PRIME DIVISORS OF IRREDUCIBLE CHARACTER DEGREES

HUNG P. TONG-VIET

Abstract. Let $G$ be a finite group. We denote by $\rho(G)$ the set of primes which divide some character degrees of $G$ and by $\sigma(G)$ the largest number of distinct primes which divide a single character degree of $G$. We show that $|\rho(G)| \leq 2\sigma(G) + 1$ when $G$ is an almost simple group. For arbitrary finite groups $G$, we show that $|\rho(G)| \leq 2\sigma(G) + 1$ provided that $\sigma(G) \leq 2$.

1. Introduction

Throughout this paper, all groups are finite and all characters are complex characters. The set of all complex irreducible characters of $G$ is denoted by $\text{Irr}(G)$ and we let $\text{cd}(G)$ be the set of all complex irreducible character degrees of $G$. We define $\rho(G)$ to be the set of primes which divide some character degree of $G$. For $\chi \in \text{Irr}(G)$, let $\pi(\chi)$ be the set of all prime divisors of $\chi(1)$ and let $\sigma(\chi) = |\pi(\chi)|$. Moreover, $\sigma(G)$ is defined to be the maximum value of $\sigma(\chi)$ when $\chi$ runs over the set $\text{Irr}(G)$. Huppert’s $\rho - \sigma$ Conjecture proposed by B. Huppert in [H] states that if $G$ is a solvable group, then $|\rho(G)| \leq 2\sigma(G)$; and if $G$ is an arbitrary group, then $|\rho(G)| \leq 3\sigma(G)$. For solvable groups, this conjecture has been verified by Manz [Man1] and Gluck [G] when $\sigma(G) = 1$ and 2, respectively; and in general, it is proved by Manz and Wolf [MW] that $|\rho(G)| \leq 3\sigma(G) + 2$. For arbitrary groups, Manz [Man2] showed that $|\rho(G)| = 3$ if $G$ is nonsolvable and $\sigma(G) = 1$. Recently, it has been proved by Casolo and Dolfi [CD] that $|\rho(G)| \leq 7\sigma(G)$ for any arbitrary groups $G$. In [MW], Manz and Wolf proposed that for any group $G$,

\[ |\rho(G)| \leq 2\sigma(G) + 1.\]

We call this new conjecture the Strengthened Huppert’s $\rho - \sigma$ conjecture. Obviously, this new conjecture is stronger than the original one. In this paper, we first improve the result due to Alvis and Barry in [AB] by proving the following.

Theorem A. Let $G$ be an almost simple group. Then $|\rho(G)| \leq 2\sigma(G)$ unless $G \cong \text{PSL}_2(2^f)$ with $f \geq 2$ and $|\pi(2^f - 1)| = |\pi(2^f + 1)|$. For the exceptions, we have $|\rho(G)| = 2\sigma(G) + 1$.

Date: October 12, 2015.

2000 Mathematics Subject Classification. Primary 20C15.

Key words and phrases. character degrees; Huppert’s $\rho - \sigma$ conjecture.
Theorem B. Let $G$ be a finite group. If $\sigma(G) \leq 2$, then $|\rho(G)| \leq 2\sigma(G) + 1$.

Notice that Theorem B is also a generalization to [1], Theorem A.

Notation. For a positive integer $n$, we denote the set of all prime divisors of $n$ by $\pi(n)$. If $G$ is a group, then we write $\pi(G)$ instead of $\pi(|G|)$ for the set of all prime divisors of the order of $G$. If $N \leq G$ and $\theta \in \text{Irr}(N)$, then the inertia group of $\theta$ in $G$ is denoted by $I_G(\theta)$. We write $\text{Irr}(G|\theta)$ for the set of all irreducible constituents of $\theta^G$. Moreover, if $\chi \in \text{Irr}(G)$, then $\text{Irr}(\chi_N)$ is the set of all irreducible constituents of $\chi$ when restricted to $N$. Recall that a group $G$ is said to be an almost simple group with socle $S$ if there exists a nonabelian simple group $S$ such that $S \leq G \leq \text{Aut}(S)$. The greatest common divisor of two integers $a$ and $b$ is $\gcd(a, b)$. Denote by $\Phi_k := \Phi_k(q)$ the value of the $k$th cyclotomic polynomial evaluated at $q$. Other notation is standard.

2. Proof of Theorem A

If $G$ is an almost simple group, then $G$ has no normal abelian Sylow subgroup and so by Ito-Michler’s Theorem [Mich, Theorem 5.4], $\rho(G) = \pi(G)$. This fact will be used without any further reference.

Lemma 2.1. Let $S$ be a sporadic simple group, the Tits group or an alternating group of degree at least 7. If $G$ is an almost simple group with socle $S$, then

$$|\pi(G)| = |\pi(S)| \leq 2\sigma(G).$$

Proof. Observe first that if $S$ is one of the simple groups in the lemma, and $G$ is any almost simple group with socle $S$, then $\pi(G) = \pi(S)$. Since $S \leq G$, we see that $\sigma(S) \leq \sigma(G)$. Thus it suffices to show that $|\pi(S)| \leq 2\sigma(S)$. By using [Atlas], we can easily check that $|\pi(S)| \leq 2\sigma(S)$ when $S$ is a sporadic simple group, the Tits group or an alternating group of degree $n$ with $7 \leq n \leq 14$. Finally, if $S \cong A_n$ with $n \geq 15$, then the result in [BW] yields that $|\pi(S)| = \sigma(S)$. This completes the proof. □

For $\epsilon = \pm$, we use the convention that $\text{PSL}_n^\epsilon(q)$ is $\text{PSL}_n(q)$ if $\epsilon = +$ and $\text{PSU}_n(q)$ if $\epsilon = -$. Let $q \geq 2$ and $n \geq 3$ be integers with $(n, q) \neq (6, 2)$. A prime $\ell$ is called a primitive prime divisor of $q^n - 1$ if $\ell \mid q^n - 1$ but $\ell \nmid q^m - 1$ for any $m < n$. By Zsigmondy’s Theorem [Z], the primitive prime divisors of $q^n - 1$ always exist. We denote by $\ell_n(q)$ the smallest primitive prime divisor of $q^n - 1$. In Table [1] which is taken from [Mal], we give the orders of two maximal tori $T_i$ and the corresponding two primitive prime divisors $\ell_i$, for $i = 1, 2$, of classical groups.
We consider the following cases.

and

\[
\chi \text{ characters}
\]

So, we can assume that \( q > 2 \). The cases when \( q = 3 \) or 4 can be checked directly using [Atlas]. So, we can assume that \( q \geq 5 \). By [S5], \( S \) possesses irreducible characters \( \chi_i, i = 1, 2, \) with degree

\[
\chi_1(1) = (q - c_1)^2(q + c_1) \quad \text{and} \quad \chi_2(1) = q(q^2 + \epsilon q + 1).
\]
Let \( d = \gcd(3, q - \epsilon 1) \). Then
\[
|S| = \frac{1}{d} q^3 (q^2 - 1) (q^3 - \epsilon 1) = \frac{1}{d} q^3 (q - \epsilon 1)^2 (q + \epsilon 1)(q^2 + \epsilon q + 1)
\]
and so
\[
\pi(S) = \pi(\chi_1) \cup \pi(\chi_2).
\]
Therefore, \(|\pi(S)| \leq 2\sigma(S)\) as wanted.

**Case 3:** \( S \cong \text{PSp}_4(q) \) with \( q = p^f > 2 \).

By [E, S], \( S \) has two irreducible characters \( \chi_i, i = 1, 2 \), with degree \( \Phi_1^2 \Phi_2^2 \) and \( q\Phi_1 \Phi_4 \), respectively. Since
\[
|S| = \frac{1}{d} q^4 \Phi_2^2 \Phi_4
\]
where \( d = \gcd(2, q - 1) \), we deduce that
\[
\pi(S) = \pi(\chi_1) \cup \pi(\chi_2),
\]
and thus \(|\pi(S)| \leq 2\sigma(S)\).

**Case 4:** \( S \) is one of the remaining simple groups in the list \( \mathcal{C} \).

Using [Atlas], it is routine to check that \(|\pi(S)| \leq 2\sigma(S)\) in all these cases.

**Case 5:** \( S \) is not in the list \( \mathcal{C} \).

We consider the following setup. Let \( \mathcal{G} \) be a simple simply connected algebraic group defined over a field of size \( q \) in characteristic \( p \) and let \( F \) be a Frobenius map on \( \mathcal{G} \) such that \( S \cong L/Z \), where \( L := \mathcal{G}^F \) and \( Z := Z(L) \). Let the pair \((\mathcal{G}^*, F^*)\) be dual to \((\mathcal{G}, F)\) and let \( L^* := \mathcal{G}^{F^*} \). By Lusztig theory, the irreducible characters of \( \mathcal{G}^F \) are partitioned into rational series \( \mathcal{E}(\mathcal{G}^F, (s)) \) which are indexed by \((\mathcal{G}^{F^*})\)-conjugacy classes \( (s) \) of semisimple elements \( s \in \mathcal{G}^{F^*} \). Furthermore, if \( \gcd(|s|, |Z|) = 1 \), then every \( \chi \in \mathcal{E}(\mathcal{G}^F, (s_i)) \) is trivial at \( Z \) and thus \( \chi \in \text{Irr}(S) = \text{Irr}(L/Z) \). (See [MT] p. 349). Notice also that \( \chi(1) \) is divisible by \( |L^*: C_{L^*}(s)|_{p'} \).

For simple classical groups of Lie type, the restriction on \( S \) guarantees that both primitive prime divisors \( \ell_i \) in Table 1 exist. Let \( s_i \in \mathcal{G}^{F^*} \) with \( |s_i| = \ell_i, i = 1, 2 \). Then \( C_{L^*}(s_i) = T_i \) for \( i = 1, 2 \), where \( T_i \) are maximal tori of \( L^* \). Similarly, for each simple exceptional group of Lie type \( S \), by [MT] Lemma 2.3 one can find two semisimple elements \( s_i \in \mathcal{G}^{F^*} \) with \( |s_i| = \ell_i, i = 1, 2 \). In both cases, we have that \((\ell_i, |Z|) = 1 \) for \( i = 1, 2 \) and if \( a := \gcd(|C_{L^*}(s_1)|, |C_{L^*}(s_2)|) \), then either \( a = 1 \) or if a prime \( r \) divides \( a \), then \( r \) also divides \( |L^*: C_{L^*}(s)|_{p'} \) for some \( i \). Let \( \chi_i \in \mathcal{E}(\mathcal{G}^F, (s_i)), i = 1, 2 \) such that \( \chi_i(1) = |L^*: C_{L^*}(s_i)|_{p'} \). Then \( \chi_i \in \text{Irr}(S) \) for \( i = 1, 2 \) and
\[
\pi(S) = \{p\} \cup \pi(\chi_1) \cup \pi(\chi_2).
\]
Notice that $p$ is relatively prime to both $\chi_i(1)$ for $i = 1, 2$. So
\[
|\pi(S)| = |\{p\} \cup \pi(\chi_1) \cup \pi(\chi_2)|
= 1 + |\pi(\chi_1)| + |\pi(\chi_2)| - |\pi(\chi_1) \cap \pi(\chi_2)|
= \sigma(\chi_1) + \sigma(\chi_2) - (|\pi(\chi_1) \cap \pi(\chi_2)| - 1)
\leq 2\sigma(S) - (|\pi(\chi_1) \cap \pi(\chi_2)| - 1).
\]
If we can show that $|\pi(\chi_1) \cap \pi(\chi_2)| \geq 1$, then clearly $|\pi(S)| \leq 2\sigma(S)$ and we are done. By way of contradiction, assume that $\pi(\chi_1) \cap \pi(\chi_2)$ is empty. Then $\gcd(\chi_1(1), \chi_2(1)) = 1$ and so
\[
\gcd(|L^*: C_{L^*}(s_1)|_{p'}, |L^*: C_{L^*}(s_2)|_{p'}) = 1.
\]
It follows that $|L^*|_{p'}$ must divide $|C_{L^*}(s_1)|_{p'} \cdot |C_{L^*}(s_2)|_{p'}$. However, we can check by using [MT, Lemma 2.3] and Table 1 that this is not the case. The proof is now complete. \qed

We now prove Theorem A which we restate here.

**Theorem 2.3.** Let $G$ be an almost simple group. Then $|\rho(G)| \leq 2\sigma(G)$ unless $G \cong \text{PSL}_2(2^f)$ with $|\pi(2^f - 1)| = |\pi(2^f + 1)|$. For the exceptions, we have $|\rho(G)| = 2\sigma(G) + 1$.

**Proof.** Let $G$ be an almost simple group with simple socle $S$. Since $S \leq G$, we obtain that $\sigma(S) \leq \sigma(G)$.

**Case 1:** $S \cong \text{PSL}_2(q)$ with $q = 2^f \geq 4$.

It is well known that $|S| = q(q^2 - 1)$, $\gcd(2^f - 1, 2^f + 1) = 1$ and
\[
\text{cd}(S) = \{1, q - 1, q, q + 1\}.
\]
If $|\pi(q - 1)| = |\pi(q + 1)|$, then
\[
\pi(S) = \{2\} \cup \pi(q - 1) \cup \pi(q + 1)
\]
and thus $|\pi(S)| = 2\sigma(S) + 1$ as $\sigma(S) = |\pi(2^f \pm 1)|$. Assume that $|\pi(q - 1)| \neq |\pi(q + 1)|$. Then $|\pi(2^f + \delta)| > |\pi(2^f - \delta)|$ for some $\delta \in \{\pm 1\}$. Hence, $\sigma(S) = |\pi(2^f + \delta)|$ and thus
\[
|\pi(S)| = |\{2\} \cup \pi(2^f - \delta) \cup \pi(2^f + \delta)| = 1 + |\pi(2^f - \delta)| + |\pi(2^f + \delta)|.
\]
Since $|\pi(2^f + \delta)| \geq |\pi(2^f - \delta)| + 1$, we obtain that
\[
|\rho(G)| \leq 2|\pi(2^f + \delta)| \leq 2\sigma(G).
\]
Thus the result holds when $G = S$. Assume now that $|G : S|$ is nontrivial. We know that $\text{Aut}(S) = S \cdot \langle \varphi \rangle$, where $\varphi$ is a field automorphism of $S$ of order $f$. Thus $G = S \cdot \langle \psi \rangle$, with $\psi \in \langle \varphi \rangle$. If $f = 2$, then $G \cong A_5 \cdot 2$ and obviously $|\pi(G)| = 2\sigma(G)$. Hence we can assume that $f > 2$. Clearly, if $f \equiv 2 \pmod{4}$ and $G = S \cdot \langle \varphi \rangle$, then
\[ |G : S| > 2. \] So by [W] Theorem A, \( G \) has two irreducible characters \( \chi_i \in \text{Irr}(G) \), \( i = 1, 2 \), with \( \chi_1(1) = |G : S|(q - 1) \) and \( \chi_2(1) = |G : S|(q + 1) \). Obviously
\[
\pi(G) = \{2\} \cup \pi(\chi_1) \cup \pi(\chi_2)
\]
and
\[
\pi(\chi_1) \cap \pi(\chi_2) = \pi(|G : S|) \neq \emptyset.
\]
If \( |G : S| \) is even, then
\[
|\rho(G)| = |\pi(\chi_1) \cup \pi(\chi_2)| \leq |\pi(\chi_1)| + |\pi(\chi_2)| \leq 2\sigma(G).
\]
If \( |G : S| \) is odd, then
\[
|\rho(G)| = |\{2\} \cup \pi(\chi_1) \cup \pi(\chi_2)|
= 1 + |\pi(\chi_1)| + |\pi(\chi_2)| - |\pi(\chi_1) \cap \pi(\chi_2)|
\leq \sigma(\chi_1) + \sigma(\chi_2) - (|\pi(|G : S|)| - 1)
\leq 2\sigma(G).
\]

**Case 2:** \( S \) is a sporadic simple group, the Tits group or an alternating group of degree at least 7. By Lemma 2.1, we obtain that \( |\rho(G)| \leq 2\sigma(G) \).

**Case 3:** \( S \) is a finite simple group of Lie type in characteristic \( p \) and \( S \) is not the Tits groups nor \( \text{PSL}_2(2^f) \) with \( f \geq 2 \).

**Subcase 3a:** \( \pi(\gamma) = \pi(S) \).

By Lemma 2.2, we have that \( |\pi(S)| \leq 2\sigma(S) \). Thus
\[
|\rho(G)| = |\pi(S)| \leq 2\sigma(S) \leq 2\sigma(G).
\]

**Subcase 3b:** \( \pi := \pi(G) - \pi(S) \) is nonempty.

Let \( A \) be the subgroup of the group of coprime outer automorphisms of \( S \) induced by the action of \( G \) on \( S \). By [MT] Lemma 2.10, \( A \) is cyclic and central in \( \text{Out}(S) \). Moreover, \( A \) is generated by a fixed field automorphism \( \gamma \in \text{Out}(S) \). It follows that the group \( S \cdot A \) is normal in \( G \) and \( \pi(S \cdot A) = \pi(G) \). Thus we can assume that \( G = S \cdot A \) with \( A = \langle \gamma \rangle \) and \( \gamma \) a field automorphism of \( S \). Furthermore, \( \pi(\gamma) = \pi \). Replacing \( A \) by a normal subgroup if necessary, we can also assume that \( |A| = |\gamma| \) is the product of all distinct primes in \( \pi \).

As in the proof of Lemma 2.2, let \( \mathcal{G} \) be a simple simply connected algebraic group defined over a field of size \( q = p^f \) in characteristic \( p \) and let \( F \) be a Frobenius map of \( \mathcal{G} \) such that \( S \cong L/Z \), where \( L := \mathcal{G}^F \) and \( Z := Z(L) \). Let the pair \( (\mathcal{G}, F^*) \) be dual to \( (\mathcal{G}, F) \) and let \( L^* := \mathcal{G}^{*F^*} \). As \( \pi \subseteq \pi(F) \), where \( \pi = \pi(G) - \pi(S) \), it is easy to check that both the primitive prime divisors in [MT] Lemmas 2.3, 2.4 exist and thus
one can find two semisimple elements $s_i \in G^{s_F}$ with $|s_i| = \ell_i$ such that $(\ell_i, |Z|) = 1$ for $i = 1, 2$. Arguing as in the proof of Lemma \ref{lem:2.2}, we obtain that
\[
\pi(S) = \{p\} \cup \pi(\chi_1) \cup \pi(\chi_2),
\]
where $\chi_i \in \mathcal{E}(G, (s_i))$ such that $\chi_i(1) = |L^*: C_{L^*}(s_i)|_p$ and $\chi_i$ can be considered as characters of $S$, for $i = 1, 2$.

We next claim that the inertia group for both $\chi_i, i = 1, 2$ in $G$ is exactly $S$. It suffices to show that no field automorphism of $S$ of prime order can fix $\chi_i$ for $i = 1, 2$. Let $\tau$ be a field automorphism of $S$ of prime order $s$. We can extend $\tau$ to an automorphism of $G^s$ and $G^{s_F}$ which we also denote by $\tau$. Notice that $C_{G^s}(\tau)$ is a finite group of Lie type of the same type as that of $G^s$ but defined over a field of size $q^{1/s}$. Now it is straightforward to check that both $\ell_i, i = 1, 2$, are relatively prime to $|C_{G^s}(\tau)|$. Hence $G^{s_F}$-conjugacy classes $(s_i)$ of $s_i$ in $G^{s_F}$ are not $\tau$-invariant for $i = 1, 2$. (See \cite[Proposition 2.6]{MT}.) Then $s_i$ and $s_1$ are not $G^{s_F}$-conjugate for $i = 1, 2$, and thus $\chi_i \in \mathcal{E}(G, (s_i)), i = 1, 2$ are not $\tau$-invariant. (See \cite[Theorem 2.7]{MT}. Therefore, we obtain that $\chi_i^G \in \text{Irr}(G)$ for $i = 1, 2$, hence $\chi_i^G(1) = |G:S|\chi_i(1) \in \text{cd}(G)$. Since
\[
\pi(S) = \{p\} \cup \pi(\chi_1) \cup \pi(\chi_2)
\]
we obtain that
\[
\pi(G) = \{p\} \cup \pi(|G:S|\chi_1(1)) \cup \pi(|G:S|\chi_2(1)) = \{p\} \cup \pi(G_1^G) \cup \pi(G_2^G).
\]
Moreover, $p \nmid |G:S|\chi_i(1) = \chi_i^G(1)$ for $i = 1, 2$, and
\[
|\pi(G_1^G) \cap \pi(G_2^G)| \geq 1.
\]
Therefore,
\[
|\pi(G)| = 1 + \sigma(\chi_1^G) + \sigma(\chi_2^G) - |\pi(G_1^G) \cap \pi(G_2^G)| \\
\leq 2\sigma(G) - (|\pi(G_1^G) \cap \pi(G_2^G)| - 1) \\
\leq 2\sigma(G).
\]
The proof is now complete. \hfill $\square$

The next results will be needed in the proof of Theorem B.

\begin{lemma}
Let $S$ be a nonabelian simple group. If $\sigma(S) \leq 2$, then $S$ is one of the following groups.
\begin{enumerate}
\item $S \cong \text{PSL}_2(2^f)$ with $|\pi(2^f \pm 1)| \leq 2$ and so $|\pi(S)| \leq 5$.
\item $S \cong \text{PSL}_2(q)$ with $q > 5$ odd and $|\pi(q \pm 1)| \leq 2$ and so $|\pi(S)| \leq 4$.
\item $S \in \{M_{11}, A_7, 2B_2(8), 2B_2(32), \text{PSL}_2^+(3), \text{PSL}_3^+(4), \text{PSL}_3(8)\}$ and $|\pi(S)| = 4$.
\end{enumerate}
\end{lemma}

\begin{proof}
As $S$ is a nonabelian simple group, we have that $|\pi(S)| \geq 3$. If $S \cong \text{PSL}_2(q)$ with $q \geq 4$, then the lemma follows easily as the character degree set of $S$ is known. Now assume that $S \not\cong \text{PSL}_2(q)$. Then Lemmas \ref{lem:2.2} and \ref{lem:2.1} imply that $|\pi(S)| \leq 2\sigma(S)$.
\end{proof}
that 

By Theorem A and thus $4 \leq |\pi(G)| = |\pi(S)|$. 

By checking the list of nonabelian simple groups with at most four prime divisors in [HL], we deduce that only those nonabelian simple groups appearing in (3) above satisfy the hypotheses of the lemma. 

**Lemma 2.5.** Let $G$ be an almost simple group with simple socle $S$. If $\sigma(G) \leq 2$, then $\pi(G) = |\pi(S)|$, where $S$ is one of the simple groups in Lemma 2.4.

**Proof.** Since $\sigma(S) \leq \sigma(G) \leq 2$, we deduce that $S$ is isomorphic to one of the simple groups in the conclusion of Lemma 2.4. If $|\pi(S)| = 3$, then $S$ is one of the simple groups in [HL] Table 1 and we can check that $\pi(G) = |\pi(S)|$ in these cases. Thus we assume that $|\pi(S)| \geq 4$. Now if $G = S$, then we have nothing to prove. So, we assume that $G \neq S$. In particular, $G \not\cong \text{PSL}_2(2^f)$ with $f \geq 2$. Then $|\pi(G)| \leq 2\sigma(G) \leq 4$ by Theorem A and thus $4 \leq |\pi(S)| \leq |\pi(G)| \leq 4$, which forces $|\pi(S)| = |\pi(G)|$ and hence $\pi(G) = |\pi(S)|$ as wanted. 

\[ \square \]

3. **Proof of Theorem B**

The following two lemmas are obvious.

**Lemma 3.1.** Let $A$ and $B$ be groups such that $|\rho(A)| \geq 3$ and $|\rho(B)| \geq 3$. If $\sigma(A \times B) \leq 2$, then $\sigma(A) = 1 = \sigma(B)$.

**Lemma 3.2.** Let $N$ be a normal subgroup of a group $G$. If $\rho(G/N) = |\pi(G/N)|$, then $3 \leq \rho(G) - \rho(G/N) \leq \rho(N)$.

Recall that the solvable radical of a group $G$ is the largest normal solvable subgroup of $G$.

**Lemma 3.3.** Let $G$ be a nonsolvable group and let $N$ be the solvable radical of $G$. Suppose that $\sigma(G) \leq 2$ and $|\rho(G)| \geq 5$. Then $G/N$ is an almost simple group.

**Proof.** We first claim that if $M/N$ is a chief factor of $G$, then $M/N$ is a nonabelian simple group.

Let $M$ be a normal subgroup of $G$ such that $M/N$ is a chief factor of $G$. Since $N$ is the largest normal solvable subgroup of $G$, we deduce that $M/N$ is nonsolvable so that $M/N \cong S^k$ for some integer $k \geq 1$ and some nonabelian simple group $S$. Let $C/N = C_{G/N}(M/N)$. Then $G/C$ embeds into $\text{Aut}(S^k)$.

Assume first that $k \geq 3$. Since $|\rho(S)| = |\pi(S)| \geq 3$, there exist three distinct prime divisors $r_i, 1 \leq i \leq 3$, and three characters $\psi_i \in \text{Irr}(S)$ for $1 \leq i \leq 3$ with $r_i | \psi_i(1)$. Let 

$$\varphi = \psi_1 \times \psi_2 \times \psi_3 \times 1 \times \cdots \times 1 \in \text{Irr}(S^k).$$

Then $\sigma(\varphi) \geq 3$, which is a contradiction since 

$$\sigma(S^k) = \sigma(M/N) \leq \sigma(M) \leq \sigma(G) \leq 2.$$
Thus $k \leq 2$.

Now assume that $k = 2$. Let $B/C = (G/C) \cap \text{Aut}(S)^2$. Then $G/B$ is a nontrivial subgroup of the symmetric group of degree 2 and thus $|G : B| = 2$. Since $S^2 \cong MC/C \leq B/C \leq G/C$ and $\sigma(G) \leq 2$, we deduce that $\sigma(S^2) \leq 2$ and thus $\sigma(S) = 1$ by Lemma 3.1. By [Man2, Satz 8], we know that $S$ is isomorphic to either PSL$_2$(4) or PSL$_2$(8). In both cases, we obtain that $\pi(\text{Aut}(S)) = \pi(S)$, hence $\pi(B/C) = \pi(S)$. Moreover, as $|G : B| = 2$, we deduce that $\pi(G/C) = \pi(S)$. As $G/C$ has no nontrivial normal abelian Sylow subgroups, Ito-Michler’s Theorem yields that $\rho(G/C) = \pi(G/C) = \pi(S)$. Since $|\pi(G/C)| = |\pi(S)| = 3$ and $|\rho(G)| \geq 5$, there exists $r \in \rho(G) - \pi(G/C)$. Then $r > 2$ and $r \in \rho(C)$ by Lemma 3.2. Let $\theta \in \text{Irr}(C)$ such that $r | \theta(1)$. Let $L$ be a normal subgroup of $MC$ such that $L/C \cong S$. Notice that $MC/C \cong S^2$. By applying [1] Lemma 4.2, $\theta$ extends to $\theta_0 \in \text{Irr}(L)$. By Gallagher’s Theorem [1] Corollary 6.17, $\theta_0 \mu \in \text{Irr}(L)$ for all $\mu \in \text{Irr}(L/C)$. Let $\mu_0 \in \text{Irr}(L/C)$ with $2 \mid \mu_0(1)$ and let $\varphi = \theta_0 \mu_0 \in \text{Irr}(L)$. Then $\pi(\varphi(1)) = \{2, r\}$ with $r > 2$. As $MC/L \cong S$, we can apply [1] Lemma 4.2 again to obtain that $\varphi$ extends to $\varphi_0 \in \text{Irr}(MC)$ and then by applying Gallagher’s Theorem, $\varphi_0 \mu \in \text{Irr}(MC)$ for all $\mu \in \text{Irr}(MC/L)$. Clearly, $MC/L \cong S$ has an irreducible character $\tau \in \text{Irr}(MC/L)$ with $s \mid \tau(1)$, where $s \notin \{2, r\}$. We now have that $\varphi_0 \tau \in \text{Irr}(MC)$. But then this is a contradiction as $\pi(\varphi_0(1)\tau(1)) = \{2, s, r\}$. This contradiction shows that $k = 1$ as wanted.

Let $M/N$ be a chief factor of $G$ and let $C/N = \text{C}_G(N)(M/N)$. We claim that $C = N$ and thus $G/N$ is an almost simple group as required. By the claim above, we know that $M/N \cong S$ for some nonabelian simple group $S$. Hence, $G/C$ is an almost simple group with socle $MC/C \cong M/N$. Suppose by contradiction that $C \neq N$. Now let $L/N$ be a chief factor of $G$ with $N \leq L \leq C$. By the claim above, we deduce that $L/N$ is isomorphic to some nonabelian simple group. In particular, $|\rho(C/N)| \geq |\pi(L/N)| \geq 3$. We have that $MC/N \cong C/N \times M/N$. Since $\sigma(MC/N) \leq \sigma(MC) \leq \sigma(G) \leq 2$, we deduce that $\sigma(C/N \times M/N) \leq 2$ and thus by Lemma 3.1, $\sigma(C/N) = 1 = \sigma(M/N)$. By [Man2], we have $C/N \cong T \times A$, where $A$ is abelian, $T$ is a nonabelian simple group and $S, T \in \{\text{PSL}_2(4), \text{PSL}_2(8)\}$. Since $C \leq G$ and the solvable radical $W$ of $C$ is characteristic in $C$, we obtain that $W \leq G$ and thus $W \leq N$ as $N$ is the largest normal solvable subgroup of $G$. Clearly, $N \leq W$ as $N$ is also a solvable normal subgroup of $C$, so $W = N$. Therefore, $C/N$ has no nontrivial normal abelian subgroup. Thus $A = 1$ and hence $C/N \cong T$. Since $\pi(G/C) = \pi(M/N)$ and $G/N$ has no normal abelian Sylow subgroup, we obtain that

$$\rho(G/N) = \pi(G/N) = \pi(C/N) \cup \pi(M/N) = \pi(S) \cup \pi(T).$$

It follows that

$$|\rho(G/N)| = |\pi(S) \cup \pi(T)| \leq |\pi(\text{PSL}_2(4)) \cup \pi(\text{PSL}_2(8))| = 4.$$
Hence, $\rho(G) - \rho(G/N)$ is nonempty. Now let $r \in \rho(G) - \rho(G/N)$. As $\{2, 3\} \subseteq \rho(G/N)$, we obtain that $r \notin \{2, 3\}$. By Lemma 3.2, $r \in \rho(N)$ and hence $r \mid \theta(1)$ for some $\theta \in \text{Irr}(N)$. Since $\sigma(M) \leq \sigma(G) \leq 2$ and $M/N \cong S$, by [1] Lemma 4.2 we deduce that $\theta$ extends to $\theta_0 \in \text{Irr}(M)$. Now let $\lambda \in \text{Irr}(M/N)$ with $2 \mid \lambda(1)$. By Gallagher’s Theorem, $\varphi = \theta_0\lambda \in \text{Irr}(M)$ with $\pi(\varphi(1)) = \{2, r\}$. Notice that $r \geq 5$ since $r \notin \{2, 3\}$. Now let $K = MC \leq G$. Then $K/M \cong T$ and $\sigma(K) \leq 2$. Applying the same argument as above, we deduce that $\varphi$ extends to $\varphi_0 \in \text{Irr}(K)$. Clearly, $K/M \cong T$ has an irreducible character $\mu$ with $3 \mid \mu(1)$ and thus by Gallagher’s Theorem again, $\psi = \varphi_0\mu \in \text{Irr}(K)$ and obviously $\sigma(\psi) \geq 3$, which is a contradiction.

We are now ready to prove Theorem B which we state here.

**Theorem 3.4.** Let $G$ be a group. If $\sigma(G) \leq 2$, then $|\rho(G)| \leq 2\sigma(G) + 1$.

**Proof.** Let $G$ be a counterexample to the theorem with minimal order. Then $\sigma(G) \leq 2$ but $|\rho(G)| > 2\sigma(G) + 1$. If $G$ is solvable or $G$ is nonsolvable with $\sigma(G) = 1$, then $|\rho(G)| \leq 2\sigma(G) + 1$ by [3, G, 1, Man2], which is a contradiction. Thus we can assume that $G$ is nonsolvable, $\sigma(G) = 2$ and $|\rho(G)| \geq 6$. Let $N$ be the solvable radical of $G$. By Lemma 3.3, $G/N$ is an almost simple group with simple socle $M/N$. Since $\sigma(M/N) \leq \sigma(G/N) \leq \sigma(G) = 2$, we deduce from Lemmas 2.5 and 2.4 that $|\pi(G/N)| = |\pi(M/N)| \leq 5$.

As $|\rho(G)| \geq 6$, we have that $\rho(G) - \rho(G/N)$ is nonempty and let $r \in \rho(G) - \rho(G/N)$. By Lemma 3.2, $r \mid \theta(1)$ for some $\theta \in \text{Irr}(N)$. Since $\sigma(M) \leq 2$, by applying [1, Lemma 4.2], we deduce that $\theta$ extends to $\theta_0 \in \text{Irr}(M)$. Using Gallagher’s Theorem, we must have that $\sigma(M/N) = 1$ and hence $M/N \cong \text{PSL}_2(4)$ or $\text{PSL}_2(8)$. Thus $|\pi(G/N)| = |\pi(M/N)| = 3$, hence $|\tau| \geq 3$, with $\tau = \rho(G) - \rho(G/N)$. By Lemma 3.2, we have that $\tau \subseteq \rho(N)$ and since $N$ is solvable, by applying Pálfy’s Condition [P, Theorem], there exists $\psi \in \text{Irr}(N)$ such that $\psi(1)$ is divisible by two distinct primes in $\tau$. Now by applying [1, Lemma 4.2] again, we obtain a contradiction. This contradiction shows that $|\rho(G)| \leq 2\sigma(G) + 1$ as wanted.

**Acknowledgment**

The author is grateful to the referee for the careful reading of the manuscript and for his or her corrections and suggestions.

**References**


D. White, Character degrees of extensions of $\text{PSL}_2(q)$ and $\text{SL}_2(q)$, *J. Group Theory* **16** (2013), 1–33.


H.P. Tong-Viet, Department of Mathematical Sciences, Kent State University, Kent, Ohio 44242, USA

E-mail address: htongvie@kent.edu