} \times [0,2\pi] \to R$. The forward jump operator is defined by $\sigma(r) = \inf\{s \in \mathbb{T}, s > r\}$, where $r \in \mathbb{T}$.

The set of functions $p(r,\varphi) : \mathbb{T} \times [0,2\pi] \to \mathbb{R}$ that are rd-continuous in $r \in \mathbb{T}$ and continuous in the variable $\varphi \in [0,2\pi]$ will be denoted by C_{rd}. The set of functions $p(r,\varphi)$ such that their n-th delta derivative with respect to the variable r exists and is rd-continuous for $r \in \mathbb{T}$, and their m-th derivative with respect to $\varphi \in [0,2\pi]$ exists and is continuous on $[0,2\pi]$ is denoted by $C_{rd}^{(n,m)}$. We say that the real-valued function

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$p(r,\varphi)$ is regressive on $\mathbb{T}\times[0,2\pi]$ if $1+\mu(r)p(r,\varphi)\neq 0$ for all $r\in\mathbb{T},\varphi\in[0,2\pi]$. The set of regressive functions on $\mathbb{T}\times[0,2\pi]$ that belong to C_{rd} is denoted by $\mathcal{R}$.

If $1+\mu p\neq 0$, then the (generalized) exponential function $e_p(t,t_0)$ is the unique solution of the initial value problem

$$x^\Delta = p(t)x(t), \quad x(t_0)=1,$$

and is given by the formula (see [4, 1])

$$e_p(t,t_0)=\exp\left[\int_{t_0}^t \lim_{q\searrow\mu(s)} \frac{Log(1+qp(s))}{q}\Delta s\right] \tag{1.3}$$

where Log is the principal logarithmic function.

The set $\mathcal{R}$ along with the addition $\oplus$ defined by

$$p\oplus q := p+q+\mu pq \tag{1.4}$$

forms an Abelian group called the regressive group (see [1].) By $\mathcal{R}^{(n,m)}$ we denote the set of regressive functions that belong to $C_{rd}^{(n,m)}$. Note that (see [1])

$$e_p(r,r_0)e_q(r,r_0)=e_{p\oplus q}(r,r_0), \quad \frac{e_p(r,r_0)}{e_q(r,r_0)}=e_{p\ominus q}(r,r_0), \quad p\ominus q=\frac{p-q}{1+\mu q}. \tag{1.5}$$

Define the auxiliary functions

$$A(r)=\exp\int_{r_0}^r \left[\lim_{q\searrow\mu(y)}\left(\frac{\ln(q/y)}{q}\right)\right]\Delta y, \quad B(r)=\sum_{n=2}^\infty \frac{1}{n^{\int_r^{r_0}\lim_{q\searrow\mu(y)}\left(\frac{1}{q}\right)\Delta y}}, \tag{1.6}$$

and

$$K(r)=\exp\int_r^{r_0}\left[\lim_{q\searrow\mu(y)}\left(\frac{\ln(1+q/y)}{q}\right)\right]\Delta y. \tag{1.7}$$

Note that $B(r)$ is the analogue of the shifted Riemann zeta function over the real.

Theorem 1.1. *Assume*

$$\int_r^{r_0}\lim_{q\searrow\mu(y)}\left(\frac{1}{q}\right)\Delta y=\int_r^{r_0}\frac{\Delta y}{\mu(y)}>1, \quad r<r_0. \tag{1.8}$$

Then the solutions $u\in C_{rd}^{(2,2)}$ *of the Dirichlet problems* (1.1), (1.2) *satisfy Poisson's inequality*

$$\left|u(r,\varphi)-\int_0^{2\pi}P_1(r,\alpha)f(\alpha)d\alpha\right|\leq\frac{A(r)B(r)}{\pi}\int_0^{2\pi}|f(\alpha)|d\alpha, \tag{1.9}$$

where

$$P_1(r,\alpha)=\frac{K^2(r)-1}{2\pi[(K(r)-1)^2+4K(r)\sin^2((\alpha-\varphi)/2)]} \tag{1.10}$$

is the Poisson kernel.

Remark 1.2. For the continuous time scale we have $\mu(r)\to 0, \quad A(r)\to 0, \quad K(r)\to r_0/r$ and hence we get Poisson's formula

$$u(r,\varphi)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(r_0^2-r^2)f(\alpha)d\alpha}{r_0^2-2r_0r\cos(\alpha-\varphi)+r^2}. \tag{1.11}$$

Example 1.3. Consider the discrete time scale

$$\mathbb{T}_1 = \left\{0, \frac{r_0}{n}, \frac{2r_0}{n}, \dots, \frac{r_0 n}{n}\right\} = \{t_k\}_{k=0}^n, \quad y = t_k = \frac{kr_0}{n} = k\mu, \quad k = 0, \dots, n. \tag{1.12}$$

Then

$$\sigma(t) = t + \frac{r_0}{n}, \quad \mu(t) = \frac{r_0}{n}, \quad r = \frac{mr_0}{n}, \quad m \geq 1. \tag{1.13}$$

By taking $f(y) = \frac{1}{\mu}\ln(1 + \mu/y)$ we get

$$K(r) = \exp\int_r^{r_0} f(y)\Delta y = \exp\sum_{k=m}^{n-1}\mu(t)f(t_k) = \prod_{k=m}^{n-1}\left(1 + \frac{1}{k}\right), \tag{1.14}$$

$$A(r) = \exp\sum_{k=m}^{n-1}\ln(1/k) = \prod_{k=m}^{n-1}\frac{1}{k}, \quad B(r) = \sum_{n=2}^{\infty} n^{m-n}. \tag{1.15}$$

Assumption (1.8) is equivalent to $n - m = n - \frac{nr}{r_0} > 1$ or $r_0 > r$. Thus (1.9) gives

$$\left|u(r,\varphi) - \int_0^{2\pi} P_1(r,\alpha)f(\alpha)d\alpha\right| \leq \frac{A(r)B(r)}{\pi}\int_0^{2\pi}|f(\alpha)|d\alpha. \tag{1.16}$$

Or for $m < n - 1$,

$$\left|u(r,\varphi) - \int_0^{2\pi} P_1(r,\alpha)f(\alpha)d\alpha\right| \leq \frac{1}{\pi}\sum_{k=2}^{\infty} n^{m-n}\left(\prod_{k=m}^{n-1}\frac{1}{k}\right)\int_0^{2\pi}|f(\alpha)|d\alpha, \tag{1.17}$$

where the Poisson kernel P_1 is given by

$$P_1 = \frac{1}{2\pi}\frac{\prod_{k=m}^{n-1}\left(1+\frac{1}{k}\right)^2 - 1}{\left(\prod_{k=m}^{n-1}\left(1+\frac{1}{k}\right) - 1\right)^2 + 4\prod_{k=m}^{n-1}\left(1+\frac{1}{k}\right)\sin^2((\alpha-\varphi)/2)}. \tag{1.18}$$

2. **Proof**

By seeking a solution of (1.1) in the form of $u = R(r)\Phi(\varphi)$ we get from (1.1)

$$R^{\Delta\Delta}(r)\Phi(\varphi) + \frac{R^{\Delta}(r)\Phi(\varphi)}{\sigma(r)} + \frac{R(r)\Phi''(\varphi)}{r\sigma(r)} = 0. \tag{2.1}$$

Separating the variables leads to

$$\frac{r\sigma(r)R^{\Delta\Delta}(r)}{R(r)} + \frac{rR^{\Delta}(r)}{R(r)} = -\frac{\Phi''(\varphi)}{\Phi(\varphi)} = n^2 = const, \tag{2.2}$$

or

$$r\sigma(r)\frac{R^{\Delta\Delta}(r)}{R(r)} + r\frac{R^{\Delta}}{R(r)} - n^2 = 0, \tag{2.3}$$

and

$$\Phi''(\varphi) = -n^2\Phi(\varphi). \tag{2.4}$$

Solutions of (2.4) are given by

$$\Phi_n(\varphi) = a_n\sin(n\varphi) + b_n\cos(n\varphi), \quad n = 0, 1, 2, \dots.$$

We seek solutions of (2.3) in the form of the exponential function on a time scale $R(r) = e_{\lambda/r}(r, r_0)$ (see (1.3)). A substitution of $R(r)$ in (2.3) gives

$$\lambda^2 - \lambda + \lambda - n^2 = 0, \quad \Rightarrow \lambda_1 = -n, \quad \lambda_2 = n. \tag{2.5}$$

Ignoring solutions corresponding to $\lambda_2 = -n$, (since $e_{-n/r}(r)$ could be unbounded as $r \to 0$) we get $R(r) = R_n(r) = e_{n/r}(r, r_0)$, and

$$u(r, \varphi) = \sum_{n=0}^{\infty} R_n(r)\Phi_n(\varphi) = \sum_{n=0}^{\infty} [a_n \sin(n\varphi) + b_n \cos(n\varphi)] e_{n/r}(r, r_0). \tag{2.6}$$

From the boundary condition (1.2) we have

$$f(\varphi) = \sum_{n=0}^{\infty} [a_n \sin(n\varphi) + b_n \cos(n\varphi)] = b_0 + \sum_{n=1}^{\infty} [a_n \sin(n\varphi) + b_n \cos(n\varphi)]. \tag{2.7}$$

Multiplying (2.7) by $\sin(m\varphi), \cos(m\varphi), \quad m = 0, \pm 1, \pm 2, \cdots$ and then integrating each corresponding expression yields to

$$\int_0^{2\pi} f(\varphi) \sin(m\varphi) d\varphi = a_m \int_0^{2\pi} \sin^2(m\varphi) d\varphi = a_m \pi,$$
$$\int_0^{2\pi} f(\varphi) \cos(m\varphi) d\varphi = b_m \int_0^{2\pi} \cos^2(m\varphi) d\varphi = b_m \pi,$$

and

$$a_m = \frac{1}{\pi} \int_0^{2\pi} f(\alpha) \sin(m\alpha) d\alpha, \quad b_m = \frac{1}{\pi} \int_0^{2\pi} f(\alpha) \cos(m\alpha) d\alpha, \quad m = 1, 2, \ldots,$$
$$b_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha) d\alpha.$$

Substitution of these formulas in (2.6) gives

$$u(r, \varphi) = \sum_{n=1}^{\infty} \left(\frac{e_{n/r}(r, r_0)}{\pi} \int_0^{2\pi} f(\alpha) [\sin(n\alpha) \sin(n\varphi) + \cos(n\alpha) \cos(n\varphi)] d\alpha \right)$$
$$+ \frac{1}{2\pi} \int_0^{2\pi} f(\alpha) d\alpha$$
$$u(r, \varphi) = \sum_{n=1}^{\infty} \frac{e_{n/r}(r, r_0)}{\pi} \int_0^{2\pi} f(\alpha) \cos(n\alpha - n\varphi) d\alpha + \frac{1}{2\pi} \int_0^{2\pi} f(\alpha) d\alpha$$

or

$$u(r, \varphi) = \int_0^{2\pi} f(\alpha) P(r, \varphi, \alpha) d\alpha, \tag{2.8}$$

where

$$P(r, \varphi, \alpha) = \frac{1}{\pi} \left(\frac{1}{2} + \sum_{n=1}^{\infty} e_{n/r}(r, r_0) \cos(n\alpha - n\varphi) \right). \tag{2.9}$$

Define the Poisson kernel by the formula

$$P_1(r, \varphi, \alpha) = \frac{1}{\pi} \left(\frac{1}{2} + \sum_{n=1}^{\infty} e_{1/r}^n(r, r_0) \cos(n\alpha - n\varphi) \right). \tag{2.10}$$

From (2.9) we get

$$P(r,\varphi,\alpha) = P_1(r,\varphi,\alpha) + Q_2(r,\varphi,\alpha), \tag{2.11}$$

where

$$Q_2(r,\varphi,\alpha) = \sum_{n=2}^{\infty} S_n(r)\cos(n\alpha - n\varphi), \quad S_n(r) = \frac{e_{n/r}(r,r_0) - e^n_{1/r}(r,r_0)}{\pi}. \tag{2.12}$$

Thus from (2.8) we get

$$u(r,\varphi) - \int_0^{2\pi} P_1(r,\varphi,\alpha)f(\alpha)d\alpha = \int_0^{2\pi} Q_2(r,\varphi,\alpha)f(\alpha)d\alpha. \tag{2.13}$$

Further since $e_{1/r}(r,r_0) = \frac{1}{K(r)} < 1$ from (2.10) we get

$$P_1(r,\varphi,\alpha) = \frac{1}{\pi}\left(\frac{1}{2} + \Re\left[\sum_{n=1}^{\infty}\left(\frac{e^{i(\alpha-\varphi)}}{K(r)}\right)^n\right]\right). \tag{2.14}$$

$$\pi P_1(r,\varphi,\alpha) = \Re\sum_{n=0}^{\infty}\frac{e^{in(\alpha-\varphi)}}{K^n(r)} - \frac{1}{2} = \Re\sum_{n=0}^{\infty}\left(\frac{e^{i(\alpha-\varphi)}}{K(r)}\right)^n - \frac{1}{2}.$$

By the geometric progression sum formula we get

$$2\pi P_1(r,\varphi,\alpha) = 2\Re\left[\frac{1}{1-\frac{e^{i(\alpha-\varphi)}}{K(r)}}\right] - 1,$$

$$\begin{aligned} 2\pi P_1(r,\varphi,\alpha) &= \frac{K(r)}{K(r)-e^{i(\alpha-\varphi)}} + \frac{K(r)}{K(r)-e^{-i(\alpha-\varphi)}} - 1 \\ &= \frac{K^2(r)-1}{1-2K(r)\cos(\alpha-\varphi)+K^2(r)}, \end{aligned}$$

and

$$P_1(r,\varphi,\alpha) = \frac{K^2(r)-1}{2\pi(K^2(r)-2K(r)\cos(\alpha-\varphi)+1)} \tag{2.15}$$

or (1.10).

Lemma 2.1.

$$e^n_{1/r}(r,r_0) = e_{p_n}(r,r_0), \quad p_n = \frac{(1+\mu/r)^n - 1}{\mu}, \quad n = 1,2,\ldots, \tag{2.16}$$

$$e^n_{1/r}(r,r_0) \le e_{n/r}(r,r_0), \quad n = 1,2,\ldots, \quad r \le r_0, \tag{2.17}$$

$$e_{n/r}(r,r_0) \le \begin{cases} A(r)n^{-\int_r^{r_0}\frac{\Delta y}{\mu(y)}}, & \mu(y) > 0, \\ \left(\frac{r}{r_0}\right)^n, & \mu(y) \equiv 0, \end{cases}, \quad n = 2,3,\ldots, \quad r \le r_0, \tag{2.18}$$

$$\pi|S_n| = \pi S_n \le \begin{cases} A(r)n^{-\int_r^{r_0}\frac{\Delta y}{\mu(y)}} - K^{-n}, & \mu(y) > 0, \\ 0, & \mu(y) \equiv 0. \end{cases} \tag{2.19}$$

Moreover, by assuming condition (1.8) we get

$$|\pi Q_2| \le \begin{cases} A(r)B(r) - \frac{1}{K(r)(K(r)-1)}, & \mu(y) > 0, \\ 0, & \mu(y) \equiv 0. \end{cases} \tag{2.20}$$

Proof of Lemma 2.1: In view of (1.3), expression (2.16) is true when $n = 1$. On the other hand, if we assume (2.16), then in view of (1.5) we have

$$e_{1/r}^{n+1}(r, r_0) = e_{\frac{1}{r}}(r, r_0) e_{p_n}(r, r_0) = e_q(r, r_0),$$

where

$$\begin{aligned} q &= \frac{1}{r} + \frac{1}{\mu}[(1+\mu/r)^n - 1] + \frac{\mu}{r\mu}[(1+\mu/r)^n - 1] \\ &= \frac{1}{\mu}(1+\mu/r)^n + \frac{1}{r}(1+\mu/r)^n - \frac{1}{\mu} \\ &= \frac{1}{\mu}[(1+\mu/r)^{n+1} - 1] = p_{n+1}. \end{aligned}$$

The rest of the proof follows by induction.

The inequality (2.17) follows from (2.16) and hence $p_n \geq n/r$. To prove the inequality (2.18) we note that if $\mu(y) > 0$ then (1.3) implies that

$$\begin{aligned} e_{n/r}(r, r_0) &= \exp \int_{r_0}^{r} \frac{\ln(n) + \ln(1/n + \mu/r)}{\mu} \Delta r \\ &= n^{\int_{r_0}^{r} \Delta r/\mu} \exp \int_{r_0}^{r} \frac{\ln(1/n + \mu/r)\Delta r}{\mu} \\ &\leq n^{\int_{r_0}^{r} \frac{\Delta y}{\mu(y)}} \exp \int_{r_0}^{r} \frac{\ln(\mu/r)\Delta r}{\mu} \\ &= A(r) n^{\int_{r_0}^{r} \frac{\Delta y}{\mu(y)}}. \end{aligned}$$

For the case $\mu(y) \equiv 0$, inequalities (2.18), (2.19) are obvious. If $\mu(y) > 0$ we obtain (2.19) by using (2.18). To see this, note that

$$\pi|S_n| = \pi S_n = e_{n/r}(r, r_0) - e_{1/r}^{n}(r, r_0) = e_{n/r}(r, r_0) - K^{-n} \leq A(r) n^{\int_r^{r_0} \frac{\Delta y}{\mu(y)}} - K^{-n}.$$

Using the assumption (1.8) we get (2.20). That is

$$|\pi Q_2| \leq \sum_{n=2}^{\infty} \pi|S_n| \leq A(r) \sum_{n=2}^{\infty} n^{\int_r^{r_0} \frac{\Delta y}{\mu(y)}} - \sum_{n=2}^{\infty} K^{-n}(r) = A(r)B(r) - \frac{1}{K(r)(K(r)-1)}.$$

Finally, from (2.13),(2.19) we get

$$\pi \left| u(r, \varphi) - \int_0^{2\pi} P_1(r, \varphi, \alpha) f(\alpha) d\alpha \right| \leq \int_0^{2\pi} \pi |Q_2(r, \varphi, \alpha) f(\alpha)| d\alpha \leq$$

$$\left(A(r)B(r) - \frac{1}{K(r)(K(r)-1)} \right) \int_0^{2\pi} |f(\alpha)| d\alpha \leq A(r)B(r) \int_0^{2\pi} |f(\alpha)| d\alpha,$$

which gives (1.9). □

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