CHARACTER DEGREE SUMS IN FINITE NONSOLVABLE GROUPS

KAY MAGAARD AND HUNG P. TONG-VIET

Abstract. Let N be a minimal normal nonabelian subgroup of a finite group G. We will show that there exists a nontrivial irreducible character of N of degree at least 5 which is extendible to G. This result will be used to settle two open questions raised by Berkovich and Mann, and Berkovich and Zhmud́.

1. Introduction and notations

All groups are finite. Let G be a group. Denote by $Irr(G)$ the set of all complex irreducible characters of G. Let N be a normal subgroup of G. Let $\theta \in Irr(N)$ be an irreducible character of N. We say that θ is extendible to G if there exists $\chi \in Irr(G)$ such that the restriction of χ to N is θ, that is $\chi_N = \theta$. There are many papers devoted to finding a sufficient conditions for θ to be extendible to G (see Gallagher [7], Gagola [6] and [8, Chapter 8 and 11]). In this paper, we are interested in the existence problem, that is, assume N is a normal subgroup of G, is there any non-trivial irreducible character of N that extends to G? We are mostly concerned with nonsolvable groups. Suppose that N is a minimal normal nonabelian subgroup of a group G. In [3, Lemma 5], it is shown that there exists a nontrivial irreducible character θ of N which is extendible to G. In Theorem 1.1 below, we will show that θ can be chosen with $\theta(1) \geq 5$. Using this result, we answer two open problems raised by Berkovich and Mann, and Berkovich and Zhmud́.

Theorem 1.1. *Suppose that N is a minimal normal nonabelian subgroup of a group G. Then there exists an irreducible character θ of N such that θ is extendible to G with $\theta(1) \geq 5$.*

Let $T(G)$ be the sum of degrees of complex irreducible characters of G, i.e $T(G) = \sum_{\chi \in Irr(G)} \chi(1)$. Let $k(G)$ be the number of conjugacy classes of G and let $b(G)$ be the largest irreducible character degree of G. Let N be a normal subgroup of G. Denote by $Irr(G, N)$ the set of all complex irreducible characters χ of G such that $N \nleq Ker\chi$ and by $T(G, N)$ the corresponding sum of degrees of all characters in $Irr(G, N)$. It is obvious that $Irr(G) = Irr(G/N) \cup Irr(G, N)$ and $T(G) = T(G/N) + T(G, N)$. In [1, Theorem 8], Y. Berkovich and A. Mann showed that if G is nonsolvable then $T(G) > 2|G : G'|$ and they asked whether or not $T(G) > 2T(G/N)$, where N is a nonsolvable normal subgroup of G. Here is our first result.

Theorem 1.2. *Let N be a nonsolvable normal subgroup of a group G. Then $T(G) \geq 6T(G/N)$.*

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This settles Question 4 in [1] or Problem 138 in [2]. By Schwarz inequality, it is easy to see that $T(G)^2 \leq |G|k(G)$. Hence $T(G)$ can be used to estimate $k(G)$. Moreover, as $T(G) \leq k(G)b(G)$, we can also get a lower bound for $b(G)$ in terms of $T(G)$ and $k(G)$. The reason that we are interested in the character degree sums comes from a question of Jan Saxl ([9, Problem 9.56]) which asked for the classification of groups in which the square of every irreducible character is multiplicity free. In fact if we could prove that $T(G/F(G)) < b(G/F(G))^2$, provided G is nonsolvable, where $F(G)$ is the Fitting subgroup of G, then such a group in Jan Saxl's question is solvable. This will limit the possibilities for such groups. The proof for this fact is quite straight forward. Let G be a minimal counter-example to the assertion that the square of every irreducible character of G is multiplicity free but G is not solvable. We first observe that if $N \trianglelefteq G$ and $\chi \in Irr(G/N)$ then $\chi \in Irr(G)$ and every irreducible constituent of χ^2 in G is also an irreducible character of G/N so that χ^2 is multiplicity free in G/N since it is multiplicity free in G. Thus G/N satisfies Saxl's condition. Secondly, for any $\chi \in Irr(G)$, as χ^2 is multiplicity free, it follows that $\chi(1)^2 = \chi^2(1) \leq T(G)$. Now combining these two observations for the quotient group $G/F(G)$, we obtain $b(G/F(G))^2 \leq T(G/F(G))$, where $b(G/F(G))$ is the largest character degree of $G/F(G)$. As G is nonsolvable and the Fitting subgroup $F(G)$ is solvable, $G/F(G)$ is nonsolvable. Then the inequality mentioned above would provide a contradiction.

Denote by $T_1(G)$ the sum of degrees of nonlinear irreducible characters of G. Let $Irr_2(G) = \{\chi \in Irr(G) \mid \chi(1) > 2\}$ and let $T_2(G)$ be the sum of degrees of characters in $Irr_2(G)$. Observe that if G does not have any irreducible characters of degree 2 then $T_1(G) = T_2(G)$, for example, this is the case if G is a nonabelian simple group. The following result is a generalization of [1, Theorem 8].

Theorem 1.3. *If G is nonsolvable then $T_2(G) \geq 5|G : G'|$.*

It is well known that a group G is abelian if and only if $T(G) = k(G)$. The following theorem shows that the structure of G is very restricted when $T(G)$ is small in terms of $k(G)$.

Theorem 1.4. *If $T(G) \leq 2k(G)$ then G is solvable.*

This gives a positive answer to Problem 24 in [2]. We note that this property does not characterize the solvability of groups. In fact, let $G \cong 3^2 : 2S_4$, which is a maximal parabolic subgroup of $PSL(3,3)$. We have $T(G) = 50$, $k(G) = 11$, $T(G) > 4k(G)$ and G is solvable. Now if $G \cong A_5$, then $T(G) = 16, k(G) = 5$, $T(G) < 4k(G)$ and G is nonsolvable. We conjecture that a group G is solvable provided that $T(G) \leq 3k(G)$.

2. Preliminaries

Lemma 2.1. *Let T be a non-abelian simple group. Then there exists a nontrivial irreducible character φ of T that extends to $Aut(T)$ with $\varphi(1) \geq 5$.*

This is essentially Lemma 4.2 in [10] or [3, Theorems 2, 3, 4]. However, the fact that $\varphi(1) \geq 5$ is not explicitly stated there so that we will give a proof for completeness.

Proof. According to the Theorem of Classification of Finite Simple Groups, every nonabelian simple group is isomorphic to the alternating group of degree $n \geq 7$, a sporadic group or a finite group of Lie type. We will consider the Tits group ${}^2F_4(2)$ as a sporadic rather than a finite group of Lie type. For alternating group $A_n, n \geq 7$, the irreducible character φ corresponding to the partition $(n-1, 1)$ extends to S_n and $\varphi(1) = n - 1 \geq 6$. If T is a sporadic group, Tits group or A_5, by inspecting [4], we can see that there exists an irreducible character φ of T that extends to $Aut(T)$ with $\varphi(1) \geq 5$. Finally assume T is a finite group of Lie type defined over a field of size $q = p^f$, where p is prime. Choose φ to be the Steinberg character of T of degree $|T|_p$, the order of the p-Sylow subgroup of T. Then φ is extendible to $Aut(T)$ (see [5]). Moreover, we can easily check that $|T|_p > 5$ provided that $T \not\cong L_2(4) \cong L_2(5) \cong A_5$. Thus $\varphi(1) \geq 5$. This completes the proof. □

Lemma 2.2. (Gallagher [8, Corollary 6.17]). *Let $N \trianglelefteq G$ and let $\chi \in Irr(G)$ be such that $\chi_N = \vartheta \in Irr(N)$. Then the characters $\beta\chi$ for $\beta \in Irr(G/N)$ are irreducible, distinct for distinct β and are all of the irreducible constituents of ϑ^G.*

3. Proof of the Main Results

Proof of Theorem 1.1. Since N is a minimal normal nonabelian subgroup of G, there exists a nonabelian simple group T such that $N = T_1 \times T_2 \times \cdots \times T_k$, where $T_i \cong T, i = 1, \cdots, k$. Let φ be an irreducible character of T obtained from Lemma 2.1 and let $\theta = \varphi \times \varphi \times \cdots \times \varphi$. By [3, Lemma 5], $\theta \in Irr(N)$ and it is extendible to G. As $\varphi(1) \geq 5$, we have $\theta(1) = \varphi(1)^k \geq 5$. The proof is now complete. □

Proof of Theorem 1.2. We argue by induction on the order of G. Assume first that N is a minimal normal subgroup of G. By Theorem 1.1, there exists an irreducible character φ of N which extends to an irreducible character χ of G with $\chi(1) \geq 5$. Now by Lemma 2.2, there is an injective map from $Irr(G/N)$ to $Irr(G, N)$ which maps $\beta \in Irr(G/N)$ to $\beta\chi \in Irr(G, N)$, so that $\chi(1)T(G/N) \leq T(G, N)$. Therefore

$$T(G) = T(G/N) + T(G, N) \geq (1 + \chi(1))T(G/N) \geq 6T(G/N).$$

Now assume that N is not a minimal normal subgroup of G. Let K be a minimal normal subgroup of G which is contained in N. Then K is a proper subgroup of N. If K is solvable, then N/K is nonsolvable, and by inductive hypothesis, we have

$$T(G) \geq T(G/K) \geq 6T((G/K)/(N/K)) = 6T(G/N).$$

If K is nonsolvable, then we can apply the result proved in the first paragraph to deduce that $T(G) \geq 6T(G/K)$. As $K \leq N$, we have $T(G/K) \geq T(G/N)$ so that $T(G) \geq 6T(G/K) \geq 6T(G/N)$. The proof is now complete. □

Proof of Theorem 1.3. Let N be the last term of the derived series of G and let K be maximal among the normal subgroups of G that are contained in N. Since $N = N' \leq G'$, it suffices to prove the result for G/K so that we can assume that $K = 1$ and hence N is a minimal normal nonabelian subgroup of G. By Theorem 1.1, there exists an irreducible character $\chi \in Irr(G)$, with $\chi(1) \geq 5$, and $\chi_N = \varphi \in Irr(N)$. Let $\psi = \chi_{G'}$. As $\psi_N = \varphi \in Irr(N)$, it follows that $\psi \in Irr(G')$ and hence $\chi_{G'} = \psi \in Irr(G')$ with $\chi \in Irr(G)$ and $\chi(1) \geq 5$. Now by Lemma 2.2, there is an injective map from $Irr(G/G')$ to $Irr_2(G)$ which maps $\beta \in Irr(G/G')$ to $\beta\chi \in Irr_2(G)$, so that $\chi(1)|G : G'| \leq T_2(G)$. Thus $T_2(G) \geq 5|G : G'|$. This finishes the proof. ☐

Proof of Theorem 1.4. By way of contradiction, assume that G is nonsolvable. Let a be the number of linear characters of G, let b be the number of irreducible characters of G of degree 2 and finally let c be the number of irreducible characters of degree greater than 2. We have

$$k(G) = a + b + c \tag{1}$$

$$T(G) = a + 2b + T_2(G) \tag{2}$$

$$T(G) \geq a + 2b + 3c \tag{3}$$

Since $T(G) \leq 2k(G)$, it follows from (1) and (3) that

$$a + 2b + 3c \leq 2a + 2b + 2c$$

and hence

$$c \leq a \tag{4}$$

Since $T(G) \leq 2k(G)$, it follows from (1) and (2) that

$$a + 2b + T_2(G) \leq 2a + 2b + 2c$$

and so

$$T_2(G) \leq a + 2c.$$

Combining with (4), we obtain

$$T_2(G) \leq 3a = 3|G : G'|. \tag{5}$$

However, this contradicts Theorem 1.3. Thus G must be solvable. This completes the proof. ☐

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E-mail address: K.Magaard@bham.ac.uk (K. Magaard)

E-mail address: tongviet@maths.bham.ac.uk (H.P. Tong-Viet)

School of Mathematics, University of Birmingham, Birmingham, B15 2TT, UK