PRIME DIVISORS OF IRREDUCIBLE CHARACTER DEGREES

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Abstract. Let G be a finite group. We denote by $\rho(G)$ the set of primes which divide some character degrees of G and by $\sigma(G)$ the largest number of distinct primes which divide a single character degree of G. We show that $|\rho(G)| \leq 2\sigma(G)+1$ when G is an almost simple group. For arbitrary finite groups G, we show that $|\rho(G)| \leq 2\sigma(G)+1$ provided that $\sigma(G) \leq 2$.

1. Introduction

Throughout this paper, all groups are finite and all characters are complex characters. The set of all complex irreducible characters of G is denoted by $\mathrm{Irr}(G)$ and we let $\mathrm{cd}(G)$ be the set of all complex irreducible character degrees of G. We define $\rho(G)$ to be the set of primes which divide some character degree of G. For $\chi \in \mathrm{Irr}(G)$, let $\pi(\chi)$ be the set of all prime divisors of $\chi(1)$ and let $\sigma(\chi) = |\pi(\chi)|$. Moreover, $\sigma(G)$ is defined to be the maximum value of $\sigma(\chi)$ when χ runs over the set $\mathrm{Irr}(G)$. Huppert's $\rho - \sigma$ Conjecture proposed by B. Huppert in [H] states that if G is a solvable group, then $|\rho(G)| \leq 2\sigma(G)$; and if G is an arbitrary group, then $|\rho(G)| \leq 3\sigma(G)$. For solvable groups, this conjecture has been verified by Manz [Man1] and Gluck [G] when $\sigma(G) = 1$ and 2, respectively; and in general, it is proved by Manz and Wolf [MW] that $|\rho(G)| \leq 3\sigma(G)+2$. For arbitrary groups, Manz [Man2] showed that $|\rho(G)| = 3$ if G is nonsolvable and $\sigma(G) = 1$. Recently, it has been proved by Casolo and Dolfi [CD] that $|\rho(G)| \leq 7\sigma(G)$ for any arbitrary groups G. In [MW], Manz and Wolf proposed that for any group G,

$$|\rho(G)| \leq 2\sigma(G) + 1.$$

We call this new conjecture the Strengthened Huppert's $\rho-\sigma$ conjecture. Obviously, this new conjecture is stronger than the original one. In this paper, we first improve the result due to Alvis and Barry in [AB] by proving the following.

Theorem A. *Let G be an almost simple group. Then $|\rho(G)| \leq 2\sigma(G)$ unless $G \cong \mathrm{PSL}_2(2^f)$ with $f \geq 2$ and $|\pi(2^f - 1)| = |\pi(2^f + 1)|$. For the exceptions, we have $|\rho(G)| = 2\sigma(G) + 1$.*

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This verifies the Strengthened Huppert's $\rho-\sigma$ conjecture for almost simple groups. In the next theorem, we verify this new conjecture for groups G with $\sigma(G) \leq 2$.

Theorem B. *Let G be a finite group. If $\sigma(G) \leq 2$, then $|\rho(G)| \leq 2\sigma(G)+1$.*

Notice that Theorem B is also a generalization to [T, Theorem A].

Notation. For a positive integer n, we denote the set of all prime divisors of n by $\pi(n)$. If G is a group, then we write $\pi(G)$ instead of $\pi(|G|)$ for the set of all prime divisors of the order of G. If $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$, then the inertia group of θ in G is denoted by $I_G(\theta)$. We write $\mathrm{Irr}(G|\theta)$ for the set of all irreducible constituents of θ^G. Moreover, if $\chi \in \mathrm{Irr}(G)$, then $\mathrm{Irr}(\chi_N)$ is the set of all irreducible constituents of χ when restricted to N. Recall that a group G is said to be an almost simple group with socle S if there exists a nonabelian simple group S such that $S \trianglelefteq G \leq \mathrm{Aut}(S)$. The greatest common divisor of two integers a and b is $\gcd(a,b)$. Denote by $\Phi_k := \Phi_k(q)$ the value of the kth cyclotomic polynomial evaluated at q. Other notation is standard.

2. Proof of Theorem A

If G is an almost simple group, then G has no normal abelian Sylow subgroup and so by Ito-Michler's Theorem [Mich, Theorem 5.4], $\rho(G) = \pi(G)$. This fact will be used without any further reference.

Lemma 2.1. *Let S be a sporadic simple group, the Tits group or an alternating group of degree at least 7. If G is an almost simple group with socle S, then*

$$|\pi(G)| = |\pi(S)| \leq 2\sigma(G).$$

Proof. Observe first that if S is one of the simple groups in the lemma, and G is any almost simple group with socle S, then $\pi(G) = \pi(S)$. Since $S \trianglelefteq G$, we see that $\sigma(S) \leq \sigma(G)$. Thus it suffices to show that $|\pi(S)| \leq 2\sigma(S)$. By using [Atlas], we can easily check that $|\pi(S)| \leq 2\sigma(S)$ when S is a sporadic simple group, the Tits group or an alternating group of degree n with $7 \leq n \leq 14$. Finally, if $S \cong \mathrm{A}_n$ with $n \geq 15$, then the result in [BW] yields that $|\pi(S)| = \sigma(S)$. This completes the proof. $\square$

For $\epsilon = \pm$, we use the convention that $\mathrm{PSL}_n^\epsilon(q)$ is $\mathrm{PSL}_n(q)$ if $\epsilon = +$ and $\mathrm{PSU}_n(q)$ if $\epsilon = -$. Let $q \geq 2$ and $n \geq 3$ be integers with $(n,q) \neq (6,2)$. A prime ℓ is called a *primitive prime divisor* of $q^n - 1$ if $\ell \mid q^n - 1$ but $\ell \nmid q^m - 1$ for any $m < n$. By Zsigmondy's Theorem [Z], the primitive prime divisors of $q^n - 1$ always exist. We denote by $\ell_n(q)$ the smallest primitive prime divisor of $q^n - 1$. In Table 1 which is taken from [Mal], we give the orders of two maximal tori T_i and the corresponding two primitive prime divisors ℓ_i, for $i = 1, 2$, of classical groups.

TABLE 1. Two tori for classical groups

$G = G(q)$	$\|T_1\|$	$\|T_2\|$	ℓ_1	ℓ_2
A_n	$(q^{n+1}-1)/(q-1)$	q^n-1	$\ell_{n+1}(q)$	$\ell_n(q)$
${}^2\mathrm{A}_n$, $(n \equiv 0(4))$	$(q^{n+1}+1)/(q+1)$	q^n-1	$\ell_{2n+2}(q)$	$\ell_n(q)$
${}^2\mathrm{A}_n$, $(n \equiv 1(4))$	$(q^{n+1}-1)/(q+1)$	q^n+1	$\ell_{(n+1)/2}(q)$	$\ell_{2n}(q)$
${}^2\mathrm{A}_n$, $(n \equiv 2(4))$	$(q^{n+1}+1)/(q+1)$	q^n-1	$\ell_{2n+2}(q)$	$\ell_{n/2}(q)$
${}^2\mathrm{A}_n$, $(n \equiv 3(4))$	$(q^{n+1}-1)/(q+1)$	q^n+1	$\ell_{n+1}(q)$	$\ell_{2n}(q)$
$\mathrm{B}_n, \mathrm{C}_n$ $(n \geq 3$ odd$)$	q^n+1	q^n-1	$\ell_{2n}(q)$	$\ell_n(q)$
$\mathrm{B}_n, \mathrm{C}_n$ $(n \geq 2$ even$)$	q^n+1	$(q^{n-1}+1)(q+1)$	$\ell_{2n}(q)$	$\ell_{2n-2}(q)$
D_n, $(n \geq 5$ odd$)$	$(q^{n-1}+1)(q+1)$	q^n-1	$\ell_{2n-2}(q)$	$\ell_n(q)$
D_n, $(n \geq 4$ even$)$	$(q^{n-1}+1)(q+1)$	$(q^{n-1}-1)(q-1)$	$\ell_{2n-2}(q)$	$\ell_{n-1}(q)$
${}^2\mathrm{D}_n$	q^n+1	$(q^{n-1}+1)(q-1)$	$\ell_{2n}(q)$	$\ell_{2n-2}(q)$

Let $\mathcal{C}$ be the set consisting of the following simple groups:

$$\begin{array}{lllll} \mathrm{PSL}_2(q), & \mathrm{PSL}_3(q), & \mathrm{PSU}_3(q), & \mathrm{PSp}_4(q) & \mathrm{PSL}_4(2), \\ \mathrm{PSL}_6(2), & \mathrm{PSL}_7(2), & \mathrm{PSU}_4(2), & \mathrm{PSU}_4(3), & \mathrm{PSU}_6(2), \\ \mathrm{Sp}_4(2)', & \mathrm{Sp}_6(2), & \mathrm{Sp}_8(2), & \Omega_7(3), & \Omega_8^+(2), \\ \Omega_8^-(2), & {}^3\mathrm{D}_4(2), & \mathrm{G}_2(2)', & \mathrm{G}_2(3), & \mathrm{G}_2(4). \end{array}$$

Lemma 2.2. *Let S be a finite simple group of Lie type in characteristic p which is not the Tits groups nor $\mathrm{PSL}_2(2^f)$ with $f \geq 2$. Then $|\pi(S)| \leq 2\sigma(S)$.*

Proof. We consider the following cases.

Case 1: $S \cong \mathrm{PSL}_2(q)$, where $q = p^f \geq 5$ is odd.

Since $\mathrm{PSL}_2(5) \cong \mathrm{PSL}_2(4)$, we can assume that $q > 5$. In this case, all character degrees of S divide $q, q-1$ or $q+1$. Observe that

$$\pi(S) = \{p\} \cup \pi(q-1) \cup \pi(q+1), \{p\} \cap \pi(q \pm 1) = \emptyset$$

and

$$\pi(q-1) \cap \pi(q+1) = \{2\}.$$

Hence, we obtain that

$$\begin{aligned} |\pi(S)| &= 1 + \sigma(q+1) + \sigma(q-1) - |\pi(q-1) \cap \pi(q+1)| \\ &= \sigma(q+1) + \sigma(q-1) \leq 2\sigma(S). \end{aligned}$$

Case 2: $S \cong \mathrm{PSL}_3^\epsilon(q)$ with $q = p^f$ and $\epsilon = \pm$. As $\mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$ and $\mathrm{PSU}_3(2)$ is not simple, we can assume that $q > 2$. The cases when $q = 3$ or 4 can be checked directly using [Atlas]. So, we can assume that $q \geq 5$. By [SF], S possesses irreducible characters $\chi_i, i = 1, 2$, with degree

$$\chi_1(1) = (q - \epsilon 1)^2(q + \epsilon 1) \text{ and } \chi_2(1) = q(q^2 + \epsilon q + 1).$$

Let $d = \gcd(3, q - \epsilon 1)$. Then

$$|S| = \frac{1}{d}q^3(q^2-1)(q^3-\epsilon 1) = \frac{1}{d}q^3(q-\epsilon 1)^2(q+\epsilon 1)(q^2+\epsilon q+1)$$

and so

$$\pi(S) = \pi(\chi_1) \cup \pi(\chi_2).$$

Therefore, $|\pi(S)| \leq 2\sigma(S)$ as wanted.

Case 3: $S \cong \mathrm{PSp}_4(q)$ with $q = p^f > 2$.

By [E, S], S has two irreducible characters $\chi_i, i = 1, 2$, with degree $\Phi_1^2\Phi_2^2$ and $q\Phi_1\Phi_4$, respectively. Since

$$|S| = \frac{1}{d}q^4\Phi_1^2\Phi_2^2\Phi_4$$

where $d = \gcd(2, q-1)$, we deduce that

$$\pi(S) = \pi(\chi_1) \cup \pi(\chi_2),$$

and thus $|\pi(S)| \leq 2\sigma(S)$.

Case 4: S is one of the remaining simple groups in the list $\mathcal{C}$.

Using [Atlas], it is routine to check that $|\pi(S)| \leq 2\sigma(S)$ in all these cases.

Case 5: S is not in the list $\mathcal{C}$.

We consider the following setup. Let $\mathscr{G}$ be a simple simply connected algebraic group defined over a field of size q in characteristic p and let F be a Frobenius map on $\mathscr{G}$ such that $S \cong L/Z$, where $L := \mathscr{G}^F$ and $Z := \mathrm{Z}(L)$. Let the pair $(\mathscr{G}^*, F^*)$ be dual to $(\mathscr{G}, F)$ and let $L^* := \mathscr{G}^{*F^*}$. By Lusztig theory, the irreducible characters of $\mathscr{G}^F$ are partitioned into rational series $\mathscr{E}(\mathscr{G}^F, (s))$ which are indexed by $(\mathscr{G}^{*F^*})$-conjugacy classes (s) of semisimple elements $s \in \mathscr{G}^{*F^*}$. Furthermore, if $\gcd(|s|, |Z|) = 1$, then every $\chi \in \mathscr{E}(\mathscr{G}^F, (s_i))$ is trivial at Z and thus $\chi \in \mathrm{Irr}(S) = \mathrm{Irr}(L/Z)$. (See [MT, p. 349]). Notice also that $\chi(1)$ is divisible by $|L^* : \mathbf{C}_{L^*}(s)|_{p'}$.

For simple classical groups of Lie type, the restriction on S guarantees that both primitive prime divisors ℓ_i in Table 1 exist. Let $s_i \in \mathscr{G}^{*F^*}$ with $|s_i| = \ell_i, i = 1, 2$. Then $\mathbf{C}_{L^*}(s_i) = T_i$ for $i = 1, 2$, where T_i are maximal tori of L^*. Similarly, for each simple exceptional group of Lie type S, by [MT, Lemma 2.3] one can find two semisimple elements $s_i \in \mathscr{G}^{*F^*}$ with $|s_i| = \ell_i, i = 1, 2$. In both cases, we have that $(\ell_i, |Z|) = 1$ for $i = 1, 2$ and if $a := \gcd(|\mathbf{C}_{L^*}(s_1)|, |\mathbf{C}_{L^*}(s_2)|)$, then either $a = 1$ or if a prime r divides a, then r also divides $|L^* : \mathbf{C}_{L^*}(s_i)|_{p'}$ for some i. Let $\chi_i \in \mathscr{E}(\mathscr{G}^F, (s_i)), i = 1, 2$ such that $\chi_i(1) = |L^* : \mathbf{C}_{L^*}(s_i)|_{p'}$. Then $\chi_i \in \mathrm{Irr}(S)$ for $i = 1, 2$ and

$$\pi(S) = \{p\} \cup \pi(\chi_1) \cup \pi(\chi_2).$$

Notice that p is relatively prime to both $\chi_i(1)$ for $i = 1, 2$. So

$$\begin{aligned}
|\pi(S)| &= |\{p\} \cup \pi(\chi_1) \cup \pi(\chi_2)| \\
&= 1 + |\pi(\chi_1)| + |\pi(\chi_2)| - |\pi(\chi_1) \cap \pi(\chi_2)| \\
&= \sigma(\chi_1) + \sigma(\chi_2) - (|\pi(\chi_1) \cap \pi(\chi_2)| - 1) \\
&\leq 2\sigma(S) - (|\pi(\chi_1) \cap \pi(\chi_2)| - 1).
\end{aligned}$$

If we can show that $|\pi(\chi_1) \cap \pi(\chi_2)| \geq 1$, then clearly $|\pi(S)| \leq 2\sigma(S)$ and we are done. By way of contradiction, assume that $\pi(\chi_1) \cap \pi(\chi_2)$ is empty. Then $\gcd(\chi_1(1), \chi_2(1)) = 1$ and so

$$\gcd(|L^* : \mathbf{C}_{L^*}(s_1)|_{p'}, |L^* : \mathbf{C}_{L^*}(s_2)|_{p'}) = 1.$$

It follows that $|L^*|_{p'}$ must divide $|\mathbf{C}_{L^*}(s_1)|_{p'} \cdot |\mathbf{C}_{L^*}(s_2)|_{p'}$. However, we can check by using [MT, Lemma 2.3] and Table 1 that this is not the case. The proof is now complete. $\square$

We now prove Theorem A which we restate here.

Theorem 2.3. *Let G be an almost simple group. Then $|\rho(G)| \leq 2\sigma(G)$ unless $G \cong \mathrm{PSL}_2(2^f)$ with $|\pi(2^f - 1)| = |\pi(2^f + 1)|$. For the exceptions, we have $|\rho(G)| = 2\sigma(G) + 1$.*

Proof. Let G be an almost simple group with simple socle S. Since $S \trianglelefteq G$, we obtain that $\sigma(S) \leq \sigma(G)$.

Case 1: $S \cong \mathrm{PSL}_2(q)$ with $q = 2^f \geq 4$.

It is well know that $|S| = q(q^2 - 1)$, $\gcd(2^f - 1, 2^f + 1) = 1$ and

$$\mathrm{cd}(S) = \{1, q - 1, q, q + 1\}.$$

If $|\pi(q - 1)| = |\pi(q + 1)|$, then

$$\pi(S) = \{2\} \cup \pi(q - 1) \cup \pi(q + 1)$$

and thus $|\pi(S)| = 2\sigma(S) + 1$ as $\sigma(S) = |\pi(2^f \pm 1)|$. Assume that $|\pi(q-1)| \neq |\pi(q+1)|$. Then $|\pi(2^f + \delta)| > |\pi(2^f - \delta)|$ for some $\delta \in \{\pm 1\}$. Hence, $\sigma(S) = |\pi(2^f + \delta)|$ and thus

$$|\pi(S)| = |\{2\} \cup \pi(2^f - \delta) \cup \pi(2^f + \delta)| = 1 + |\pi(2^f - \delta)| + |\pi(2^f + \delta)|.$$

Since $|\pi(2^f + \delta)| \geq |\pi(2^f - \delta)| + 1$, we obtain that

$$|\rho(G)| \leq 2|\pi(2^f + \delta)| \leq 2\sigma(G).$$

Thus the result holds when $G = S$. Assume now that $|G : S|$ is nontrivial. We know that $\mathrm{Aut}(S) = S \cdot \langle \varphi \rangle$, where φ is a field automorphism of S of order f. Thus $G = S \cdot \langle \psi \rangle$, with $\psi \in \langle \varphi \rangle$. If $f = 2$, then $G \cong \mathrm{A}_5 \cdot 2$ and obviously $|\pi(G)| = 2\sigma(G)$. Hence we can assume that $f > 2$. Clearly, if $f \equiv 2 \pmod 4$ and $G = S \cdot \langle \varphi \rangle$, then

$|G:S|>2$. So by [W, Theorem A], G has two irreducible characters $\chi_i \in \mathrm{Irr}(G)$, $i=1,2$, with $\chi_1(1)=|G:S|(q-1)$ and $\chi_2(1)=|G:S|(q+1)$. Obviously

$$\pi(G)=\{2\}\cup\pi(\chi_1)\cup\pi(\chi_2)$$

and

$$\pi(\chi_1)\cap\pi(\chi_2)=\pi(|G:S|)\neq\emptyset.$$

If $|G:S|$ is even, then

$$|\rho(G)|=|\pi(\chi_1)\cup\pi(\chi_2)|\leq|\pi(\chi_1)|+|\pi(\chi_2)|\leq 2\sigma(G).$$

If $|G:S|$ is odd, then

$$\begin{array}{rcl}|\rho(G)| & = & |\{2\}\cup\pi(\chi_1)\cup\pi(\chi_2)| \\ & = & 1+|\pi(\chi_1)|+|\pi(\chi_2)|-|\pi(\chi_1)\cap\pi(\chi_2)| \\ & = & \sigma(\chi_1)+\sigma(\chi_2)-(|\pi(|G:S|)|-1) \\ & \leq & \sigma(\chi_1)+\sigma(\chi_2) \\ & \leq & 2\sigma(G).\end{array}$$

Case 2: S is a sporadic simple group, the Tits group or an alternating group of degree at least 7.

By Lemma 2.1, we obtain that $|\rho(G)|\leq 2\sigma(G)$.

Case 3: S is a finite simple group of Lie type in characteristic p and S is not the Tits groups nor $\mathrm{PSL}_2(2^f)$ with $f\geq 2$.

Subcase 3a: $\pi(G)=\pi(S)$.

By Lemma 2.2, we have that $|\pi(S)|\leq 2\sigma(S)$. Thus

$$|\rho(G)|=|\pi(S)|\leq 2\sigma(S)\leq 2\sigma(G).$$

Subcase 3b: $\pi:=\pi(G)-\pi(S)$ is nonempty.

Let A be the subgroup of the group of coprime outer automorphisms of S induced by the action of G on S. By [MT, Lemma 2.10], A is cyclic and central in $\mathrm{Out}(S)$. Moreover, A is generated by a fixed field automorphism $\gamma\in\mathrm{Out}(S)$. It follows that the group $S\cdot A$ is normal in G and $\pi(S\cdot A)=\pi(G)$. Thus we can assume that $G=S\cdot A$ with $A=\langle\gamma\rangle$ and γ a field automorphism of S. Furthermore, $\pi(\gamma)=\pi$. Replacing A by a normal subgroup if necessary, we can also assume that $|A|=|\gamma|$ is the product of all distinct primes in π.

As in the proof of Lemma 2.2, let $\mathscr{G}$ be a simple simply connected algebraic group defined over a field of size $q=p^f$ in characteristic p and let F be a Frobenius map of $\mathscr{G}$ such that $S\cong L/Z$, where $L:=\mathscr{G}^F$ and $Z:=\mathrm{Z}(L)$. Let the pair $(\mathscr{G}^*,F^*)$ be dual to $(\mathscr{G},F)$ and let $L^*:=\mathscr{G}^{*F^*}$. As $\pi\subseteq\pi(f)$, where $\pi=\pi(G)-\pi(S)$, it is easy to check that both the primitive prime divisors in [MT, Lemmas 2.3, 2.4] exist and thus one can find two semisimple elements $s_i \in \mathscr{G}^{*F^*}$ with $|s_i| = \ell_i$ such that $(\ell_i, |Z|) = 1$ for $i = 1, 2$. Arguing as in the proof of Lemma 2.2, we obtain that

$$\pi(S) = \{p\} \cup \pi(\chi_1) \cup \pi(\chi_2),$$

where $\chi_i \in \mathscr{E}(\mathscr{G}^F, (s_i))$ such that $\chi_i(1) = |L^* : \mathbf{C}_{L^*}(s_i)|_{p'}$ and χ_i can be considered as characters of S, for $i = 1, 2$.

We next claim that the inertia group for both $\chi_i, i = 1, 2$ in G is exactly S. It suffices to show that no field automorphism of S of prime order can fix χ_i for $i = 1, 2$. Let τ be a field automorphism of S of prime order s. We can extend τ to an automorphism of $\mathscr{G}^F$ and $\mathscr{G}^{*F^*}$ which we also denote by τ. Notice that $\mathbf{C}_{\mathscr{G}^{*F^*}}(\tau)$ is a finite group of Lie type of the same type as that of $\mathscr{G}^{*F^*}$ but defined over a field of size $q^{1/s}$. Now it is straight forward to check that both $\ell_i, i = 1, 2$, are relatively prime to $|\mathbf{C}_{\mathscr{G}^{*F^*}}(\tau)|$. Hence $\mathscr{G}^{*F^*}$-conjugacy classes (s_i) of s_i in $\mathscr{G}^{*F^*}$ are not τ-invariant for $i = 1, 2$. (See [MT, Proposition 2.6].) Then $\tau(s_i)$ and s_i are not $\mathscr{G}^{*F^*}$-conjugate for $i = 1, 2$, and thus $\chi_i \in \mathscr{E}(\mathscr{G}^F, (s_i)), i = 1, 2$ are not τ-invariant. (See [MT, Theorem 2.7].) Therefore, we obtain that $\chi_i^G \in \mathrm{Irr}(G)$ for $i = 1, 2$, hence $\chi_i^G(1) = |G : S|\chi_i(1) \in \mathrm{cd}(G)$. Since

$$\pi(S) = \{p\} \cup \pi(\chi_1) \cup \pi(\chi_2) \text{ and } \pi(G) = \pi(S) \cup \pi(|G : S|),$$

we obtain that

$$\pi(G) = \{p\} \cup \pi(|G : S|\chi_1(1)) \cup \pi(|G : S|\chi_2(1)) = \{p\} \cup \pi(\chi_1^G) \cup \pi(\chi_2^G).$$

Moreover, $p \nmid |G : S|\chi_i(1) = \chi_i^G(1)$ for $i = 1, 2$, and

$$|\pi(\chi_1^G) \cap \pi(\chi_2^G)| \geq 1.$$

Therefore,

$$\begin{aligned} |\pi(G)| &= 1 + \sigma(\chi_1^G) + \sigma(\chi_2^G) - |\pi(\chi_1^G) \cap \pi(\chi_2^G)| \\ &\leq 2\sigma(G) - (|\pi(\chi_1^G) \cap \pi(\chi_2^G)| - 1) \\ &\leq 2\sigma(G). \end{aligned}$$

The proof is now complete. □

The next results will be needed in the proof of Theorem B.

Lemma 2.4. *Let S be a nonabelian simple group. If $\sigma(S) \leq 2$, then S is one of the following groups.*

(1) $S \cong \mathrm{PSL}_2(2^f)$ *with* $|\pi(2^f \pm 1)| \leq 2$ *and so* $|\pi(S)| \leq 5$.
(2) $S \cong \mathrm{PSL}_2(q)$ *with* $q > 5$ *odd and* $|\pi(q \pm 1)| \leq 2$ *and so* $|\pi(S)| \leq 4$.
(3) $S \in \{\mathrm{M}_{11}, \mathrm{A}_7, {}^2\mathrm{B}_2(8), {}^2\mathrm{B}_2(32), \mathrm{PSL}_3^{\pm}(3), \mathrm{PSL}_3^{\pm}(4), \mathrm{PSL}_3(8)\}$ *and* $|\pi(S)| = 4$.

Proof. As S is a nonabelian simple group, we have that $|\pi(S)| \geq 3$. If $S \cong \mathrm{PSL}_2(q)$ with $q \geq 4$, then the lemma follows easily as the character degree set of S is known. Now assume that $S \ncong \mathrm{PSL}_2(q)$. Then Lemmas 2.2 and 2.1 imply that $|\pi(S)| \leq 2\sigma(S)$.

So, $3 \leq |\pi(S)| \leq 4$. By checking the list of nonabelian simple groups with at most four prime divisors in [HL], we deduce that only those nonabelian simple groups appearing in (3) above satisfy the hypotheses of the lemma. □

Lemma 2.5. *Let G be an almost simple group with simple socle S. If $\sigma(G) \leq 2$, then $\pi(G) = \pi(S)$, where S is one of the simple groups in Lemma 2.4.*

Proof. Since $\sigma(S) \leq \sigma(G) \leq 2$, we deduce that S is isomorphic to one of the simple groups in the conclusion of Lemma 2.4. If $|\pi(S)| = 3$, then S is one of the simple groups in [HL, Table 1] and we can check that $\pi(G) = \pi(S)$ in these cases. Thus we assume that $|\pi(S)| \geq 4$. Now if $G = S$, then we have nothing to prove. So, we assume that $G \neq S$. In particular, $G \not\cong \mathrm{PSL}_2(2^f)$ with $f \geq 2$. Then $|\pi(G)| \leq 2\sigma(G) \leq 4$ by Theorem A and thus $4 \leq |\pi(S)| \leq |\pi(G)| \leq 4$, which forces $|\pi(S)| = |\pi(G)|$ and hence $\pi(G) = \pi(S)$ as wanted. □

3. Proof of Theorem B

The following two lemmas are obvious.

Lemma 3.1. *Let A and B be groups such that $|\rho(A)| \geq 3$ and $|\rho(B)| \geq 3$. If*

$$\sigma(A \times B) \leq 2,$$

then $\sigma(A) = 1 = \sigma(B)$.

Lemma 3.2. *Let N be a normal subgroup of a group G. If $\rho(G/N) = \pi(G/N)$, then*

$$\rho(G) - \rho(G/N) \subseteq \rho(N).$$

Recall that the solvable radical of a group G is the largest normal solvable subgroup of G.

Lemma 3.3. *Let G be a nonsolvable group and let N be the solvable radical of G. Suppose that $\sigma(G) \leq 2$ and $|\rho(G)| \geq 5$. Then G/N is an almost simple group.*

Proof. We first claim that if M/N is a chief factor of G, then M/N is a nonabelian simple group.

Let M be a normal subgroup of G such that M/N is a chief factor of G. Since N is the largest normal solvable subgroup of G, we deduce that M/N is nonsolvable so that $M/N \cong S^k$ for some integer $k \geq 1$ and some nonabelian simple group S. Let $C/N = \mathbf{C}_{G/N}(M/N)$. Then G/C embeds into $\mathrm{Aut}(S^k)$.

Assume first that $k \geq 3$. Since $|\rho(S)| = |\pi(S)| \geq 3$, there exist three distinct prime divisors $r_i, 1 \leq i \leq 3$, and three characters $\psi_i \in \mathrm{Irr}(S)$ for $1 \leq i \leq 3$ with $r_i \mid \psi_i(1)$. Let

$$\varphi = \psi_1 \times \psi_2 \times \psi_3 \times 1 \times \cdots \times 1 \in \mathrm{Irr}(S^k).$$

Then $\sigma(\varphi) \geq 3$, which is a contradiction since

$$\sigma(S^k) = \sigma(M/N) \leq \sigma(M) \leq \sigma(G) \leq 2.$$

Thus $k \leq 2$.

Now assume that $k = 2$. Let $B/C = (G/C) \cap \mathrm{Aut}(S)^2$. Then G/B is a nontrivial subgroup of the symmetric group of degree 2 and thus $|G : B| = 2$. Since $S^2 \cong MC/C \trianglelefteq B/C \trianglelefteq G/C$ and $\sigma(G) \leq 2$, we deduce that $\sigma(S^2) \leq 2$ and thus $\sigma(S) = 1$ by Lemma 3.1. By [Man2, Satz 8], we know that S is isomorphic to either $\mathrm{PSL}_2(4)$ or $\mathrm{PSL}_2(8)$. In both cases, we obtain that $\pi(\mathrm{Aut}(S)) = \pi(S)$, hence $\pi(B/C) = \pi(S)$. Moreover, as $|G : B| = 2$, we deduce that $\pi(G/C) = \pi(S)$. As G/C has no nontrivial normal abelian Sylow subgroups, Ito-Michler's Theorem yields that $\rho(G/C) = \pi(G/C) = \pi(S)$. Since $|\pi(G/C)| = |\pi(S)| = 3$ and $|\rho(G)| \geq 5$, there exists $r \in \rho(G) - \pi(G/C)$. Then $r > 2$ and $r \in \rho(C)$ by Lemma 3.2. Let $\theta \in \mathrm{Irr}(C)$ such that $r \mid \theta(1)$. Let L be a normal subgroup of MC such that $L/C \cong S$. Notice that $MC/C \cong S^2$. By applying [T, Lemma 4.2], θ extends to $\theta_0 \in \mathrm{Irr}(L)$. By Gallagher's Theorem [I, Corollary 6.17], $\theta_0\mu \in \mathrm{Irr}(L)$ for all $\mu \in \mathrm{Irr}(L/C)$. Let $\mu_0 \in \mathrm{Irr}(L/C)$ with $2 \mid \mu_0(1)$ and let $\varphi = \theta_0\mu_0 \in \mathrm{Irr}(L)$. Then $\pi(\varphi(1)) = \{2, r\}$ with $r > 2$. As $MC/L \cong S$, we can apply [T, Lemma 4.2] again to obtain that φ extends to $\varphi_0 \in \mathrm{Irr}(MC)$ and then by applying Gallagher's Theorem, $\varphi_0\mu \in \mathrm{Irr}(MC)$ for all $\mu \in \mathrm{Irr}(MC/L)$. Clearly, $MC/L \cong S$ has an irreducible character $\tau \in \mathrm{Irr}(MC/L)$ with $s \mid \tau(1)$, where $s \notin \{2, r\}$. We now have that $\varphi_0\tau \in \mathrm{Irr}(MC)$. But then this is a contradiction as $\pi(\varphi_0(1)\tau(1)) = \{2, s, r\}$. This contradiction shows that $k = 1$ as wanted.

Let M/N be a chief factor of G and let $C/N = \mathbf{C}_{G/N}(M/N)$. We claim that $C = N$ and thus G/N is an almost simple group as required. By the claim above, we know that $M/N \cong S$ for some nonabelian simple group S. Hence, G/C is an almost simple group with socle $MC/C \cong M/N$. Suppose by contradiction that $C \neq N$. Now let L/N be a chief factor of G with $N \leq L \leq C$. By the claim above, we deduce that L/N is isomorphic to some nonabelian simple group. In particular, $|\rho(C/N)| \geq |\pi(L/N)| \geq 3$. We have that $MC/N \cong C/N \times M/N$. Since $\sigma(MC/N) \leq \sigma(MC) \leq \sigma(G) \leq 2$, we deduce that $\sigma(C/N \times M/N) \leq 2$ and thus by Lemma 3.1, $\sigma(C/N) = 1 = \sigma(M/N)$. By [Man2], we have $C/N \cong T \times A$, where A is abelian, T is a nonabelian simple group and $S, T \in \{\mathrm{PSL}_2(4), \mathrm{PSL}_2(8)\}$. Since $C \trianglelefteq G$ and the solvable radical W of C is characteristic in C, we obtain that $W \trianglelefteq G$ and thus $W \leq N$ as N is the largest normal solvable subgroup of G. Clearly, $N \leq W$ as N is also a solvable normal subgroup of C, so $W = N$. Therefore, C/N has no nontrivial normal abelian subgroup. Thus $A = 1$ and hence $C/N \cong T$. Since $\pi(G/C) = \pi(M/N)$ and G/N has no normal abelian Sylow subgroup, we obtain that

$$\rho(G/N) = \pi(G/N) = \pi(C/N) \cup \pi(M/N) = \pi(S) \cup \pi(T).$$

It follows that

$$|\rho(G/N)| = |\pi(S) \cup \pi(T)| \leq |\pi(\mathrm{PSL}_2(4)) \cup \pi(\mathrm{PSL}_2(8))| = 4.$$

Hence, $\rho(G)-\rho(G/N)$ is nonempty. Now let $r \in \rho(G)-\rho(G/N)$. As $\{2,3\} \subseteq \rho(G/N)$, we obtain that $r \notin \{2,3\}$. By Lemma 3.2, $r \in \rho(N)$ and hence $r \mid \theta(1)$ for some $\theta \in \mathrm{Irr}(N)$. Since $\sigma(M) \leq \sigma(G) \leq 2$ and $M/N \cong S$, by [T, Lemma 4.2] we deduce that θ extends to $\theta_0 \in \mathrm{Irr}(M)$. Now let $\lambda \in \mathrm{Irr}(M/N)$ with $2 \mid \lambda(1)$. By Gallagher's Theorem, $\varphi = \theta_0\lambda \in \mathrm{Irr}(M)$ with $\pi(\varphi(1)) = \{2, r\}$. Notice that $r \geq 5$ since $r \notin \{2,3\}$. Now let $K = MC \trianglelefteq G$. Then $K/M \cong T$ and $\sigma(K) \leq 2$. Applying the same argument as above, we deduce that φ extends to $\varphi_0 \in \mathrm{Irr}(K)$. Clearly, $K/M \cong T$ has an irreducible character μ with $3 \mid \mu(1)$ and thus by Gallagher's Theorem again, $\psi = \varphi_0\mu \in \mathrm{Irr}(K)$ and obviously $\sigma(\psi) \geq 3$, which is a contradiction. $\square$

We are now ready to prove Theorem B which we state here.

Theorem 3.4. *Let G be a group. If $\sigma(G) \leq 2$, then $|\rho(G)| \leq 2\sigma(G) + 1$.*

Proof. Let G be a counterexample to the theorem with minimal order. Then $\sigma(G) \leq 2$ but $|\rho(G)| > 2\sigma(G) + 1$. If G is solvable or G is nonsolvable with $\sigma(G) = 1$, then

$$|\rho(G)| \leq 2\sigma(G) + 1$$

by [Man1, G, Man2], which is a contradiction. Thus we can assume that G is nonsolvable, $\sigma(G) = 2$ and $|\rho(G)| \geq 6$. Let N be the solvable radical of G. By Lemma 3.3, G/N is an almost simple group with simple socle M/N. Since $\sigma(M/N) \leq \sigma(G/N) \leq \sigma(G) = 2$, we deduce from Lemmas 2.5 and 2.4 that

$$|\pi(G/N)| = |\pi(M/N)| \leq 5.$$

As $|\rho(G)| \geq 6$, we have that $\rho(G) - \rho(G/N)$ is nonempty and let $r \in \rho(G) - \rho(G/N)$. By Lemma 3.2, $r \mid \theta(1)$ for some $\theta \in \mathrm{Irr}(N)$. Since $\sigma(M) \leq 2$, by applying [T, Lemma 4.2], we deduce that θ extends to $\theta_0 \in \mathrm{Irr}(M)$. Using Gallagher's Theorem, we must have that $\sigma(M/N) = 1$ and hence $M/N \cong \mathrm{PSL}_2(4)$ or $\mathrm{PSL}_2(8)$. Thus $|\pi(G/N)| = |\pi(M/N)| = 3$, hence $|\tau| \geq 3$, with $\tau = \rho(G) - \rho(G/N)$. By Lemma 3.2, we have that $\tau \subseteq \rho(N)$ and since N is solvable, by applying Pálfy's Condition [P, Theorem], there exists $\psi \in \mathrm{Irr}(N)$ such that $\psi(1)$ is divisible by two distinct primes in τ. Now by applying [T, Lemma 4.2] again, we obtain a contradiction. This contradiction shows that $|\rho(G)| \leq 2\sigma(G) + 1$ as wanted. $\square$

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