# A SHORT SUMMARY OF SEQUENCES AND SERIES

by John Alexopoulos

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# Chapter 1

# Sequences

## 1.1 Definitions, notation and examples

A sequence is a function whose domain is the set of positive (or the non–negative) integers. If $a$ denotes such an object, and $n$ is a positive integer, we write $a(n) = a_n$ and we denote the sequence $a$ itself by $a = \{a_n\}$.

**Examples:** $\left\{\frac{1}{n}\right\}$, $\left\{\left(\frac{2}{3}\right)^n\right\}$, $\left\{\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ldots,\ \frac{n}{n+1},\ \ldots\right\}$, $\{0,\ 2,\ 0,\ 2,\ \ldots\}$ are all examples of sequences.

We can talk about *limits* of sequences as $n$ tends to infinity. If $\{a_n\}$ is a sequence, we denote its limit by $\lim_{n\to\infty} a_n$ or simply $\lim a_n$. If $\lim a_n$ exists (in a *finite* sense) we say that the sequence $\{a_n\}$ is a *convergent* sequence. Otherwise we say that the sequence $\{a_n\}$ is *divergent.* In the special case of $\lim a_n = 0$, we say that the (convergent) sequence $\{a_n\}$ is *null.*

**Examples:**

1. $\left\{\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ldots,\ \frac{n}{n+1},\ \ldots\right\}$ is an example of a convergent sequence since $\lim \frac{n}{n+1} = 1$.

2. $\left\{\frac{1}{n}\right\}$, $\left\{\left(\frac{2}{3}\right)^n\right\}$ are examples of null sequences since $\lim \frac{1}{n} = 0$ and $\lim \left(\frac{2}{3}\right)^n = 0$.

3. $\{0,\ 2,\ 0,\ 2,\ \ldots\}$, $\{(-1)^n\}$, $\{e^n\}$ are all examples of divergent sequences since their limits do not exist.

## 1.2 More definitions and terms

1. We now give the definition of the limit of a sequence:

$\lim a_n = L$ if and olny if for every $\varepsilon > 0$ there is a positive integer $N$ such that $|a_n - L| < \varepsilon$ whenever $n \geq N$.

2. We say that a sequence $\{a_n\}$ satisfies a given property *eventually* if there is a positive integer $N$ such that the given property is satisfied for all terms $a_n$ for which $n \geq N$. That is, the given property is satisfied for all but finitely many terms of the sequence. As an example of the usage of the word "eventually", refer to the definition of the limit which can be reformulated as follows:

   $\lim a_n = L$ if and olny if for every $\varepsilon > 0$, $|a_n - L| < \varepsilon$ eventually.

3. We say that a sequence $\{a_n\}$ is *increasing* if $a_n \leq a_{n+1}$ for all $n$.

   We say that a sequence $\{a_n\}$ is *decreasing* if $a_n \geq a_{n+1}$ for all $n$.

   In any of these cases we say that $\{a_n\}$ is *monotonic.*

4. We say that a sequence $\{a_n\}$ is *bounded* if there is a positive number $M$ such that $|a_n| \leq M$ for all $n$.

**Examples:**

1. We will use the definition of the limit to establish the following:

   (a) $\lim \left(\frac{3n+1}{2n+5}\right) = \frac{3}{2}$ : Let $\varepsilon > 0$ and choose a positive integer $N$ such that $N > \frac{13}{4\varepsilon}$. Then

$$
\begin{aligned}
n \geq N &\Longrightarrow n > \frac{13}{4\varepsilon} \Longrightarrow \frac{13}{4n} < \varepsilon \Longrightarrow \frac{13}{4n+10} < \frac{13}{4n} < \varepsilon \Longrightarrow \left|\frac{-13}{4n+10}\right| < \varepsilon \\
&\Longrightarrow \left|\frac{6n-6n+2-15}{4n+10}\right| < \varepsilon \Longrightarrow \left|\frac{2\,(3n+1)-3\,(2n+5)}{2\,(2n+5)}\right| < \varepsilon \\
&\Longrightarrow \left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| < \varepsilon
\end{aligned}
$$

   and so $\lim \left(\frac{3n+1}{2n+5}\right) = \frac{3}{2}$ by definition.

   (b) $\lim \left(\frac{n^2-1}{2n^2+3}\right) = \frac{1}{2}$ : Let $\varepsilon > 0$ and choose a positive integer $N$ such that $N > \frac{2}{\sqrt{5\varepsilon}}$. Then

$$
\begin{aligned}
n \geq N &\Longrightarrow n > \frac{\sqrt{5}}{2\sqrt{\varepsilon}} \Longrightarrow \frac{\sqrt{5}}{n} < 2\sqrt{\varepsilon} \Longrightarrow \frac{5}{n^2} < 4\varepsilon \Longrightarrow \frac{5}{4n^2} < \varepsilon \\
&\Longrightarrow \frac{5}{4n^2+6} < \frac{5}{4n^2} < \varepsilon \Longrightarrow \left|\frac{-5}{2\,(2n^2+3)}\right| < \varepsilon \Longrightarrow \left|\frac{2n^2-2-2n^2-3}{2\,(2n^2+3)}\right| < \varepsilon \\
&\Longrightarrow \left|\frac{2\left(n^2-1\right)-\left(2n^2+3\right)}{2\,(2n^2+3)}\right| < \varepsilon \Longrightarrow \left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| < \varepsilon
\end{aligned}
$$

   and so $\lim \left(\frac{n^2-1}{2n^2+3}\right) = \frac{1}{2}$ by definition.

(c) $\lim\left(\frac{(-1)^n n}{n^2+1}\right) = 0$ : Let $\varepsilon > 0$ and choose a positive integer $N$ such that $N > \frac{1}{\varepsilon}$. Then

$$n \geq N \Longrightarrow n > \frac{1}{\varepsilon} \Longrightarrow \frac{1}{n} < \varepsilon \Longrightarrow \frac{n}{n^2} < \varepsilon \Longrightarrow \frac{n}{n^2+1} < \frac{n}{n^2} < \varepsilon$$
$$\Longrightarrow |(-1)^n| \cdot \left|\frac{n}{n^2+1}\right| < \varepsilon \Longrightarrow \left|\frac{(-1)^n n}{n^2+1}\right| < \varepsilon \Longrightarrow \left|\frac{(-1)^n n}{n^2+1} - 0\right| < \varepsilon$$

and so $\lim\left(\frac{(-1)^n n}{n^2+1}\right) = 0$ by definition.

2. $\left\{\frac{1}{n}\right\}$, $\{-n\}$ are monotonic decreasing sequences, while $\left\{\frac{n}{n+1}\right\}$ and $\{2^n\}$ are monotonic increasing.

3. The sequences $\{1,\ 1,\ 2,\ 2,\ \ 3,\ 3,\ 4,\ 4, \ldots\}$, $\{5\}$

   and $\{\ \frac{1}{2},\ \frac{1}{2},\ \frac{2}{3},\ \frac{2}{3},\ \frac{3}{4},\ \frac{3}{4}, \ldots\}$ are also increasing

   while $\{-1,\ -1,\ -2,\ -2,\ \ -3,\ -3,\ -4,\ -4, \ldots\}$, $\{5\}$

   and $\{\ -\frac{1}{2},\ -\frac{1}{2},\ -\frac{2}{3},\ -\frac{2}{3},\ -\frac{3}{4},\ -\frac{3}{4}, \ldots\}$ are decreasing.

4. The sequences $\left\{\frac{(-1)^n n^2}{n^2+1}\right\}$, $\left\{\frac{(-1)^n}{n}\right\}$ and $\left\{1 + \frac{(-1)^n}{n}\right\}$ are not monotonic.

5. The sequences $\left\{\frac{(-1)^n n^2}{n^2+1}\right\}$, $\left\{\frac{1}{n}\right\}$, $\{0,\ 2,\ 0,\ 2,\ \ldots\}$ and $\{(-1)^n\}$ are all bounded while $\{2^n\}$, $\{-n\}$ and $\left\{\frac{n+(-1)^n n}{2}\right\}$ are not bounded.

## 1.3 Some important theorems and facts

1. Convergent sequences are always bounded.

   The converse is not true: That is, bounded sequences need not be convergent.

   Think of the sequence $\{0,\ 2,\ 0,\ 2,\ \ldots\}$. It is bounded (by two) yet it is not convergent.

2. Every bounded monotonic sequence converges.

   The converse is not true: That is, convergent sequences need not be monotonic.

   Think of sequences like $\left\{\frac{(-1)^n}{n}\right\}$ or $\left\{1 + \frac{(-1)^n}{n}\right\}$. These sequences are convergent since $\lim \frac{(-1)^n}{n} = 0$ and $\lim\left(1 + \frac{(-1)^n}{n}\right) = 1$, yet they are not monotonic.

# Chapter 2

# Series

## 2.1 Definitions, terminology and basic examples

Let $\{a_n\}$ be a sequence. For each positive integer $n$ define $s_n = a_1 + a_2 + \cdots + a_n$ or more concisely written $s_n = \sum_{k=1}^{n} a_k$. The sequence of *partial sums* $\{s_n\}$ is called an *infinite series* and it is denoted by $\sum_{n=1}^{\infty} a_n$ or simply $\sum a_n$. The sequence $\{a_n\}$ is called the *sequence of terms* of the series $\sum a_n$.

**Example:** $\sum \frac{1}{n} = \left\{1,\ 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3},\ \ldots,\ \sum_{k=1}^{n} \frac{1}{k}\ ,\ \ldots\right\}$ is an infinite series. The sequence $\left\{\frac{1}{n}\right\} = \left\{1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4}, \ldots,\ \frac{1}{n},\ \ldots\right\}$ is the sequence of terms of this series. $\sum \frac{1}{n}$ is a special series called *the harmonic series.*

**Convergent and Divergent series:** Since after all series are sequences, it makes sense to ask whether or not they converge or diverge. That is, if $\lim s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$ exists (in the finite sense) then we say that $\sum a_n$ *is convergent*. Otherwise we say that $\sum a_n$ *diverges*. If a series converges then the value $s = \lim s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$ is called *the sum* of the series and it is denoted (watch out!) by $s = \sum_{n=1}^{\infty} a_n$ . In most instances, the context determines whether the meaning of the symbol $\sum_{n=1}^{\infty} a_n$ refers to the series itself or its sum.

## 2.2 Computing sums of infinite series

Computing the sum of a (convergent) series is in general a hard task. Nevertheless it is reasonably easy for the following two cases:

1. **Telescoping series:** This term refers to series whose partial sums support substantial cancellation. Here is a couple of examples:

   (a) $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$. For this series we have that $s_n = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right) = 1-\frac{1}{n+1}$.
   Thus the sum of the series is $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \lim s_n = \lim\left(1-\frac{1}{n+1}\right) = 1$

   (b) $\sum_{n=2}^{\infty}\frac{1}{(n-1)(n+2)}$ . Notice that by partial fractions decomposition we have $\frac{1}{(n-1)(n+2)} = \frac{1}{3(n-1)} - \frac{1}{3(n+2)}$.
   Thus

$$\begin{aligned} s_n &= \frac{1}{3}\left[\left(1-\frac{1}{4}\right)+\left(\frac{1}{2}-\frac{1}{5}\right)+\left(\frac{1}{3}-\frac{1}{6}\right)+\cdots+\left(\frac{1}{n-3}-\frac{1}{n}\right)+\left(\frac{1}{n-2}-\frac{1}{n+1}\right)+\left(\frac{1}{n-1}-\frac{1}{n+2}\right)\right] \\ &= \frac{1}{3}\left[1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}\right] \end{aligned}$$

   Thus the sum of the series is

$$\begin{aligned} \sum_{n=2}^{\infty}\frac{1}{(n-1)(n+2)} = \lim s_n &= \lim_{n\to\infty}\frac{1}{3}\left[1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}\right] \\ &= \frac{1}{3}\left[1+\frac{1}{2}+\frac{1}{3}\right] = \frac{11}{18} \end{aligned}$$

2. **Geometric series:** These are series of the form $\sum_{n=0}^{\infty} r^n$ where $r$ *is a real constant called the ratio of the geometric series.*

   For this series we have that $s_n = 1+r+r^2+\cdots+r^n = \frac{1-r^{n+1}}{1-r}$ (check it!) as long as $r \neq 1$. Now if $|r|<1$ we have that the sum of a geometric series of ratio $r$ is

$$\sum_{n=0}^{\infty} r^n = \lim s_n = \lim\frac{1-r^{n+1}}{1-r} = \frac{1}{1-r}$$

### 2.2.1 Computing sums of geometric series. Some examples.

1. (a)

$$\sum_{n=0}^{\infty}\frac{2^n}{3^n} = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{1-\frac{2}{3}} = 3$$

   (b)

$$\begin{aligned} \sum_{n=3}^{\infty}\frac{5^{n+1}}{3^{2n}} &= 5\sum_{n=3}^{\infty}\frac{5^n}{(3^2)^n} = 5\sum_{n=3}^{\infty}\frac{5^n}{9^n} = 5\sum_{n=3}^{\infty}\left(\frac{5}{9}\right)^n \\ &= 5\left(\frac{5}{9}\right)^3\sum_{n=3}^{\infty}\left(\frac{5}{9}\right)^{n-3} = \frac{625}{729}\sum_{n=0}^{\infty}\left(\frac{5}{9}\right)^n \\ &= \frac{625}{729}\cdot\frac{1}{1-\frac{5}{9}} = \frac{625}{729}\cdot\frac{9}{4} = \frac{625}{324} \end{aligned}$$

(c) The following technique is called summation by parts. It is similar to the "cyclic" integration by parts:

i.

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{n}{2^n} &= \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots \\
&= \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right) + \left(\frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \cdots\right) \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n} + \sum_{n=1}^{\infty} \frac{n}{2^{n+1}} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{n}{2^n} \\
&= \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{2^n} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{n}{2^n} \\
&= 1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{n}{2^n}
\end{aligned}
$$

Thus if $x = \sum_{n=1}^{\infty} \frac{n}{2^n}$ we have that $x = 1 + \frac{1}{2}x$ and so $x = \sum_{n=1}^{\infty} \frac{n}{2^n} = 2$.

ii. Compute the sum of the series $\sum_{n=1}^{\infty} \frac{n^2 3^n}{5^{n+1}}$:

First notice that

$$
\sum_{n=1}^{\infty} \frac{n^2 3^n}{5^{n+1}} = \frac{1^2 \cdot 3}{5^2} + \sum_{n=2}^{\infty} \frac{n^2 3^n}{5^{n+1}} = \frac{3}{25} + \sum_{n=2}^{\infty} \frac{n^2 3^n}{5^{n+1}}
$$

Furthermore

$$
\begin{aligned}
\sum_{n=2}^{\infty} \frac{n^2 3^n}{5^{n+1}} &= \sum_{n=1}^{\infty} \frac{(n+1)^2\, 3^{n+1}}{5^{n+2}} = \frac{3}{5}\sum_{n=1}^{\infty} \frac{\left(n^2+2n+1\right) 3^n}{5^{n+1}} \\
&= \frac{3}{5}\left[\sum_{n=1}^{\infty} \frac{n^2 3^n}{5^{n+1}} + 2\sum_{n=1}^{\infty} \frac{n 3^n}{5^{n+1}} + \sum_{n=1}^{\infty} \frac{3^n}{5^{n+1}}\right] \\
&= \frac{3}{5}\left[\sum_{n=1}^{\infty} \frac{n^2 3^n}{5^{n+1}} + 2\sum_{n=1}^{\infty} \frac{n 3^n}{5^{n+1}} + \frac{3}{25}\sum_{n=1}^{\infty} \frac{3^{n-1}}{5^{n-1}}\right] \\
&= \frac{3}{5}\left[\sum_{n=1}^{\infty} \frac{n^2 3^n}{5^{n+1}} + 2\sum_{n=1}^{\infty} \frac{n 3^n}{5^{n+1}} + \frac{3}{25}\sum_{n=0}^{\infty} \frac{3^n}{5^n}\right] \\
&= \frac{3}{5}\left[\sum_{n=1}^{\infty} \frac{n^2 3^n}{5^{n+1}} + 2\sum_{n=1}^{\infty} \frac{n 3^n}{5^{n+1}} + \frac{3}{25} \cdot \frac{1}{1-\frac{3}{5}}\right] \\
&= \frac{3}{5}\left[\sum_{n=1}^{\infty} \frac{n^2 3^n}{5^{n+1}} + 2\sum_{n=1}^{\infty} \frac{n 3^n}{5^{n+1}} + \frac{3}{10}\right]
\end{aligned}
$$

Now following the footsteps of the previous example we have that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n3^n}{5^{n+1}} &= \left(\frac{1\cdot 3}{5^2} + \frac{2\cdot 3^2}{5^3} + \frac{3\cdot 3^3}{5^4} + \cdots\right) \\
&= \left(\frac{3}{5^2} + \frac{3^2}{5^3} + \frac{3^3}{5^4} + \cdots\right) + \left(\frac{1\cdot 3^2}{5^3} + \frac{2\cdot 3^3}{5^4} + \frac{2\cdot 3^4}{5^5} + \cdots\right) \\
&= \sum_{n=1}^{\infty} \frac{3^n}{5^{n+1}} + \sum_{n=1}^{\infty} \frac{n3^{n+1}}{5^{n+2}} \\
&= \frac{3}{25}\sum_{n=0}^{\infty} \frac{3^n}{5^n} + \frac{3}{5}\sum_{n=1}^{\infty} \frac{n3^n}{5^{n+1}} \\
&= \frac{3}{10} + \frac{3}{5}\sum_{n=1}^{\infty} \frac{n3^n}{5^{n+1}} \text{ and so } \frac{2}{5}\sum_{n=1}^{\infty} \frac{n3^n}{5^{n+1}} = \frac{3}{10} \\
\therefore \sum_{n=1}^{\infty} \frac{n3^n}{5^{n+1}} &= \frac{5}{2}\cdot\frac{3}{10} = \frac{3}{4}
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{n^2 3^n}{5^{n+1}} &= \frac{3}{5}\left[\sum_{n=1}^{\infty} \frac{n^2 3^n}{5^{n+1}} + 2\sum_{n=1}^{\infty} \frac{n3^n}{5^{n+1}} + \frac{3}{10}\right] \\
&= \frac{3}{5}\left[\sum_{n=1}^{\infty} \frac{n^2 3^n}{5^{n+1}} + 2\cdot\frac{3}{4} + \frac{3}{10}\right] \\
&= \frac{3}{5}\left[\sum_{n=1}^{\infty} \frac{n^2 3^n}{5^{n+1}} + \frac{9}{5}\right]
\end{aligned}$$

and so if $x = \sum_{n=1}^{\infty} \frac{n^2 3^n}{5^{n+1}}$ then

$$\begin{aligned}
x &= \frac{3}{25} + \sum_{n=2}^{\infty} \frac{n^2 3^n}{5^{n+1}} = \frac{3}{25} + \frac{3}{5}\left[\sum_{n=1}^{\infty} \frac{n^2 3^n}{5^{n+1}} + \frac{9}{5}\right] = \frac{3}{25} + \frac{3}{5}\left[x + \frac{9}{5}\right] \\
\therefore\ x &= \frac{3}{25} + \frac{3}{5}\left[x + \frac{9}{5}\right] \text{ and so } x = 3 \\
\therefore \sum_{n=1}^{\infty} \frac{n^2 3^n}{5^{n+1}} &= 3
\end{aligned}$$

## 2.3 Convergence and divergence of three important classes of series

1. Geometric series: These are series of the form $\sum_{n=0}^{\infty} r^n$ where $r$ *is a real constant called the ratio of the geometric series.*We have already defined geometric series and we know that they converge if $|r| < 1$ and diverge if $|r| \geq 1$. It is important to note that we know how to compute the sum of geometric series.

2. P-series: These are series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where $p > 0$ is a positive constant. They converge if $p > 1$ and

they diverge if $p \leq 1$. In order to see this apply the *integral test* or *Cauchy's Condensation Test* (see below). It is worth noting that the harmonic series is a p-series with $p = 1$.

3. Log-p-series: These are series of the form $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^p}$ where $p > 0$ is a positive constant. They converge if $p > 1$ and they diverge if $p \leq 1$. In order to see this apply the *integral test* or *Cauchy's Condensation Test* (see below).

## 2.4 Tests for the convergence or divergence of series. What the theorems say and what they don't

### 2.4.1 The test for divergence

**What does the theorem say**:

All three of the following statements are true and logically equivalent to each other:

1. If $\sum a_n$ converges then $\lim a_n = 0$
2. If $\sum a_n$ converges then $\{a_n\}$, the sequence of terms of the series, is null.
3. If $\{a_n\}$ is not a null sequence (i.e. $\lim a_n \neq 0$) then $\sum a_n$ diverges.

**Examples:**

1. Suppose that the series $\sum 2^n w_n$ is convergent. What can you say about the sequence of terms $\{2^n w_n\}$?

   Well... $\{2^n w_n\}$ is a null sequence. That is $\lim 2^n w_n = 0$.

2. Suppose that $\lim 2^n w_n = 0$. What can you say about the convergence or divergence of $\sum 2^n w_n$ ?

   Absolutely NOTHING. We simply don't have enough information to decide.

3. Does the series $\sum (-1)^n \frac{n}{2n+1}$ converge or diverge?

   Well... it diverges because $\lim \frac{n}{2n+1} = \frac{1}{2}$ and thus $\lim (-1)^n \frac{n}{2n+1}$ does not exist. In particular the sequence $\left\{(-1)^n \frac{n}{2n+1}\right\}$ is not a null sequence.

**Common abuses:** The theorem NEVER claimed the convergence of $\sum a_n$ whenever $\lim a_n = 0$. Such a statement is plainly false.

Just consider the harmonic series $\sum \frac{1}{n}$: We know that this series diverges YET $\lim \frac{1}{n} = 0$

### 2.4.2 The Integral Test

**What does the theorem say**:

Suppose that $f$ is (eventually) decreasing continuous and positive. Then $\int_m^\infty f(x)dx$ converges if and only if $\sum_{n=m}^\infty f(n)$ converges.

In other words $\int_m^\infty f(x)dx$ and $\sum_{n=m}^\infty f(n)$ either *both* converge or *both* diverge.

**Example:** The integral test is utilized to determine the convergence or divergence of p-series and log-p-series. Once these facts are established the Integral Test has a limited use one reason being that integration is not always easy (or even possible).

**Common abuses:**

People often times forget to check the hypotheses of the theorem. In particular *make sure that you check that* $f$ *is decreasing.* In non–obvious cases a sign–chart for the derivative of $f$ is just what the doctor ordered.

### 2.4.3 Cauchy's Condensation Test

**What does the theorem say**:

Let $\{a_n\}$ be an (eventually) decreasing positive sequence. Then $\sum a_n$ converges if and only if $\sum 2^n a_{2^n}$ converges.

In other words $\sum a_n$ and $\sum 2^n a_{2^n}$ either *both* converge or *both* diverge.

**Examples:** Cauchy's condensation test provides in many instances a cleaner alternative to the Integral Test. For example lets see how this test treats p-series and log-p-series:

1. Consider the p-series $\sum \frac{1}{n^p}$ and use the condensation test to obtain the series $\sum 2^n \cdot \frac{1}{(2^n)^p} = \sum \frac{2^n}{(2^p)^n} = \sum \left(\frac{2}{2^p}\right)^n$. If $0 < p \leq 1$ then $2^p \leq 2$ and thus $\frac{2}{2^p} \geq 1$ and so the geometric series $\sum \left(\frac{2}{2^p}\right)^n$ diverges. On the other hand if $p > 1$ then $2^p > 2$ and so $\frac{2}{2^p} < 1$ making the geometric series $\sum \left(\frac{2}{2^p}\right)^n$ convergent.

2. Consider the log-p-series $\sum \frac{1}{n(\ln n)^p}$ and use the condensation test to obtain the series $\sum 2^n \cdot \frac{1}{2^n(n \ln 2)^p} = \sum \frac{1}{n^p(\ln 2)^p} = \frac{1}{(\ln 2)^p} \sum \frac{1}{n^p}$. This series is a non–zero constant multiple of a p-series and thus it converges if

$p > 1$ and it diverges for $0 < p \leq 1$.

**Common abuses:** Just like in the integral test make sure that the hypotheses are satisfied. In particular ensure that the sequence $\{a_n\}$ is (eventually) decreasing.

### 2.4.4 The Comparison Test

**What does the theorem say**:

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of non–negative terms and suppose that (eventually) $a_n \leq b_n$. The following statements are true:

1. If $\sum b_n$ is convergent then so is $\sum a_n$
2. If $\sum a_n$ is divergent then so is $\sum b_n$

The comparison test is the "work–horse" of all tests. Its use depends on prior knowledge of the behavior of certain series.

**Examples:**

1. Determine whether or not $\sum \frac{n-1}{n^3+n+1}$ converges or diverges.

   Notice that $\frac{n-1}{n^3+n+1} < \frac{n}{n^3+n+1} < \frac{n}{n^3} = \frac{1}{n^2}$.

   We know that $\sum \frac{1}{n^2}$ converges (p-series $p = 2 > 1$). Hence so does $\sum \frac{n-1}{n^3+n+1}$ thanks to the comparison test.

2. Determine whether or not $\sum_{n=2}^{\infty} \frac{n}{n^2-n-1}$ converges or diverges.

   Notice that $\frac{n}{n^2-n-1} > \frac{n}{n^2} = \frac{1}{n}$.

   We know that $\sum \frac{1}{n}$ diverges (harmonic series). Hence so does $\sum_{n=2}^{\infty} \frac{n}{n^2-n-1}$ thanks to the comparison test.

3. Determine whether or not $\sum_{n=2}^{\infty} \frac{n}{n^3-n^2-1}$ converges or diverges.

   Here the inequalities do not work in our favor. But we know that the sequence $\left\{\frac{n}{n^3-n^2-1}\right\}$ behaves essentially like $\left\{\frac{1}{n^2}\right\}$. Based on this observation we can try the following argument:

   $\frac{n}{n^3-n^2-1} < \frac{n}{n^3-\frac{1}{2}n^3} = \frac{n}{\frac{1}{2}n^3} = \frac{2n}{n^3} = \frac{2}{n^2}$ (eventually)

   Now $\sum \frac{1}{n^2}$ converges (p-series $p = 2 > 1$). Hence $\sum \frac{2}{n^2} = 2\sum \frac{1}{n^2}$ converges and thus so does $\sum_{n=2}^{\infty} \frac{n}{n^3-n^2-1}$ by the comparison test.

**Common abuses:** Careful: the inequalities must be just right and they should be going the "right way". For instance, one could be tempted to do the following on example (3):

$\frac{n}{n^3-n^2-1} > \frac{n}{n^3} = \frac{1}{n^2}$. Even though $\sum \frac{1}{n^2}$ converges we can conclude NOTHING about the behavior of $\sum_{n=2}^{\infty} \frac{n}{n^3-n^2-1}$.

### 2.4.5 The limit comparison test

**What does the theorem say**:

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive terms and let $\lim \frac{a_n}{b_n} = L$.

1. If $L \neq 0$ and $L < \infty$ then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

   That is, $\sum a_n$ if and only if $\sum b_n$ converges.

2. If $L = 0$ and $\sum b_n$ converges then so does $\sum a_n$.

3. If $L = 0$ and $\sum a_n$ diverges then so does $\sum b_n$.

4. If $L = \infty$ and $\sum a_n$ converges then so does $\sum b_n$.

5. If $L = \infty$ and $\sum b_n$ diverges then so does $\sum a_n$.

The most commonly used part is part (1). I suggest that you don't memorize parts (2)–(5) but if you have to, rather think that if $\lim \frac{a_n}{b_n} = 0$ then eventually $a_n < b_n$ and if $\lim \frac{a_n}{b_n} = \infty$ then eventually $a_n > b_n$. Then use the comparison test.

The limit comparison test is especially useful in situations like in example (3) of the previous section.

**Examples:**

1. Determine whether or not $\sum_{n=2}^{\infty} \frac{n}{n^3-n^2-1}$ converges or diverges.

   Lets use the limit comparison test with $\sum \frac{1}{n^2}$:

   $\lim \frac{\frac{n}{n^3-n^2-1}}{\frac{1}{n^2}} = \lim \frac{n}{n^3-n^2-1} \cdot \frac{n^2}{1} = \lim \frac{n^3}{n^3-n^2-1} = \lim \frac{1}{1-\frac{1}{n}-\frac{1}{n^2}} = 1$

   Since $\sum \frac{1}{n^2}$ converges (p-series $p = 2 > 1$) then so does $\sum_{n=2}^{\infty} \frac{n}{n^3-n^2-1}$ thanks to the limit comparison test.

2. Determine whether or not $\sum \frac{1}{n^{1+\frac{1}{n}}}$ converges or diverges.

   Use the limit comparison test with the harmonic series $\sum \frac{1}{n}$:

$\lim \frac{\frac{1}{n^{1+\frac{1}{n}}}}{\frac{1}{n}} = \lim \frac{n}{n^{1+\frac{1}{n}}} = \lim \frac{1}{n^{\frac{1}{n}}} = 1$

We know that $\sum \frac{1}{n}$ diverges (harmonic series). Hence so does $\sum \frac{1}{n^{1+\frac{1}{n}}}$ thanks to the limit comparison test.

### 2.4.6 The alternating series test

So far all are tests for convergence of series dealt with series of non–negative terms. In this section we take a look at a special kind of series whose sequence of terms "alternates" from positive to negative or vice–versa:

**Definition:** Let $\{a_n\}$ be a sequence of positive terms (i.e. $a_n > 0$). The series $\sum (-1)^n a_n$ is called an *alternating series.*

**What does the theorem say**:

Let $\{a_n\}$ be a decreasing, null sequence of positive terms (i.e. $\{a_n\}$ decreases, $a_n > 0$ and $\lim a_n = 0$). Then the alternating series $\sum (-1)^n a_n$ converges.

**Examples:**

$\sum (-1)^n \frac{1}{n}$, $\sum (-1)^n \frac{1}{\ln n}$, $\sum (-1)^n \left(\frac{1}{n!}\right)$ are all convergent series thanks to the alternating series test.

**Common abuses:** Just like in the integral test and Cauchy's condensation test make sure that the hypotheses are satisfied. In particular ensure that the sequence $\{a_n\}$ is (eventually) decreasing.

### 2.4.7 Absolute and conditional convergence of series

**Definition:** If the series $\sum |a_n|$ converges then the series $\sum a_n$ is called *absolutely convergent.*

If the series $\sum |a_n|$ diverges and the series $\sum a_n$ converges then the series $\sum a_n$ is called *conditionally convergent.*

**Some facts:**

1. Absolutely convergent series are always convergent.

2. It follows from the statement above that if $\sum a_n$ diverges then so does $\sum |a_n|$.

**Examples:**

1. $\sum (-1)^n \frac{1}{n}$ is a conditionally convergent series because $\sum (-1)^n \frac{1}{n}$ converges (alternating series test)

   yet $\sum \left|(-1)^n \frac{1}{n}\right| = \sum \frac{1}{n}$ diverges (harmonic series).

2. $\sum \frac{(-1)^n}{n^3}$ is an absolutely convergent series because $\sum \left|\frac{(-1)^n}{n^3}\right| = \sum \frac{1}{n^3}$ converges (p-series $p = 2 > 1$).

### 2.4.8 The ratio test

**What does the theorem say**:

Let $\{a_n\}$ be a sequence of (eventually) non–zero terms (i.e. eventually $a_n \neq 0$) and let $\lim \left|\frac{a_{n+1}}{a_n}\right| = L$. Then

1. If $L < 1$ then $\sum a_n$ is absolutely convergent.
2. If $L > 1$ then $\sum a_n$ is divergent.
3. If $L = 1$ then no conclusions can be drawn from this test.

**Examples:**

1. $\sum \frac{1}{n!}$ is (absolutely) convergent:

   $\lim \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim \frac{n!}{(n+1)!} = \lim \frac{1}{n} = 0 < 1$

   So $\sum \frac{1}{n!}$ is (absolutely) convergent by the ratio test

2. $\sum \frac{2^n n!}{n^n}$ is (absolutely) convergent:

$$\begin{aligned}\lim \frac{\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}}{\frac{2^n n!}{n^n}} &= \lim \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n n!} = \lim 2 \cdot \frac{n^n}{(n+1)^n} \\ &= 2\lim\left(\frac{n}{n+1}\right)^n = 2\lim \frac{1}{\left(\frac{n+1}{n}\right)^n} \\ &= 2\lim \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{2}{e} < 1\end{aligned}$$

   So $\sum \frac{2^n n!}{n^n}$ is (absolutely) convergent by the ratio test.

3. $\sum \frac{n!}{10^n}$ diverges:

   $\lim \frac{\frac{(n+1)!}{10^{n+1}}}{\frac{n!}{10^n}} = \lim \frac{(n+1)!}{10^{n+1}} \frac{10^n}{n!} = \lim \frac{n+1}{10} = \infty > 1$

   So $\sum \frac{n!}{10^n}$ diverges by the ratio test.

### 2.4.9 The root test

**What does the theorem say**:

Let $\{a_n\}$ be a sequence of terms and let $\lim \sqrt[n]{|a_n|} = \lim |a_n|^{\frac{1}{n}} = L$. Then

1. If $L < 1$ then $\sum a_n$ is absolutely convergent.

2. If $L > 1$ then $\sum a_n$ is divergent.

3. If $L = 1$ then no conclusions can be drawn from this test.

**Examples:**

1. $\sum \frac{r^n}{n^n}$ is (absolutely) convergent for any constant $r$:

   $\lim \sqrt[n]{\left|\frac{r^n}{n^n}\right|} = \lim \frac{|r|}{n} = 0 < 1$

   $\therefore \sum \frac{r^n}{n^n}$ is (absolutely) convergent for any constant $r$, by the root test.

2. $\sum n^p r^n$ is (absolutely) convergent for any constants $r$ with $|r| < 1$ and $p > 0$:

   $\lim \sqrt[n]{|n^p r^n|} = \lim |r| \left( \sqrt[n]{n} \right)^p = |r| < 1$

   $\therefore \sum n^p r^n$ is (absolutely) convergent for any constants $r$ with $|r| < 1$ and $p > 0$ thanks to the root test.

**Comments about the ratio and root tests:** Even though the ratio and root tests appear to be very powerful they are nothing more but fancy comparison tests to geometric series. Consequently these tests will yield no information when they are used to determine the behavior of p-series or log-p-series (try it!). Their utility becomes more apparent in the study of power–series.

# Chapter 3

# Power series

## 3.1 Definitions and basic concepts

**Definitions:** Let $\{a_n\}$ be a sequence and $a$ any constant. The function $f$ defined by

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

is called a *power–series centered at* $x = a$. Adopt the conventions $0^0 = 1$, $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$ and let $R = \frac{1}{\lim\left|\frac{a_{n+1}}{a_n}\right|}$ or $R = \frac{1}{\lim |a_n|^{\frac{1}{n}}}$ (whichever makes sense. If both do, then they are the same anyway). *$R$ is called the radius of convergence of the power–series $\sum_{n=0}^{\infty} a_n (x-a)^n$.* By the use of the ratio or root test it follows that the function $f$ is defined in the interval $(a - R,\ a + R)$ centered at $x = a$. This interval is called *the open interval of convergence of the power–series $\sum_{n=0}^{\infty} a_n (x-a)^n$.* The open interval of convergence together with any (or both) of its endpoints at which the resulting series converges is called *the interval of convergence of the power–series $\sum_{n=0}^{\infty} a_n (x-a)^n$.* The interval of convergence is the domain of the function $f$.

**Example:** Find the radius, open interval and the interval of convergence of the power–series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)5^n}$.

$$R = \frac{1}{\lim \left|\frac{1}{(n+1)5^n}\right|^{\frac{1}{n}}} = \frac{5}{\lim \frac{1}{\sqrt[n]{n+1}}} = 5$$

and so the open interval of convergence is $(-3, 7)$. For $x = -3$ we have that

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)\,5^n} = \sum_{n=0}^{\infty} \frac{(-5)^n}{(n+1)\,5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n 5^n}{(n+1)\,5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)}$$

which converges (by the alternating series test). For $x = 7$ we have that

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)\,5^n} = \sum_{n=0}^{\infty} \frac{5^n}{(n+1)\,5^n} = \sum_{n=0}^{\infty} \frac{1}{n+1}$$

which diverges (limit compare with the harmonic series $\sum \frac{1}{n}$). Thus the interval of convergence of the power–series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)5^n}$ is $[-3, 7)$.

## 3.2 Some important facts

1. It turns out that power–series are infinitely differentiable functions within their open interval of convergence. Moreover their derivatives have the same open interval of convergence.

2. Power–series are differentiated and integrated term–by–term. That is

$$\frac{d}{dx} \sum_{n=0}^{\infty} a_n (x-a)^n = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1} = \sum_{n=0}^{\infty} (n+1)\, a_{n+1} (x-a)^n$$

   and

$$\int \left( \sum_{n=0}^{\infty} a_n (x-a)^n \right) dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-a)^{n+1} = C + \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} (x-a)^n$$

3. If $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ then $\boxed{f^{(n)}(a) = a_n \cdot n!}$ and thus $\boxed{a_n = \frac{f^{(n)}(a)}{n!}}$. This means that if an infinite differentiable function can be expressed as a power series centered at $x = a$ then such a power series representation is unique. That is, for $x$ in the open interval of convergence $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$.

   **Examples:**

   (a) If $f(x) = e^x$ then for each $n$ we know that $f^{(n)}(x) = e^x$ and so $f^{(n)}(0) = 1$. Hence if $f$ can be expressed as a power–series centered at $x = 0$ this series has no choice but be $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. In fact, we will see later that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x$.

   (b) If $f(x) = \frac{1}{1-x}$ then we know that for each $x$ in $(-1, 1)$, $f(x) = \sum_{n=0}^{\infty} x^n$. Thus $f^{(n)}(0) = n!$

   One may ask what is $f^{(n)}(3)$ ? Well...here is some magic:

$$f(x) = \frac{1}{1-x} = \frac{1}{1-x+3-3} = \frac{1}{-2-(x-3)} = \frac{-1}{2+(x-3)} = \frac{\frac{-1}{2}}{1+\frac{(x-3)}{2}} = -\frac{1}{2} \cdot \frac{1}{1+\frac{1}{2}(x-3)}$$

$$= -\frac{1}{2} \sum_{n=0}^{\infty} \left( -\frac{1}{2}(x-3) \right)^n = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} (x-3)^n = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}} (x-3)^n$$

By the uniqueness of such representation we conclude that $f^{(n)}(3) = \frac{(-1)^{n+1}}{2^{n+1}} \cdot n!$

(c) Let $f(x) = \sum_{n=0}^{\infty} \frac{n+1}{2^n}(x-1)^{3n+1}$. Then notice that

i. $f(1) = 0$

ii. $f^{(1)}(1) = 1$ ( $3n+1 = 1$ when $n = 0$, and the coefficient $\frac{n+1}{2^n}\Big|_{n=0} = 1$)

iii. $f^{(2)}(1) = 0$ ($3n+1 \neq 2$ and so the coefficient of $(x-1)^2$ MUST be 0.)

iv. $f^{(3)}(1) = 0$ ($3n+1 \neq 3$ and so the coefficient of $(x-1)^3$ MUST be 0.)

v. $f^{(4)}(1) = 4! = 24$ ($3n+1 = 4$ only when $n = 1$ and so the coefficient of $(x-1)^3$ is $\frac{n+1}{2^n}\Big|_{n=1} = 1$

vi. In general

$$f^{(n)}(1) = \begin{cases} \frac{k+1}{2^k}k! = \frac{(k+1)!}{2^k} & \text{if } n = 3k+1 \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$$

## 3.3 Taylor and Maclaurin series

**Definitions:**

1. Let $f$ be infinitely differentiable at $x = a$. As we have seen, the only possible power–series representation of $f$ centered at $x = a$ is given by $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$. This power–series is called *the Taylor series of $f$ about $x = a$.*

2. If $a = 0$ then the power–series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$ *is called the Maclaurin series of $f$.*

3. If $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ for all $x$ in some interval centered at $x = a$ then we say that $f$ *is analytic at* $x = a$

In tandem to what we mentioned in the previous section, if a function $f$ happens to equal the power–series $\sum_{n=0}^{\infty} a_n(x-a)^n$ on some interval centered at $x = a$ then this series IS THE TAYLOR SERIES OF $f$ ABOUT $x = a$ (i.e. $a_n = \frac{f^{(n)}(a)}{n!}$ for all $n$).

**Examples:** The examples that follow are very important. You can obtain these series by direct computation of the coefficients $a_n = \frac{f^{(n)}(a)}{n!}$.

1. We have already seen the Maclaurin series of $f(x) = e^x$ and we have already mentioned that it is actually equal to $e^x$ for all $x$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

2. The Maclaurin series of $f(x) = \sin x$. It too equals $\sin x$ for all $x$:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

3. The Maclaurin series of $f(x) = \cos x$. It too equals $\cos x$ for all $x$:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

There are examples of infinitely differentiable functions that DO NOT equal their Maclaurin series on any interval containing 0. Here is one of the most commonly presented examples:

**Example:** The function $f$ defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is infinitely differentiable at $x = 0$ with $f^{(n)}(0) = 0$. Hence the Maclaurin series of $f$ is identically zero YET $f(x) = 0$ ONLY WHEN $x = 0$. In other words $f$ is an example of an infinitely differentiable function at $x = 0$ that is not analytic at $x = 0$.

How does one decides whether or not the Taylor series of a function actually equals the function on some interval containing the center?

There are many possible approaches into answering this question. We are going to mention two of them.

1. Take advantage of the uniqueness of the representation. If we somehow know that $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ in some open interval containing $a$ then $\sum_{n=0}^{\infty} a_n(x-a)^n$ HAS NO CHOICE BUT TO BE THE TAYLOR SERIES OF $f$ ABOUT $x = a$ (i.e. $a_n = \frac{f^{(n)}(a)}{n!}$ for all $n$). Here are couple of examples:

   **Examples:**

   (a) We know that $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for all $x$ in $(-1, 1)$. Hence $\sum_{n=0}^{\infty} x^n$ IS the Maclaurin series of $f$.

(b) How can we find the Maclaurin series of $f(x) = \arctan x$ and at the same time KNOW that it is equal to $f$ on some interval containing 0? Here is some more magic:

Notice that $f'(x) = \frac{1}{1+x^2}$. The strategy is to find the Maclaurin series and integrate term-by-term.

Well...$f'(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} \left(-x^2\right)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ and so

$$f(x) = \int \left( \sum_{n=0}^{\infty} (-1)^n x^{2n} \right) dx = \sum_{n=0}^{\infty} (-1)^n \left( \int x^{2n} dx \right) = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Since $f(0) = \arctan 0 = 0$ we conclude that $C = 0$ and thus $f(x) = \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for all $x$ in $(-1, 1)$. Hence IS the $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ Maclaurin series of $f$.

(c) Find the Taylor series of $f(x) = \ln x$ about $x = 3$ and show that it is equal to $f$ on some interval containing 3.

Again, the strategy is to find the Taylor series of $f'$ and integrate term-by-term:

$$\begin{aligned} f'(x) &= \frac{1}{x} = \frac{1}{x-3+3} = \frac{1}{3+(x-3)} = \frac{1}{3} \cdot \frac{1}{1+\frac{(x-3)}{3}} \\ &= \frac{1}{3} \sum_{n=0}^{\infty} \left( -\frac{x-3}{3} \right)^n = \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} (x-3)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} (x-3)^n \end{aligned}$$

So

$$f(x) = \ln x = \int \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} (x-3)^n \right) dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} \int (x-3)^n dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)\,3^{n+1}} (x-3)^{n+1}$$

Since $f(3) = \ln 3$ we conclude that $C = \ln 3$. Hence the Taylor series of $f$ about $x = 3$ is $\ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)3^{n+1}} (x-3)^{n+1}$ and in fact $f(x) = \ln x = \ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)3^{n+1}} (x-3)^{n+1}$ for all $x$ in $(0, 6)$ (find the open interval of convergence of the resulting series!).

2. You can always directly compute the coefficients $a_n = \frac{f^{(n)}(a)}{n!}$, find the Taylor series and subsequently use *the Lagrange form of the remainder theorem* given below, to examine whether or not the Taylor series converges in some interval containing $a$:

**Lagrange form of the remainder theorem:** Suppose that $f$ is infinitely differentiable at $x = \dot{a}$. Then for each $n$ and $x$ there is a $c$ between $a$ and $x$ so that

$$f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$$

In particular it follows that $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ if and only if $\lim_{n\to\infty} \left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\right| = 0$ independently of the value of $c$ between $a$ and $x$.

For our examples we need the following (independently interesting) fact:

For each real number $x$, $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$.

In order to see that consider the series $\sum_{n=0}^{\infty} \frac{|x|^{n+1}}{(n+1)!}$: Use the ratio test

$$\lim_{n\to\infty} \frac{\frac{|x|^{n+2}}{(n+2)!}}{\frac{|x|^{n+1}}{(n+1)!}} = \lim_{n\to\infty} \frac{|x|^{n+2}}{(n+2)!}\frac{(n+1)!}{|x|^{n+1}} = \lim_{n\to\infty} \frac{|x|}{n+2} = 0$$

$\therefore$ $\sum_{n=0}^{\infty} \frac{|x|^{n+1}}{(n+1)!}$ converges and thus $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$.

**Examples:**

(a) Show that $f(x) = \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$ and $f(x) = \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$ for all $x$.

First notice that in either case $f^{(n+1)}(x) = \begin{cases} \pm\sin x \\ \\ \pm\cos x \end{cases}$ and so $\left|f^{(n+1)}(x)\right| \leq 1$ for all $x$. Hence

$$\left|\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right| \leq \frac{|x|^{n+1}}{(n+1)!}$$

Since $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$ we conclude that $\lim_{n\to\infty} \left|\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right| = 0$ by the squeeze theorem.

$\therefore$ $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$ and $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$ for all $x$ by Lagrange's remainder theorem.

(b) Show that $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x$.

Notice that $f^{(n+1)}(x) = e^x$ and so if $c$ is between 0 and $x$ we have:

$$\left|\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right| = \frac{e^c}{(n+1)!}|x|^{n+1} \leq \frac{e^{|x|}}{(n+1)!}|x|^{n+1} = e^{|x|}\frac{|x|^{n+1}}{(n+1)!}$$

Since $\lim_{n\to\infty} e^{|x|}\frac{|x|^{n+1}}{(n+1)!} = 0$ we conclude that $\lim_{n\to\infty} \left|\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right| = 0$ by the squeeze theorem.

$\therefore$ $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x$, thanks to the remainder theorem.

## 3.4 Computing sums of series using power series techniques

Lets review the tools that we have at our disposal for computing sums of series so far:

1. Use the definition and directly compute the limit of the partial sums. Telescoping behavior is what we hope for in this case (maybe with the aid of partial fractions).

2. If we are lucky, our series could be geometric in nature (recall that $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ whenever $|r| < 1$).

3. Maybe we can use the technique of summation by parts. Yet, as we have seen this could be quite difficult

The theory of power–series and Taylor series provide us with more tools. I will present a few examples illustrating these techniques:

**Examples:**

1. Find the sum of the series $\sum_{n=0}^{\infty} \frac{2^n}{n!}$.

   Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all values of $\dot{x}$. So

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$$

   and thats all she wrote.

2. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{36^n (2n)!}$.

   This series looks suspiciously like the cosine of something. Indeed, recall that for all $x$, $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$.

   Now lets take a closer look at our series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{36^n (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n} (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\pi}{6}\right)^{2n} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

3. Find the sum of the series $\sum_{n=1}^{\infty} \frac{n}{5^n}$.

   The first observation here is to notice that multiplication of an expression by $n$ is often times the result of differentiation (after all $\frac{d}{dx} x^n = nx^{n-1}$). So the series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ is the perfect candidate.

   Differentiating both sides we obtain $\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(x-1)^2}$ which is valid for all $x$ in $(-1, 1)$.

   Hence for $x = \frac{1}{5}$ we have that $\sum_{n=1}^{\infty} \frac{n}{5^{n-1}} = \frac{1}{\left(\frac{1}{5}-1\right)^2} = \frac{25}{16}$.

   Thus

$$\sum_{n=1}^{\infty} \frac{n}{5^n} = \frac{1}{5} \sum_{n=1}^{\infty} \frac{n}{5^{n-1}} = \frac{1}{5} \cdot \frac{25}{16} = \frac{5}{16}$$

4. Find the sum of the series $\sum_{n=1}^{\infty} \frac{2^n}{n3^n}$.

The first observation here is to notice that division of an expression by $n$ is often times the result of integration (after all $\int x^{n-1}dx = \frac{x^n}{n} + C$). So again the series $\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$ is the perfect candidate.

Integrating both sides we obtain $\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x) + C$

Since $\ln 1 = 0$ we conclude that $C = 0$ and thus $\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x)$ being true for all $x$ in $(-1, 1)$. Hence for $x = \frac{2}{3}$ we have that

$$\sum_{n=1}^{\infty} \frac{2^n}{n3^n} = \sum_{n=1}^{\infty} \frac{\left(\frac{2}{3}\right)^n}{n} = -\ln\left(1 - \frac{2}{3}\right) = -\ln\left(\frac{1}{3}\right) = \ln 3$$