ON HUPPERT’S CONJECTURE FOR $G_2(q)$, $q \geq 7$

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Abstract. Let $G$ be a finite group and let $\text{cd}(G)$ be the set of all complex irreducible character degrees of $G$. Bertram Huppert conjectured that if $H$ is a finite nonabelian simple group such that $\text{cd}(G) = \text{cd}(H)$, then $G \cong H \times A$, where $A$ is an abelian group. In this paper, we verify the conjecture for the family of simple exceptional groups of Lie type $G_2(q)$ for $q \geq 7$.

1. Introduction and notation

All groups considered are finite and all characters are complex characters. Let $G$ be a group, $\text{Irr}(G)$ be the set of all irreducible characters of $G$, and denote the set of character degrees of $G$ by $\text{cd}(G) = \{ \chi(1) : \chi \in \text{Irr}(G) \}$. In instances where the context is clear, we will refer to character degrees as degrees. In the late 1990s Bertram Huppert conjectured that the nonabelian simple groups are essentially determined by the set of their degrees. In [9], he posed the following conjecture.

Huppert’s Conjecture. Let $G$ be a group and let $H$ be a nonabelian simple group. If $\text{cd}(G) = \text{cd}(H)$, then $G \cong H \times A$, where $A$ is an abelian group.

Huppert verified the conjecture on a case-by-case basis for many nonabelian simple groups, including the Suzuki groups, many of the sporadic simple groups, and a few of the simple groups of Lie type [9]. Our goal is to verify Huppert’s Conjecture for the simple groups of Lie type of Lie rank two. In [18, 19, 20, 21, 22], Huppert’s Conjecture was verified for $L_3(q)$, $U_3(q^2)$, $S_4(q)$ with $q > 2$, $2G_2(q^2)$ for $q^2 = 3^{2m+1}$, $m \geq 1$, $2F_4(q^2)$, $q^2 = 2^{2m+1}$, $m \geq 1$ and $3D_4(q)$, $q \geq 3$. The remaining simple group of Lie type of Lie rank two is $G_2(q)$ for $q \geq 3$. In the preprint ‘Some simple groups, which are determined by the set of their character degrees VII’, Huppert verified his conjecture for the simple groups $G_2(3)$ and $G_2(4)$. In [1], the conjecture is verified for $G_2(5)$. In this paper, we will establish Huppert’s Conjecture for $G_2(q)$ when $q \geq 7$.

Theorem 1.1. Let $q \geq 7$ be a prime power. If the sets of character degrees of $G$ and $G_2(q)$ are the same, then $G \cong G_2(q) \times A$, where $A$ is an abelian group.

We will denote the commutator subgroup of $G$ by $G'$. Consequently, the derived subgroup of $G'$ will be denoted by $G''$. In [9], Huppert outlines the following five steps which can be used to verify the conjecture.

1. Show $G' = G''$. Then if $G'/M$ is a chief factor of $G$, then $G'/M \cong S^k$, where $S$ is a nonabelian simple group.

(2) Show $G'/M \cong H$.

(3) Show that if $\theta \in \text{Irr}(M)$ and $\theta(1) = 1$, then $\theta$ is stable under $G'$, which implies $[M, G'] = M'$.

(4) Show that $M = 1$.

(5) Show that $G = G' \times C_G(G')$. As $G/G' \cong C_G(G')$ is abelian and $G' \cong H$, Huppert’s Conjecture is verified.

**Notation.** Let $G$ be a group. The set of all prime divisors of $|G|$ is denoted by $\pi(G)$. If $N \leq G$ and $\lambda \in \text{Irr}(N)$, then the set of all irreducible constituents of $\lambda^G$ is denoted by $\text{Irr}(G \lambda)$. We denote the greatest common divisor of two integers $a$ and $b$ by $\gcd(a, b)$. The $n$th cyclotomic polynomial in variable $q$ is denoted by $\Phi_n := \Phi_n(q)$. We follow [6] for the notation of simple groups. Other notation is standard.

### 2. Preliminaries

We will require several lemmas to carry out the proof of Huppert’s Conjecture. We begin with the following results from Clifford Theory. These results can be found in [9, Lemmas 2,3]. Lemma 2.1(b) is often referred to as Gallagher’s Theorem.

**Lemma 2.1.** Suppose $N \leq G$ and $\chi \in \text{Irr}(G)$.

(a) If $\chi_N = \theta_1 + \theta_2 + \cdots + \theta_k$ with $\theta_j \in \text{Irr}(N)$, then $k$ divides $|G : N|$. In particular, if $\chi(1)$ is relatively prime to $|G : N|$, then $\chi_N \in \text{Irr}(N)$.

(b) If $\chi_N \in \text{Irr}(N)$, then $\chi \tau \in \text{Irr}(G)$ for every $\tau \in \text{Irr}(G/N)$.

**Lemma 2.2.** Suppose $N \leq G$ and $\theta \in \text{Irr}(N)$. Let $I = I_G(\theta)$ denote the inertia subgroup of $\theta$ in $G$.

(a) If $\theta^I = \sum_{i=1}^k \phi_i$ with $\phi_i \in \text{Irr}(I)$, then $\phi_i^G \in \text{Irr}(G)$ for all $i$. In particular, $\phi_i(1)|G : I| \in \text{cd}(G)$.

(b) If $\theta$ admits an extension $\theta_0$ to $I$, then $(\theta_0 \tau)^G \in \text{Irr}(G)$ for all $\tau \in \text{Irr}(I/N)$. In particular, $\theta(1)^2|I| = \text{cd}(G)$.

(c) If $\rho \in \text{Irr}(I)$ such that $\rho_N = e\theta$, then $\rho = \theta_0 \tau_0$, where $\theta_0$ is a character of an irreducible projective representation of $I$ of degree $\theta(1)$ while $\tau_0$ is the character of an irreducible projective representation of $I/N$ of degree $e$.

The next result will be used to verify Step 1 of Huppert’s method.

**Lemma 2.3.** ([18] Lemma 2.3). Let $G/N$ be a solvable factor group of $G$, minimal with respect to being nonabelian. Then two cases can occur.

(a) $G/N$ is an $r$-group for some prime $r$. Hence there exists $\psi \in \text{Irr}(G/N)$ such that $\psi(1) = r^b > 1$. If $\chi \in \text{Irr}(G)$ and $r \nmid \chi(1)$, then $\chi \tau \in \text{Irr}(G)$ for all $\tau \in \text{Irr}(G/N)$.

(b) $G/N$ is a Frobenius group with an elementary abelian Frobenius kernel $F/N$. Thus $f = |G : F| \in \text{cd}(G)$ while $|F : N| = r^c$ for some prime $r$ and $F/N$ is an irreducible module for the cyclic group $G/F$, hence $c$ is the smallest integer such that $r^{c-1} \equiv 0 \pmod{|G/F|}$. If $\psi \in \text{Irr}(F)$, then either $f \psi(1) \in \text{cd}(G)$ or $r^c$ divides $\psi(1)^2$. In the latter case, $r$ divides $\psi(1)$. Moreover the following hold.

(i) If no proper multiple of $f$ is in $\text{cd}(G)$, then $\chi(1)$ divides $f$ for all $\chi \in \text{Irr}(G)$ such that $r \nmid \chi(1)$.

(ii) If there exists $\chi \in \text{Irr}(G)$ such that no proper multiple of $\chi(1)$ is in $\text{cd}(G)$, then either $f$ divides $\chi(1)$ or $r^c$ divides $\chi(1)^2$. Furthermore if $\chi(1)$ is divisible by no nontrivial proper character degree in $G$, then $f = \chi(1)$ or $r^c | \chi(1)^2$. 
 maximal subgroups. If \( q \) is a maximal subgroup, then one of the following cases holds.

(1) If \( q \neq 7, 11 \), then \( K \) is the normalizer of a Sylow \( p \)-subgroup of order \( q \) in \( SL_2(q) \).

(2) If \( q = 7 \) or \( 11 \), then either \( K \) is the normalizer of a Sylow \( p \)-subgroup of order \( q \) in \( SL_2(q) \) or \( K \) is a nonabelian group of order \( q \).

Proof. Note that if \( K \) is a maximal subgroup of \( SL_2(q) \), then the center \( Z \) of \( SL_2(q) \) of order \( \gcd(2, q - 1) \) must lie in \( K \) as \( SL_2(q) \) is perfect, so \( K/Z \) is a maximal subgroup of \( SL_2(q)/Z \cong L_2(q) \). The result follows by inspecting the list of maximal subgroups of \( L_2(q) \) in [10]. Note that if \( |L_2(7) : K| = 7 \), then \( K \cong 2 \cdot S_4 \cong SL_2(3) \cdot 2 \), and there are two conjugacy classes of such maximal subgroups. If \( |L_2(11) : K| = 11 \), then \( K \cong SL_2(5) \) and there are two conjugacy classes of such maximal subgroups. If \( q = 9 \) then \( SL_2(9) \) has two conjugacy classes of maximal subgroups which are isomorphic to \( SL_2(5) \cong 2 \cdot A_5 \) both with index 6. But 6 does not divide 9 ± 1 or 9. Moreover if \( q \neq 7, 9, 11 \), then \( q + 1 \) is the smallest index of a maximal subgroup of \( SL_2(q) \), where \( q \geq 7 \).

The following lemma will be useful in proving Step 4.

Lemma 2.6. ([5] Lemma 6). Suppose \( M \leq G' = G'' \) and \( \lambda^g = \lambda \) for all \( g \in G' \) and all \( \lambda \in \text{Irr}(M) \) such that \( \lambda(1) = 1 \). Then \( M' = [M, G'] \) and \( |M : M'| \) divides the order of the Schur multiplier of \( G'/M \).

The next lemma will be used to verify Step 5.

Lemma 2.7. ([5] Theorem C). Let \( \sigma \) be an outer automorphism of the nonabelian simple group \( G \). Then there exists a conjugacy class \( C \) of \( G \) with \( C^\sigma \neq C \).

We now have the lemmas necessary to begin the verification of Huppert’s Conjecture for \( G_2(q) \), for \( q \geq 7 \).

3. Results Concerning the Character Degrees of \( G_2(q) \)

We first consider the character degrees of \( G_2(q) \), where \( q \) is a power of a prime \( p \). As listed in [5], the nontrivial character degrees of \( G_2(q) \) are

\[
\frac{1}{2}q\Phi_1^2\Phi_6, \frac{1}{6}q\Phi_2^2\Phi_3, \frac{1}{2}q\Phi_1^2\Phi_2, \frac{1}{3}q\Phi_3\Phi_6, \frac{1}{2}q\Phi_1^2\Phi_3, \frac{1}{2}q\Phi_2^2\Phi_6, \Phi_1\Phi_3\Phi_6, \Phi_2\Phi_3\Phi_6.
\]

We will use the following lemma to establish Step 2.

Lemma 2.4. ([5] Lemma 5). Let \( N \) be a minimal normal subgroup of \( G \) such that \( N \cong S^k \), where \( S \) is a nonabelian simple group. If \( \sigma \in \text{Irr}(S) \) extends to \( \text{Aut}(S) \), then \( \sigma^k \in \text{Irr}(N) \) extends to \( G \).

The following lemmas will be used in proving Step 3.

Lemma 2.5. Let \( q \geq 7 \) be a power of a prime \( p \). If \( K \) is a maximal subgroup of \( SL_2(q) \) whose index divides \( q \pm 1 \) or \( q \), then one of the following cases holds.

(1) If \( q \neq 7, 11 \), then \( K \) is the normalizer of a Sylow \( p \)-subgroup of order \( q \) in \( SL_2(q) \) of index \( q + 1 \).

(2) If \( q = 7 \) or \( 11 \), then either \( K \) is the normalizer of a Sylow \( p \)-subgroup of order \( q \) in \( SL_2(q) \) or \( K \) is a nonabelian group of index \( q \).

We first consider the character degrees of \( G_2(q) \) and all \( \lambda \in \text{Irr}(G) \) for all \( \lambda \). Suppose \( M \leq G' = G'' \) and \( \lambda^g = \lambda \) for all \( g \in G' \) and all \( \lambda \in \text{Irr}(M) \) such that \( \lambda(1) = 1 \). Then \( M' = [M, G'] \) and \( |M : M'| \) divides the order of the Schur multiplier of \( G'/M \).

The next lemma will be used to verify Step 5.

Lemma 2.7. ([5] Theorem C). Let \( \sigma \) be an outer automorphism of the nonabelian simple group \( G \). Then there exists a conjugacy class \( C \) of \( G \) with \( C^\sigma \neq C \).

We now have the lemmas necessary to begin the verification of Huppert’s Conjecture for \( G_2(q) \), for \( q \geq 7 \).
reduces to the question of which degrees can be written as $y^p$, where $p$ is a prime. The following result follows from \cite{17} \cite{12}.

**Lemma 3.1.** The number $q^2 + q + 1$ cannot be written in the form $y^n$ for $n > 1$. The number $q^2 - q + 1$ is of the form $y^n$, for $n > 1$, only for $q = 19$.

From Lemma 3.1, $q^2 + q + 1$ is never a power and $q^2 - q + 1$ is a power only when $q = 19$. In addition, as shown in \cite{16}, $q^2 + q + 1$ and $q^2 - q + 1$ can never be written in the form $3y^n$ for any $n > 1$. As $\gcd(q + 1, q^2 + q + 1) = \gcd(q - 1, q^2 - q + 1) = 1$ while $\gcd(q - 1, q^2 + q + 1)$ and $\gcd(q + 1, q^2 - q + 1)$ are either 1 or 3, we see that $(q \pm 1)(q^2 \pm q + 1)$ and $(q^2 + q + 1)(q^2 - q + 1)$ are never nontrivial powers. All the degrees of $G_2(q)$ except $q^6$ and $\frac{1}{4}q\Phi_1^2\Phi_2^2$ contain one of these products. As shown in \cite{13}, the only power of a prime among the degrees of $G_2(q)$ for $q > 3$ is $q^6$. We have proved the following lemma.

**Lemma 3.2.** For $q > 7$, the only nontrivial powers among the degrees of $G_2(q)$ are $q^6$ and possibly $\frac{1}{4}q\Phi_1^2\Phi_2^2$. The only nontrivial prime power degree of $G_2(q)$ is $q^6$.

We will also need to know which pairs of character degrees of $G_2(q)$ are consecutive integers. By examining the degrees of $G_2(q)$, it is possible to prove the following lemma.

**Lemma 3.3.** The only pair of consecutive integers among the character degrees of $G_2(q)$, for $q > 2$, is $q^6 - 1$ and $q^6$.

Excluding $q^6$ and 1 from consideration, the only pairs of relatively prime degrees of $G_2(q)$ are possibly

- $\Phi_2\Phi_6$ and $q\Phi_3\Phi_1^2/2$ for $q \equiv 4 \pmod{6}$;
- $\Phi_3\Phi_6$ and $q\Phi_1^2\Phi_2^2/3$ for $q \equiv 3 \pmod{6}$;
- $\Phi_4\Phi_3$ and $q\Phi_2^2\Phi_6/2$ for $q \equiv 2, 5 \pmod{6}$.

Hence, excluding 1 and $q^6$ from consideration, there is at most one pair of relatively prime character degrees of $G_2(q)$. We will frequently use the following well-known result found in \cite{14} \cite{17}.

**Lemma 3.4.** The only divisors of $\Phi_3$ are 3, but not $3^2$, and numbers of the form $1 + 3m$. The only divisors of $\Phi_6$ are 3, but not $3^2$, and numbers of the form $1 + 6m$.

4. Establishing $G' = G''$ when $H \cong G_2(q)$

From now on we assume that $H \cong G_2(q)$ where $q = p^a$ $\geq 7$, $p$ is a prime and $a \geq 1$. Suppose that $G' \neq G''$. Then there exists a solvable factor group $G/N$ of $G$ minimal with respect to being nonabelian. By Lemma 2.3, $G/N$ is an $r$-group or a Frobenius group.

**Case 1:** $G/N$ is an $r$-group for some prime $r$. By Lemma 2.3(a), $G/N$ has a character degree $r^b > 1$. By Lemma 3.2, the only nontrivial prime power degree of $G$ is $q^6$, hence we deduce that $r^b = q^6 \in \text{cd}(G/N)$. Now $\Phi_1^2\Phi_2^2\Phi_3$ is a character degree of $G$ and $r \nmid \Phi_1^2\Phi_2^2\Phi_3$. By Lemma 2.1 if $\chi \in \text{Irr}(G)$ with $\chi(1) = \Phi_1^2\Phi_2^2\Phi_3$, then $\chi_N \in \text{Irr}(N)$. Let $\tau \in \text{Irr}(G/N)$ with $\tau(1) = q^6$. Lemma 2.1 implies that $G$ has a character degree $\phi^6\Phi_1^2\Phi_2^2\Phi_3$, which is impossible.

**Case 2:** $G/N$ is a Frobenius group with elementary abelian Frobenius kernel $F/N$, where $|F : N| = r^c$ for some prime $r$. In addition, $f = |G : F| \in \text{cd}(G)$ and $f$ divides $r^c - 1$. 

Subcase 2(a) : $r \neq p$. Let $\chi \in \text{Irr}(G)$, $\chi(1) = q^6$. As $r \nmid \chi(1)$ and no proper multiples and no proper divisors of this degree are in $\text{cd}(G)$, we deduce from Lemma 2.3 that $f = q^6$. Now let $\psi \in \text{Irr}(G)$, with $1 < \psi(1) \neq q^6$. Then $\psi(1)_{p'} \neq 1$. If $\theta \in \text{Irr}(F)$ is an irreducible constituent of $\psi_F$, then $\psi(1)/\theta(1) \mid |G:F| = q^6$ and hence $\theta(1)_{p'} = \psi(1)_{p'} \neq 1$. It follows that $\theta(1)f$ is not a character degree of $G$ and hence $r \nmid \theta(1)$, which implies that $r \nmid \psi(1)_{p'}$ for any $\psi \in \text{Irr}(G)$ with $1 < \psi(1) \neq q^6$. If $r \nmid \Phi_1$, then as $\gcd(\Phi_2, \Phi_3) = 1$, we have that either $r \nmid q\Phi_1^2\Phi_3^2/3$ or $r \nmid q\Phi_1^2\Phi_3^2/2$, a contradiction. Thus $r \mid \Phi_1$. If $r = 2$, then as $r \neq p$, $q$ is odd. Hence $q$ is congruent to 1, 3 or 5 modulo 6. But then $2 \nmid \Phi_1$ and $2 \nmid \Phi_3$. So $2 \nmid q\Phi_3^2\Phi_3/3$. If $r = 3$, then $3 \mid \Phi_1$. Thus $q\Phi_3^2\Phi_3/6$ is not divisible by 3 since $3^2 \nmid \Phi_3$, a contradiction. Hence $r \mid \Phi_1$ and $r > 3$. Since $r \neq 2$, $r \nmid \Phi_2$. Now $\gcd(\Phi_1, \Phi_3) = 1$ or 3, so as $r \neq 3$, we have that $r \nmid q\Phi_3^2\Phi_3/6$, a contradiction. Hence $r \neq p$ is not possible.

Subcase 2(b) : $r = p$. Let $\psi \in \text{Irr}(G)$ with $\psi(1) = q\Phi_1^2\Phi_3^2/3$. As $q \geq 7$, no proper multiples and no proper divisors of $\psi(1)$ are in $\text{cd}(G)$, by Lemma 2.3(b), since $r \mid \psi(1)$, we have $f \neq \psi(1)$ therefore $r^2 \nmid \psi(1)^2$. Hence $r^2 \nmid q^2$. As $f \nmid r^2 - 1$, we have $f \leq r^2 - 1 \leq q^2 - 1$, where $1 < f \in \text{cd}(G)$, which is impossible. Thus $G' = G''$.

5. Establishing $G'/M \cong H$ when $H \cong G_2(q)$

In this section, we will establish Step 2 for the simple groups $G_2(q)$ with $q \geq 7$.

5.1. Eliminating the Tits, Sporadic, and Alternating Groups when $k > 1$.

Let $n$ be a positive integer. We call $\lambda = (\lambda_1, \ldots, \lambda_r)$ a partition of $n$, provided $\lambda_i, i = 1, \ldots, r$ are integers, with $\lambda_1 \geq \cdots \geq \lambda_r > 0$ and $\sum_{i=1}^{r} \lambda_i = n$. We collect the same parts together and write $\lambda = (\ell_1^{a_1}, \cdots, \ell_k^{a_k})$, with $\ell_i > \ell_{i+1} > 0$ for $i = 1, \cdots, k - 1; a_i > 0$ and $\sum_{i=1}^{k} a_i \ell_i = n$. It is well known that the irreducible complex characters of the symmetric group $S_n$ are parametrized by partitions of $n$. Denote by $\chi^\lambda$ the irreducible character of $S_n$ corresponding to partition $\lambda$. The irreducible characters of the alternating group $A_n$ are then obtained by restricting $\chi^\lambda$ to $A_n$. In fact, $\chi^\lambda$ is still irreducible upon restriction to the alternating group $A_n$ if and only if $\lambda$ is not self-conjugate. Otherwise, $\chi^\lambda$ splits into two irreducible characters of $A_n$ having the same degree. The character degree $\chi^\lambda(1)$ can be computed using the Hook formula.

**Lemma 5.1.** If $n \geq 7$, then $\text{Irr}(A_n)$ contains at least four nonlinear irreducible characters of different degrees which extend to $\text{Aut}(A_n)$.

**Proof.** As $n \geq 7$, the symmetric group $S_n$ possesses at least four nontrivial irreducible characters labeled by the partitions $(n - 1, 1), (n - 2, 2), (n - 2, 1^2)$ and $(n - 3, 2, 1)$, respectively. Observe that these partitions are not self-conjugate and so these irreducible characters when restricted to $A_n$ remain irreducible. The degrees of these irreducible characters are $n - 1, n(n - 3)/2, (n - 1)(n - 2)/2$ and $n(n - 2)(n - 4)/3$, respectively. As $n \geq 7$, it is easy to verify that all of these character degrees are distinct. \(\square\)

To eliminate the sporadic groups and the Tits group, we need the following result, found by checking the Atlas [6].

**Lemma 5.2.** If $S$ is a sporadic simple group or the Tits group, then there exist at least five distinct nonlinear irreducible characters of different degrees of $S$ which extend to $\text{Aut}(S)$. 


**Proposition 5.3.** If S is an alternating group $A_n$ with $n \geq 7$, a sporadic simple group or the Tits group, then $k = 1$.

**Proof.** Suppose that $k > 1$ and $S \cong A_n$ for some $n \geq 7$, a sporadic simple group or the Tits group. Lemmas 5.1 and 5.2 imply that $S$ has nonlinear irreducible characters of distinct degrees $\phi_1, \phi_2,$ and $\phi_3$, say, which extend to Aut($S$).

By Lemma 2.4, $\phi_1^k$, $\phi_2^k$, and $\phi_3^k$ extend to $G/M$. As shown in Lemma 3.2, there are at most two nontrivial powers among the character degrees of $G/M$. Thus, if $S \cong A_n$ with $n \geq 7$, a sporadic simple group, or the Tits group, then $k = 1$. □

Note that $A_5 \cong L_2(5)$ and $A_6 \cong L_2(9)$ will be considered with the simple groups of classical Lie type. We have proved that if $S \cong A_n$ with $n \geq 7$, a sporadic simple group or the Tits group, then $k = 1$. In this case, $G'/M \cong S$. We will now show that $S$ cannot be one of these groups.

### 5.2. Eliminating Sporadic Simple Groups and the Tits Group when $k = 1$.

**Proposition 5.4.** The simple group $S$ cannot be a sporadic simple group nor the Tits group.

**Proof.** Now $G$ has at most 22 distinct, nontrivial degrees. If an irreducible character of $S$ extends to Aut($S$), then it must extend to $G$ by Lemma 2.4. Consulting the Atlas [6], most of the sporadic groups have more than 22 irreducible characters of distinct degrees which extend to Aut($S$). This implies that $G$ has more than 22 distinct character degrees, a contradiction.

Thus we only need to consider the following cases of sporadic simple groups with 22 or less extendible characters of distinct degrees.

**Case 1:** $S \cong M_{11}, S \cong M_{12}, S \cong M_{23}$, $S \cong M_{24}$ or $S \cong J_1$. For each of these simple groups, irreducible characters of consecutive degrees extend. The higher degree of these consecutive degrees is not a prime power. As stated in Lemma 3.3, the only consecutive degrees of $G$ are $q^j - 1$ and $q^j$. Hence this is not possible.

**Case 2:** $S \cong M_{22}$. The simple group $M_{22}$ has irreducible characters of relatively prime degrees 45 and 154 which extend to Aut($M_{22}$). Examining the list of pairwise relatively prime degrees of $G$, we see that this is not possible.

**Case 3:** $S \cong J_2$ or $S \cong J_3$. The simple group $J_2$ has an irreducible character of degree $225 = 15^2$ which extends to Aut($J_2$) while the simple group $J_3$ has an irreducible character of degree $324 = 18^2$ which extends to Aut($J_3$). The only nontrivial powers among the degrees of $G$ are $q^j$ and possibly $\frac{1}{3}q^2\Phi_1^2\Phi_2^2$ and we see that neither of these could possibly be 225 or 324.

**Case 4:** $S \cong HS$, $S \cong O'N$ or $S \cong McL$. The simple group $HS$ has five pairs of irreducible characters of relatively prime degrees which extend to Aut($HS$), while the simple group $O'N$ has three pairs of irreducible characters of relatively prime degrees which extend to Aut($O'N$). The simple group $McL$ has irreducible characters of relatively prime degrees 22 and 5103 as well as 3520 and 5103 which extend to Aut($McL$). As $G$ has at most one pair of relatively prime degrees, we see that these cases are impossible.

**Case 5:** $S \cong He$. The simple group $He$ has irreducible characters of relatively prime degrees 1275 and 6272 which extend to Aut($He$). Examining the odd degrees of $G$ relatively prime to another degree we see that it is not possible for 1275 to be a degree of $G$. 


Case 6: $S \cong Z_{2}F_{4}(2)'$. In this case, $S$ possesses an irreducible character of degree $3^3$ which extends to $\text{Aut}(S)$ and hence $3^3 \in \text{cd}(G)$. However the only nontrivial prime power degree of $G$ is $q^6$ by Lemma 3.2. Thus $q^6 = 3^3$, which is impossible. \hfill \Box

5.3. Eliminating $A_n$ when $k = 1$.

**Proposition 5.5.** The simple group $S$ is not an alternating group $A_n$ with $n \geq 7$.

**Proof.** As shown in the proof of Lemma 5.1 if $n \geq 7$, then $A_n$ has two irreducible characters $\chi_i$, $i = 1, 2$, of degrees $n(n-3)/2$ and $n(n-3)/2 + 1 = (n-1)(n-2)/2$, corresponding to the partitions $(n-2, 2)$ and $(n-2, 1^2)$. If $n = 2m$, then $\chi_1(1) = 2m^2 - 3m$ and $\chi_2(1) = 2m^2 - 3m + 1 = (m-1)(2m-1)$. If $n = 2m + 1$, then $\chi_1(1) = 2m^2 - m - 1$ and $\chi_2(1) = m(2m-1)$. In both cases, $\chi_1(1)$ and $\chi_2(1)$ are consecutive integers, with $\chi_2(1)$ larger. But $q^6 \neq (m-1)(2m-1)$, as $q^6$ is a power of a prime while $\gcd(m-1, 2m-1) = 1$. Similarly, $q^6 \neq m(2m-1)$, as $q^6$ is a power of a prime while $\gcd(m, 2m-1) = 1$.

5.4. Eliminating the Groups of Lie Type when $k > 1$. We refer the reader to [4] for the classification of unipotent characters and the notion of symbols. It is well known that every simple group of Lie type $S$ in characteristic $p$ (excluding the Tits group) possesses an irreducible character of degree $|S|_p$, which is the size of the Sylow $p$-subgroup of $S$, and is denoted by $St_G$ and is called the Steinberg character of $S$. Moreover the Steinberg character and also the unipotent characters of $S$ extend to $\text{Aut}(S)$, apart from some explicit exceptions. By Lemma 2.4 we have that $\chi(1)^k$ is a degree of $G$, for any irreducible character $\chi$ of $S$ that extends to $\text{Aut}(S)$. As the only composite power of a prime among degrees of $G$ is $q^6$, we must have that $St_G(1)^k = q^6$. Hence, the defining characteristic of the simple group $S$ must be the same as the prime divisor of $q^6$. We define a mixed degree of $S$ to be a degree of $S$ which is divisible by $q$ but is not a power of $q$.

**Lemma 5.6.** If $S = S(q_1)$ is a simple group of Lie type, and $S \not\cong L_2(q_1)$, then $S$ possesses an irreducible character of mixed degree.

**Proof.** Examining the list of unipotent characters for the simple groups of exceptional Lie type found in [4, 13.9], we see that it is true for these groups. For the simple classical groups of Lie type, the degrees of the unipotent characters labeled by the symbols in Table 1 have mixed degrees. \hfill \Box

**Proposition 5.7.** If $S = S(q_1)$ is a simple group of Lie type and $S \not\cong L_2(q_1)$, then $k = 1$.

**Proof.** Suppose that $k \geq 2$. The Steinberg character of $S$ extends to $\text{Aut}(S)$ so $St_S(1)^k = q^6$. Write $St_S(1) = q_1^j$. Since $S \not\cong L_2(q_1)$, Lemma 5.6 implies that $S$ possesses an irreducible character of mixed degree, say $\tau$. As $G/M \cong S^k$, there is an irreducible character of $G/M$ found by multiplying $k - 1$ copies of $St_S$ with $\tau$. Then $(St_S^{k-1}\tau)(1)$ is a mixed degree of $G/M$. As the degrees of $G/M$ divide the degrees of $G$, we must have that the degree of this irreducible character divides one of the mixed degrees of $G$. The highest power of $q$ on any mixed degree of $G$ is 3. Now $q^6 = q_1^{jk}$ implies $q = q_1^{jk/6}$. The power of $q_1$ in $(St_S^{k-1}\tau)(1)$ is at least $j(k-1) + 1$. We must have that $j(k-1) + 1 \leq 3jk/6$, which reduces to $3j(k-2) + 6 \leq 0$. Thus $k < 2$, a contradiction. Hence $k = 1$ if $S \not\cong L_2(q_1)$. \hfill \Box

We must now eliminate the case when $k > 1$ and $S \cong L_2(q_1)$.
Proposition 5.8. If $S \cong L_2(q_1)$ for $q_1 \geq 4$, then $k = 1$.

Proof. Recall that $S$ has the character degree $q_1 - 1$ and Steinberg character of degree $q_1$. Repeating the argument of Proposition 5.7, noting that $St_S(1) = q_1$, so $q^6 = q_1^k$ and choosing $\tau$ to be the irreducible character of $S$ of degree $q_1 - 1$, implies that $k \leq 2$. Suppose that $k = 2$. Then $q_1^2 = q_1^6$ so $q_1 = q_3^3$. Hence $(q_3^3 - 1)^2 = \Phi_1^2 \Phi_3^2$ divides a character degree of $G$, which is impossible. □

5.5. Eliminating Simple Groups of Exceptional Lie Type when $k = 1$.

Proposition 5.9. If $G'/M \cong S$, where $S$ is a simple group of exceptional Lie type, then $S \cong G_2(q)$.

Proof. We will examine each of the families of simple groups of exceptional Lie type individually. Again, all notation is adapted from [4]. Recall that $q^6 = St_S(1)$.

Case 1: $S \cong G_2(q_1)$. The Steinberg character of $S \cong G_2(q_1)$ has degree $q_1^6$. Then $q^6 = q_1^6$ which implies $q = q_1$. Hence $S \cong G_2(q)$ as wanted.

Case 2: $S \cong 2B_2(q_1^2)$, $q_1^2 = 2^{2m+1}$, $m \geq 1$. Now
\[ \text{cd}(2B_2(q_1^2)) = \{ 1, q_1^4, q_1^4 + 1, (q_1^2 - 1)a, (q_1^2 - 1)b, (q_1^2 - 1)u \}, \]
for $q_1^2 = 2^{2m+1} \geq 8$, $u = \frac{1}{\sqrt{2}} q_1$, $a = q_1^2 + \frac{1}{\sqrt{2}} q_1 + 1$, and $b = q_1^2 - \frac{1}{\sqrt{2}} q_1 + 1$. As the Steinberg character of $S$ has degree $q_1^4$, we have that $q_1^4 = q^6$. Then $q_1^4 + 1 = q^6 + 1$ must divide a degree of $G$. But this is not the case. Hence $S \not\cong 2B_2(q_1^2)$.

Case 3: $S \cong 2G_2(q_1^2)$, $q_1^2 = 2^{2m+1}$, $m \geq 1$. Here, $q_1^6 = q^6$ so $q = q_1$. But $q$ is an integer while $q_1$ is not. Thus $S \not\cong 2G_2(q_1^2)$.

Case 4: $S \cong 2F_4(q_1^2)$, $q_1^2 = 2^{2m+1}$, $m \geq 1$. We have $St_S(1) = q_1^2$ and hence $q^6 = q_1^2$, which implies $q = q_1^4$. Consider the character $\epsilon''$ of $2F_4(q_1^2)$ of degree $q_1^{10}(q_1^4 - q_1^2 + 1)(q_1^8 - q_1^4 + 1)$. This degree must divide a mixed degree of $G$. As $q = q_1^4$, this degree must divide $q^3(q^3 \pm 1)$. But $q^3 \pm 1 = q_1^{12} \pm 1$ and $q_1^8 - q_1^4 + 1$ does not divide $q_1^{12} \pm 1$. Thus $S \not\cong 2F_4(q_1^2)$.

Case 5: $S$ is isomorphic to one of the remaining simple groups of exceptional Lie type. Recall $S = S(q_1)$ is a simple group of exceptional Lie type defined over a field of $q_1$ elements. Suppose the Steinberg character of $S$ has degree $q_1^3$. Then $q_1^6 = q_1^3$, so $q = q_1^2$. Each of the remaining possibilities for $S$ has a mixed degree whose power on $q_1$ is greater than $3j/6$. As the mixed degrees of $G$ have power at most $3j/6$ on $q_1$, we have a contradiction. Table 1 exhibits the degree of the Steinberg character of $S$ and a character of $S$ of appropriate degree which will result in a contradiction. □

5.6. Eliminating the Groups of Classical Lie Type when $k = 1$. These cases can be eliminated as follows. Assume that $G'/M \cong S$, where $S$ is a nonabelian simple classical group of Lie type. It follows that $S$ is of characteristic $p$ and is defined over a field of size $q_1$ and $|S|_p = q^8$. Let $\chi$ be a unipotent character of $S$ different from $St_S$, which is extendible to Aut($S$). We know that $\chi(1)$ is also a degree of $G$. Moreover as the $p$-part of any irreducible character of $G_2(q)$ which is different from $q^6$ is at most $q^3$, we deduce that $\chi(1)_p \leq q^3$. From this inequality, we obtain an upper bound on the Lie rank of $S$. For the remaining cases, by using various choices of the character degrees of $S$, we finally obtain a contradiction. In Table 1 we list the degree of the Steinberg character of $S$ and also the unipotent characters of $S$ different from $St_S$ whose $p$-part are large. We will write $L^*_n(q)$ to
denote $L_n(q)$ when $\epsilon = +$ and $U_n(q)$ when $\epsilon = -$. We consider each family of the simple classical groups separately.

**Case 1:** $S \cong L_{q+1}(q_1)$. By Table 1 we have that $\ell(\ell - 1)/2 \leq 3\ell(\ell + 1)/12$, which implies that $\ell \leq 3$.

**Subcase 1(a):** $\ell = 1$. Then $S \cong L_2(q_1) \cong U_2(q)^2$ for $q_1 \geq 4$ and $q^6 = q_1$. As $q \geq 7$, we deduce that $q_1 > 7$. In this case $S$ has a character degree $q_1^6 + 1 = q^6 + 1$. This degree must divide a degree of $G$. Examining the degrees of $G$, it is clear that $q^6 + 1$ does not divide any of them. Note that this eliminates the possibilities that $S \cong L_2(4) \cong L_2(5) \cong A_5$ and $S \cong A_6 \cong L_2(2)$.

**Subcase 1(b):** $\ell = 2$. Then $S \cong L_2^-(q_1)$ and $q = q^2$. But the mixed degrees of $S$ are $q^2(q^2 + 1)$ and $q^6(q^2 + q^2 + 1)$. These must divide $q^6(q^2 + 1)$ or $q^6(q^2 + q^2 + 1)$. Certainly $q^2 + 1$ does not divide $q^6 + 1$, and $q^2 + 1$ is relatively prime to $q^4 + q^2 + 1$ so it does not divide $q^6(q^2 + q^2 + 1)$ either.

**Subcase 1(c):** $\ell = 3$. Then $S \cong L_3(q_1)$ and $q = q_1$. By [4, 13.8], $S$ has a unipotent character labeled by the partition $(2, 2)$ with degree $q^2(q^2 + 1)$. However $G$ has no such degree.

**Case 2:** $S \cong O_{2\ell+1}^+(q_1)$ or $S \cong S_{2\ell}(q_1)$, where $\ell \geq 2$. Then $q_1^2 = q^6$. As shown in Table 1 we have that $\ell^2 - 2\ell \leq 3\ell^2/6$, which implies that $\ell \leq 4$. Thus $2 \leq \ell \leq 4$.

When $\ell = 2$, we have $q^4 = q^6$, so $q_1 = q_1^3/2$. This group has a unipotent degree $q^3(q_1^6 + 1)/2$ corresponding to the symbol $((1/2)^1)$. This degree divides no degree of $G$. For $\ell = 3$, we have $q^6 = q^3$. From [4, 13.8], we see that $S$ has a unipotent character $\chi_\alpha$ of degree $\chi_\alpha(1) = q^4(q^2 + 1)(q^2 + 1)/(2q + 1)$ corresponding to the symbol $((1/0)^2)$. This is a mixed degree of $S$, hence must divide a mixed degree of $G$. But it has the factor $q^2 + 1$, which does not divide any of the mixed degrees of $G$. For $\ell = 4$, we have that $q_1^3 = q^3$. The group $S$ has a unipotent character labeled by the symbol $((1/0)^2)\ell$ for $B_1$ and $((1/0)^2)\ell - 1$ for $C_\ell$ with degrees $4^1 q^1 q^3 q^3 q^3 q^3 q^3 q^3 q^3 q^3 q^3$ and $q_1^3 q_1^3 q_1^3 q_1^3 q_1^3 q_1^3 q_1^3 q_1^3 q_1^3 q_1^3 q_1^3 q_1^3 q_1^3 q_1^3$, respectively. By comparing the $p$-part of these degrees, we obtain a contradiction.

**Case 3:** $S \cong O_{2\ell}^+(q_1), \ell \geq 4$. As shown in Table 1 we have that $\ell^2 - 3\ell + 2 \leq 3\ell(\ell - 1)/6$, which implies that $\ell \leq 4$. Hence $\ell = 4$ and so $q = q_1^2$. Since $q \geq 7$, we deduce that $q_1 \geq 3$.

Suppose $S \cong O_8^+(3)$. But $O_8^+(3)$ has a character degree $\chi_{113}(1) = 716800$, which is larger than any of the degrees of $G_2(9)$.

Suppose $S \cong O_8^+(q_1), q_1 > 3$. From [4, 13.8], $S$ possesses a unipotent character $\chi$ labeled by the symbol $((1/0)^2)\ell$ of degree $\chi(1) = q_1^3(q_1 + 1)(q_1^2 - q_1 + 1)/2$. This is a mixed degree of $S$, hence must divide a mixed degree of $G$. But the power of $q_1$ on this degree of $S$ implies it must divide $q^2(q^4 + q^2 + 1)$ or $q^3(q^2 + 1)$. If it divides one of the latter, then it divides $(q^2 - 1)(q^2 + 1) = q^6 - 1 = q_1^{12} - 1$. But $(q_1 + 1)^4$ does not divide $q_1^{12} - 1$. Now $q^4 + q^2 + 1 = q_1^6 + q_1^4 + 1$ and it is clear that $(q_1 + 1)^4$ does not divide this term either. So it is not possible for this degree of $S$ to divide a degree of $G$.

Suppose $S \cong O_8^-(q_1)$. By [4, 13.8], $S$ possesses a unipotent character $\chi$ labeled by the symbol $((1/0)^2)$ of degree $\chi(1) = q_1(q_1^4 + 1) = q_1(q_1^2 + 1)$. This degree divides no character degrees of $G_2(2)$.

6. **Proving $I_{G'}(\theta) = G'$ when $H \cong G_2(q)$**

Let $\theta \in \text{Irr}(M)$ with $\theta(1) = 1$. Suppose $I_{G'}(\theta) = I \leq G'$ for some $\theta \in \text{Irr}(M)$. Let $U$ be maximal such that $I \leq U \leq G'$. If $\theta^U = \sum_{i=1}^{s} e_i \phi_i$, for $\phi_i \in \text{Irr}(I)$, then by Lemma 2.2, $\phi_i(1)G' : |I|$ is a degree of $G'$ and thus divides some degree
Lemma 6.1. The only maximal subgroups of \( G_2(q) \), for \( q \geq 7 \), whose indices divide degrees of \( G \) are the parabolic subgroups with structure \([q^3] : GL_2(q)\) and the subgroups with structure \( SL_3(q) : 2 \) and \( SU_3(q) : 2 \).

Proof. We begin by examining the case when \( q \) is odd. The index of the parabolic subgroups with structure \([q^3] : GL_2(q)\) divides the degrees:

\[
\Phi_2 \Phi_3 \Phi_6, \Phi_1 \Phi_2 \Phi_3 \Phi_6, q \Phi_2 \Phi_3 \Phi_6, \Phi_2^2 \Phi_3 \Phi_6.
\]

The index of the maximal subgroups with structure \((SL_2(q) \circ SL_2(q)) \cdot 2\) does not divide degrees of \( G \) as the power on \( q \) in the index of the maximal subgroup is too large. The same reasoning eliminates the maximal subgroups with structure \( 2^3 \cdot L_3(2) \), as \( q \) is odd. The index of the maximal subgroups with structure \( SL_3(q) : 2 \) divides the degree \( q^3(q^3 \pm 1) \) if \( q \equiv 1 \pmod{6} \). The index of the maximal subgroups with structure \( SU_3(q) : 2 \) divides the degree \( q^3(q^3 - 1) \) if \( q \equiv 5 \pmod{6} \). Next consider the indices of maximal subgroups with structure \( G_2(q_0) \). Here \( \alpha > 1 \), so this index must divide a mixed degree of \( G \). Now \( 6a - 6 > 3\alpha \) when \( \alpha > 2 \). So this index will not divide a degree of \( G \) when \( \alpha > 2 \). If \( \alpha = 2 \), we have that the index of the subgroup is \( q_0^6(q_0^6 + 6)(q_0^6 + 1) \) and this does not divide \( q_0^6(q_0^6 \pm 1) \). The index of the maximal subgroups with structure \( 2^3 G_2(q_0) \) is too large to divide the mixed degrees \( q^3(q^3 + 1) \) and \( q^3(q^3 - 1) \) and the power of \( q \) is too large to divide the other mixed degrees of \( G \). The index of the maximal subgroups with structure
\[ PGL_2(q) \] does not divide degrees of \( G \) as the power on \( q \) in the index of the maximal subgroup is too large. The index of the maximal subgroups with structure \( L_2(8) \) and \( L_2(13) \) must divide mixed degrees of \( G \). As \( q \) is odd, we see that the power on \( q \) in the index is too large to allow the index to divide a mixed degree of \( G \). If the maximal subgroup of \( G_2(q) \) has structure \( G_2(2) \), then the index must divide a mixed degree of \( G \). As \( q \) is odd, the index will not divide a mixed degree of \( G \) if \( q \neq 3 \) and \( q \neq 7 \). If \( q = 7 \), then the power on \( q \) in the index is 5 and thus too large to divide a mixed degree of \( G \). If \( q = 3 \), the mixed degrees of \( G \) have power 2 or less on \( q \), so this does not divide a degree of \( G \). Finally, if the maximal subgroup of \( G_2(q) \) has structure of \( J_1 \), then the power on 11 is too large to divide a mixed degree of \( G \).

We will now consider the maximal subgroups of \( G_2(q) \) for even \( q \). These were determined in [7]. The maximal subgroup structure is summarized in Table 3.

The index of the parabolic subgroups with structure \( [q^5] : GL_2(q) \) divides the degrees:

\[ \Phi_2 \Phi_3 \Phi_6, \quad \Phi_1 \Phi_2 \Phi_3 \Phi_6, \quad q\Phi_2 \Phi_3 \Phi_6, \quad \Phi_2^2 \Phi_3 \Phi_6. \]

The index of the maximal subgroups with structure \( SL_2(q) \times SL_2(q) \) does not divide degrees of \( G \) as the power on \( q \) in this index is too large. The index of the maximal subgroups with structure \( SL_3(q) : 2 \) divides the degree \( q^3(q^3 + 1) \) if \( q \equiv 4 \) (mod 6).

The index of the maximal subgroups with structure \( SU_3(q) : 2 \) divides the degree \( q^3(q^3 - 1) \) if \( q \equiv 2 \) (mod 6). Next consider the index of maximal subgroups with structure \( G_2(2^m) \). Here \( a/m > 1 \), so this index must divide a mixed degree of \( G \). Now \( 6a - 6m > 3a \) when \( a/m > 2 \). So this index will not divide a degree of \( G \) when \( a/m > 2 \). If \( a/m = 2 \), then \( a = 2m \) and the index of the subgroup is \( 2^m(2^{2m} + 1)(2^{2m} + 1) \). Now \( q = 2^a = 2^{2m} \) and so the only degrees of \( G \) this index could divide are \( q^3(q^3 + 1) = 2^m(2^{6m} + 1) \) and it is clearly too large to do so. Thus, the only maximal subgroups of \( G_2(q) \), for \( q \geq 7 \), whose indices divide degrees of \( G \) are the parabolic subgroups and the subgroups with structure \( SL_3(q) : 2 \) and \( SU_3(q) : 2 \).

6.1. Maximal Subgroups with Structure \( SL_3(q) : 2 \) or \( SU_3(q) : 2 \).

First, consider the maximal subgroups with structure \( SL_3(q) : 2 \) or \( SU_3(q) : 2 \), whose indices divide the degree \( q^3(q^3 + 1) \) or \( q^3(q^3 - 1) \), respectively. We have that \( \frac{1}{2}q^3(q^3 \pm 1) / U/M : I/M \langle \phi_i(1) \rangle = q^3(q^3 \pm 1) \). Thus \( U/M : I/M \langle \phi_i(1) \rangle \) divides 2. There are three cases to consider.

Case 1: \( |U/M : I/M| = 2 \). Then \( \phi_i(1) = 1 \) for all \( i \). Since \( |U/M : I/M| = 2 \), \( I/M \) is a normal subgroup of \( U/M \) of index 2. As \( \phi_i(1) = 1 \), \( \phi_i \) is an extension of \( \theta \) to \( I \). By Lemma 2.2(b), \( (\phi_i \tau)^G \in \text{Irr}(G') \) for all \( \tau \in \text{Irr}(I/M) \). Now \( (\phi_i \tau)^G(1) = |G' : I| \phi_i(1) \tau(1) \in \text{cd}(G'), \) which forces \( \tau(1) = 1 \) since \( |G' : I| = q^3(q^3 \pm 1) \) and this degree must divide a degree of \( G \). Hence \( I/M \) is abelian. But \( I/M \leq U/M \) and \( U/M \cong SL_3(q) : 2 \) or \( U/M \cong SU_3(q) : 2 \). Since \( I/M \) is abelian, it is solvable. But \( (U/M)/(I/M) \) is of order 2 and thus solvable. But then \( U/M \) is solvable, which is a contradiction. So this case is not possible.

Case 2: \( |U/M : I/M| = 1 \) and \( \phi_i(1) = 1 \) for some \( i \). In this case, \( I/M \) is a maximal subgroup of \( G_2(q) \) with structure \( SL_3(q) : 2 \) or \( SU_3(q) : 2 \). As \( \phi_i(1) = 1 \), \( \phi_i \) is an extension of \( \theta \) to \( I \). By Lemma 2.2(b), \( (\phi_i \tau)^G \in \text{Irr}(G') \) for all \( \tau \in \text{Irr}(I/M) \). Now \( (\phi_i \tau)^G(1) = |G' : I| \phi_i(1) \tau(1) \in \text{cd}(G'), \) which forces \( \tau(1) = 1 \) or \( \tau(1) = 2 \). Hence all the irreducible characters of \( I/M \) are of degree one or two. But this is
not the case, as \( I/M \) has structure \( SL_3(q) : 2 \) or \( SU_3(q) : 2 \). In particular \( I/M \) has an irreducible character of degree \( q^3 \), which is greater than 2. Hence it is not possible for \( U/M \) to have the structure \( SL_3(q) : 2 \) or \( SU_3(q) : 2 \).

**Case 3**: \( |U : I/M| = 1 \) and \( \phi_i(1) = 2 \) for all \( i \). In this case, \( I/M \) is a maximal subgroup of \( G_2(q) \) with structure \( SL_3(q) : 2 \) or \( SU_3(q) : 2 \). Since \( q \geq 7 \), we deduce that \( I/M \) is nonsolvable. As \( \theta \) is \( I \)-invariant and all the irreducible constituents of \( \theta^I \) have the same degree 2, by [15, Theorem 2.3] we deduce that \( I/M \) is solvable, a contradiction.

### 6.2. Maximal Parabolic Subgroups with Structure \([q^5] : GL_2(q)\)

Suppose \( U/M \) is a maximal parabolic subgroup of \( G' \). We have \( I/M \leq U/M \leq G'/M \).

Then \( \Phi_2 \Phi_1 \Phi_0 U/I|\phi_0(1) \) divides a degree of \( G \). So \( |U : I|\phi_0(1) \) divides \( q^2 \) or \( q \). We have \( U/M \cong [q^2] : SL_2(q) : (q - 1) \). Let \( L \) and \( V \) be subgroups of \( U \) such that \( L/M \cong [q^2] \) and \( V/M \cong SL_2(q) \). It follows that \( L \leq U, LV \leq U, \) and \( L \cap V = M \). Let \( t = |U : I| \). Recall that \( \theta \in \text{Irr}(M) \) with \( \theta(1) = 1 \), \( I = \text{Irr}(\theta) \), \( \theta^I = \sum_{i=1}^s e_i \phi_i \), where \( \phi_i \in \text{Irr}(I) \), \( e_i \geq 1 \), and \( t \phi_1(1) \) divides \( q^2 \) or \( q \), where \( 1 \leq i \leq s \). Finally let \( W = LV \leq U \).

**Case 1**: \( t \phi_1(1) \mid q \) for some \( j \). Then \( t \) is relatively prime to \( p \). As \( L \leq U \), we deduce that \( I \leq IL \leq U \) so \( t = |U : I| = |U : IL| \cdot |IL : I| \). Now \( |IL : I| = |L : L \cap I| \).

Thus if \( L \not\leq I \), then \( |L : L \cap I| > 1 \) and is divisible by \( p \) thus \( p \mid t \), which is a contradiction. Therefore we can assume that \( L \leq I \leq U \).

Let \( \lambda \in \text{Irr}(L) \) be an irreducible constituent of \( \phi_j \) when restricted to \( L \). Since \( (\phi_j)_{|M} = e_j \theta \), we deduce that \( \lambda_{|M} = e \theta \) for some integer \( e \). Now \( \lambda(1) = (\lambda(1)/\theta(1)) = e \).

By Gallagher’s Theorem, we have that \( \gamma = \lambda \tau \in \text{Irr}(L) \) and \( p \mid \gamma(1) \). Since \( L \leq I \), we have that if \( \phi \in \text{Irr}(L) \), then \( \phi(1) \mid q \). Now let \( \phi_k \) be an irreducible constituent of \( \gamma^I \).

We next show that \( \gamma \) is \( W \)-invariant. As \( W \leq I \) and \( p \mid \gamma(1) \), if \( \phi \in \text{Irr}(W|\gamma) \) then \( \phi(1) \mid q \). Suppose that \( \gamma \) is not \( W \)-invariant and let \( J = J_I(\gamma) \). Then \( L \leq J < W \).

Let \( \delta \) be an irreducible constituent of \( \gamma^J \). Then \( \delta^I(1) = |W : J| \delta(1) \mid q \), and \( |W : J| \) is divisible by the index of a maximal subgroup of \( W/L \cong SL_2(q) \), \( q \geq 7 \). By Lemma 2.5, we deduce that \( q = 7 \) or \( q = 11 \) and also \( J/L \) is isomorphic to a nonabelian subgroup of \( SL_2(q) \) of index \( q \).

Thus \( |W : J| = q \) and hence \( \delta(1) = 1 \), which is impossible as \( p \mid (\gamma(1) \gamma(1))/\delta(1) \neq 1 \). Thus \( \gamma \) is \( W \)-invariant at every irreducible constituent of \( \gamma^W \) is a power of a fixed prime \( p \), and so by [15, Theorem 2.3], \( W/L \) is solvable, which is a contradiction.

**Case 2**: \( t \phi_1(1) \mid q \) for all \( i \). Then \( t \mid q \) and all \( \phi_i(1) \) are \( p \)-powers.

We will show that \( I/M \) is nonsolvable. If \( t = 1 \), then \( I = U \) and hence \( I/M \cong [q^5] : GL_2(q) \) is obviously nonsolvable. Thus we assume that \( t > 1 \). If \( W \leq I \), then as \( W/L \cong SL_2(q) \), we deduce that \( I/M \) is nonsolvable. Hence we assume \( W \not\leq I \) and \( t > 1 \). Then \( I \leq W/I \leq U \) and so \( t = |U : W/I| \cdot |W/I : I| = |U : W/I| \cdot |W : W \cap I| \).

As \( W \not\leq I \), we have \( W \cap I \leq W \) so \( |W : W \cap I| \) is a nontrivial divisor of \( q \). Let \( X = W \cap I \). Assume that \( L \leq X \). Then \( X/L \) is a proper subgroup of \( W/L \cong SL_2(q) \) and hence \( |W : X| \) is divisible by the index of some maximal subgroup of \( SL_2(q) \).
whose index divides $q$. By Lemma 2.5 we deduce that $q = 7$ or $q = 11$, and that $X/L$ is nonabelian. In both cases, as $q$ is prime and $t > 1$, we obtain that $t = q$ and hence since $t_0(1) \mid q$, we have that $t_0(1) = 1$ for all $i$. Now Gallagher’s Theorem yields that $I/M$ is abelian, which is a contradiction as $I/M$ possesses a nonabelian section $X/L$. Thus $L \not\leq X$. Since $L \leq W$ and $X = W \cap I \leq L$, we deduce that $X \leq XL \leq W$ and $L \leq XL \leq W$. It follows that $|W : X| = |W : XL| \cdot |XL : X|$ is a nontrivial divisor of $q$. If $W = XL$, then $SL_2(q) \cong W/L \cong XL/L \cong X/X \cap L$ and hence since $M \leq X \cap L \leq X \leq I$, we deduce that $I/M$ is nonsolvable. Hence we assume $|W : XL| \neq 1$ and so $|W : XL|$ is a nontrivial divisor of $q$, $L \leq XL \leq W$ and $XL/L \cong X/X \cap L$. Arguing as in the previous case, we obtain a contradiction.

Therefore $I/M$ is nonsolvable. We now have that $\theta \in \text{Irr}(M)$, $\theta$ is $I$-invariant and for any $\chi \in \text{Irr}(I/\theta)$, we have that $\chi(1)$ is a power of a fixed prime $p$, and so by [15 Theorem 2.3] we have that $I/M$ is solvable which is a contradiction.

7. Establishing $M = 1$ when $H \cong G_2(q)$

For $q \geq 7$, the Schur multiplier of $G_2(q)$ is trivial. Thus $M' = M$ by Step 3 and Lemma 2.6. If $M$ is abelian, we are done. Suppose $M = M' \neq 1$. Let $M/N \cong T^k$ be a chief factor of $G'$, where $T$ is a nonabelian simple group. Then $T$ possesses a nonprincipal irreducible character $\lambda$ which is extendible to $\text{Aut}(T)$. By Lemma 2.4 $\lambda^k \in \text{Irr}(M/N)$ extends to $\psi \in \text{Irr}(G'/N)$. Let $\chi \in \text{Irr}(G'/M)$ be such that $\chi(1) = q^6$. By Gallagher’s Theorem, we deduce that $\psi\chi \in \text{Irr}(G')$ so $\psi(1)\chi(1) = \lambda(1)^k q^6 \in \text{cd}(G')$ and hence $\lambda(1)^k q^6$ must divide some character degree of $G$, which is impossible. Hence $M = 1$.

8. Establishing $G = G' \times C_G(G')$ when $H \cong G_2(q)$

Recall that $q = p^a \geq 7$. Suppose $G' \times C_G(G') \leq G$. Then $G$ induces on $G'$ some outer automorphism $\sigma$. By Lemma 2.7 some conjugacy class of $G'$ is not fixed by $\sigma$. As the irreducible characters of $G'$ separate the conjugacy classes of $G'$, there exists some $\chi \in \text{Irr}(G')$ such that $\chi$ is not fixed by $\sigma$. Let $\psi \in \text{Irr}(G)$ such that
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