LECTURE 2.

Section 12.3

__Def:__ the dot product of the vectors $v = \langle v_1, v_2, v_3 \rangle$ and $u = \langle u_1, u_2, u_3 \rangle$ is denoted as $u \cdot v$ and is defined to be

$$u \cdot v = u_1 \cdot v_1 + u_2 \cdot v_2 + u_3 \cdot v_3$$

(so dot product is a number)

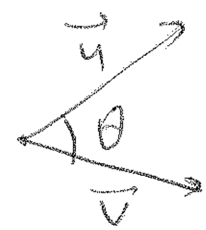
__Example:__ $u = \langle -1, 0, 1 \rangle$, $v = \langle 2, 3, 4 \rangle$

Then $u \cdot v = -1 \cdot 2 + 0 \cdot 3 + 1 \cdot 4 = -2 + 4 = 2$

Angle between two vectors

Let $u = \langle u_1, u_2, u_3 \rangle$, $v = \langle v_1, v_2, v_3 \rangle$

then $\boxed{u \cdot v = |u| \cdot |v| \cdot \cos\theta}$



$0 \le \theta \le \pi$

Therefore $\cos\theta = \dfrac{u \cdot v}{|u| \cdot |v|}$

or $\theta = \arccos\left(\dfrac{u \cdot v}{|u| \cdot |v|}\right)$

Example: Find the angle between the vectors $u = \langle 1,2,3 \rangle$, $v = \langle 1,0,1 \rangle$

Answer:

$$\theta = \arccos\left(\frac{u \cdot v}{|u| \cdot |v|}\right) = \arccos\left(\frac{1+3}{\sqrt{1^2+2^2+3^2} \cdot \sqrt{1^2+1^2}}\right) =$$

$$= \arccos\left(\frac{4}{\sqrt{14}\,\sqrt{2}}\right) = \arccos\left(\frac{4}{\sqrt{28}}\right)$$

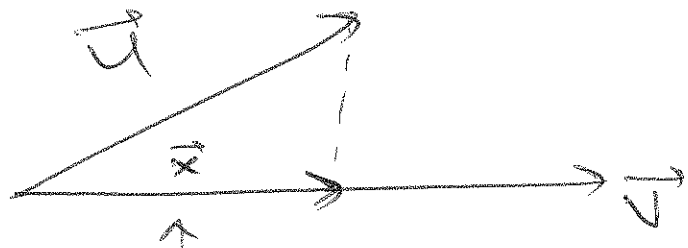
Remark: Sometimes ~~instead~~ instead of writing vector $v = \langle v_1, v_2, v_3 \rangle$ we write $v = v_1 \cdot i + v_2 \cdot j + v_3 \cdot k$ where $i = \langle 1,0,0 \rangle$, $j = \langle 0,1,0 \rangle$, $k = \langle 0,0,1 \rangle$

Indeed:

$$\underline{v_1 \cdot i + v_2 \cdot j + v_3 \cdot k} = v_1 \langle 1,0,0 \rangle + v_2 \cdot \langle 0,1,0 \rangle + v_3 \langle 0,0,1 \rangle =$$

$$= \langle v_1,0,0 \rangle + \langle 0,v_2,0 \rangle + \langle 0,0,v_3 \rangle = \langle v_1, v_2, v_3 \rangle = \underline{v}$$

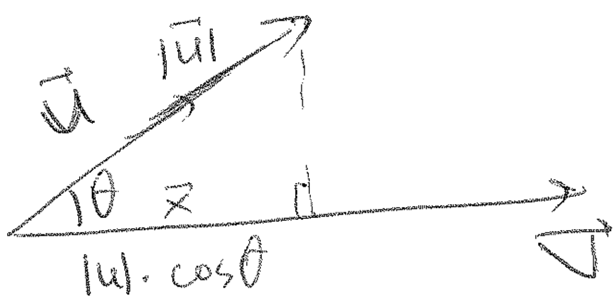
Vector projection.



Projection of vector $\vec{u}$ onto vector $\vec{v}$

$$\vec{x} \stackrel{\text{def}}{=} \text{Proj}_{\vec{v}} \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \cdot \vec{v}$$

Proof.



$|\vec{x}| = |u| \cdot \cos\theta$ and it has direction $\dfrac{\vec{v}}{|\vec{v}|}$

Therefore,

$$\underline{\vec{x}} = |x| \cdot \frac{\vec{v}}{|\vec{v}|} = |u| \cdot \cos\theta \cdot \frac{\vec{v}}{|\vec{v}|} =$$

$$= \frac{u \cdot v}{|v|} \cdot \frac{\vec{v}}{|v|} = \underline{\frac{u \cdot v}{|v|^2} \cdot \vec{v}}$$

Remark: Any vector $\vec{u}$ can be represented as unit vector times its length. Indeed:

$$\vec{u} = \underbrace{\frac{\vec{u}}{|\vec{u}|}}_{\text{unit vector}} \cdot \underbrace{|\vec{u}|}_{\text{length}}$$

Example: Let $u = i + j$, $\vec{v} = i - j$

Find $\text{Proj}_{\vec{v}}\vec{u}$ - ?

Solution:

$u = i + j = \langle 1, 1, 0 \rangle$, $v = i - j = \langle 1, -1, 0 \rangle$

$$\text{Proj}_{\vec{v}}\vec{u} = \left(\frac{u \cdot v}{|v|^2}\right) \cdot v = \frac{1 - 1}{\sqrt{1^2 + 1^2}} \cdot \langle 1, -1, 0 \rangle =$$

$$= 0 \cdot \langle 1, -1, 0 \rangle = 0.$$

Example:

Let $\vec{u} = \langle 1,2,3\rangle$

Express $\vec{u}$ as a unit vector times (direction) its length.

$$\vec{u} = \frac{\vec{u}}{|\vec{u}|}\cdot|\vec{u}| = \frac{\langle 1,2,3\rangle}{|\vec{u}|}\cdot|\vec{u}| =$$

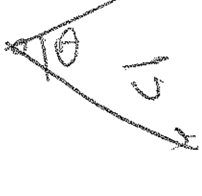
$$= \frac{\langle 1,2,3\rangle}{\sqrt{1^2+2^2+3^2}}\cdot\sqrt{1^2+2^2+3^2} = \frac{\langle 1,2,3\rangle}{\sqrt{14}}\cdot\sqrt{14} =$$

$$= \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle\cdot\sqrt{14}.$$

Remark: Nonzero vectors $\vec{u}$ and $\vec{v}$ are perpendicular if $\vec{u}\cdot\vec{v} = 0$. Indeed

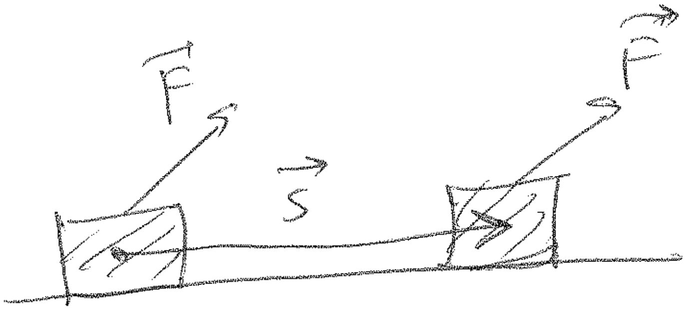
$$\vec{u}\cdot\vec{v} = |\vec{u}|\cdot|\vec{v}|\cdot\cos\theta = 0 \text{ if } \cos\theta = 0$$

or $\theta = \frac{\pi}{2}$



So 

Work



$$\boxed{\text{Work} = \vec{F} \cdot \vec{S}.}$$

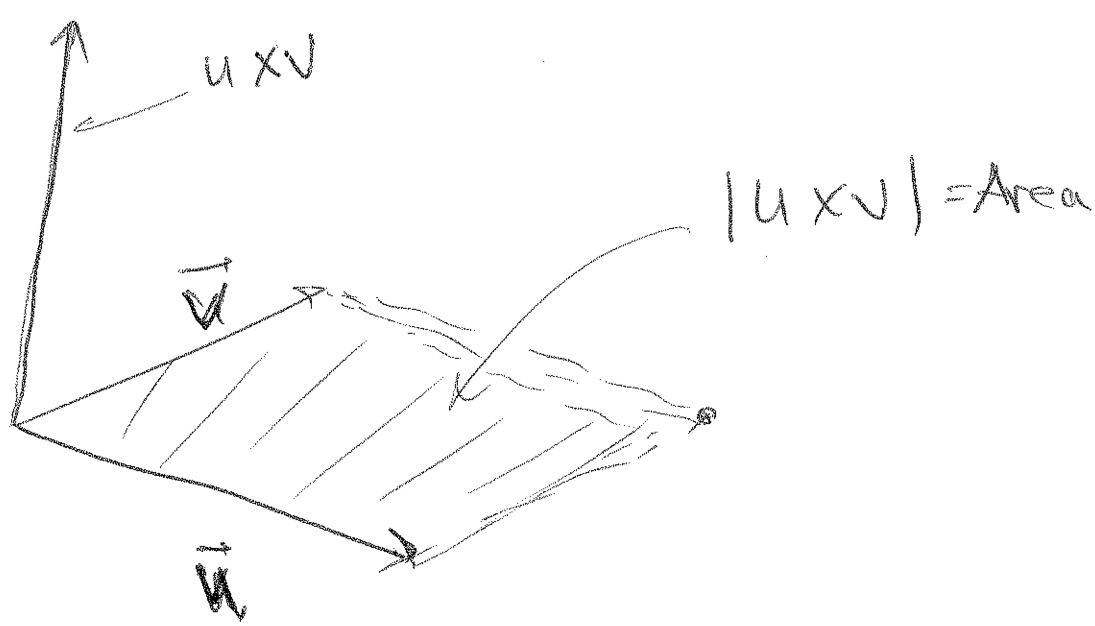
$\vec{F}$ - is a forve (vector)

and $\vec{S}$ is a vector (from which place to which place the object moved.

Section 12.4.

Cross product

If $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$ then the <u>cross product</u> is denoted as $u \times v$ and is such a vector which is perpendicular both to u and v and it has a length of area of Paralelogram constructed by $\vec{u}$ and $\vec{v}$.



Determinants:

$$\det \begin{vmatrix} a & b \\ c & d \end{vmatrix} \stackrel{\text{def}}{=} ad - bc.$$

$$\det \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \stackrel{\text{def}}{=} a_1 \cdot \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \cdot \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + $$

$$+ c_1 \cdot \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} =$$

$$= a_1 \left(b_2 c_3 - b_3 c_2 \right) - b_1 \left(a_2 c_3 - c_2 a_3 \right) +$$

$$+ c_1 \left(a_2 b_3 - b_2 a_3 \right)$$

So If $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$

then

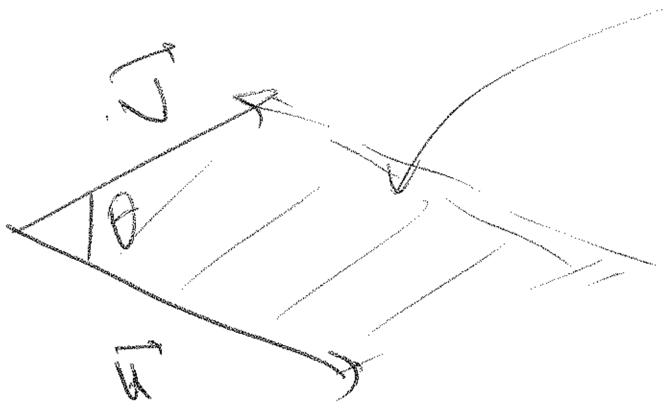
$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = i \cdot \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - j \cdot \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} +$$

$$+ k \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} =$$

$$= \left\langle u_2 v_3 - u_3 v_2, \; -u_1 v_3 + v_1 u_3, \; u_1 v_2 - u_2 v_1 \right\rangle$$

Also Remember that

$$\boxed{|u \times v| = |u| \cdot |v| \sin\theta = \text{Area}}$$



Example: $u = \langle 1,2,3 \rangle$, $v = \langle 0,0,1 \rangle$

$$\underline{u \times v} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 0 & 0 & 1 \end{vmatrix} = i \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} - j \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} + k \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} =$$

$$= i \cdot 2 - j + 0 = \underline{\langle 2, -1, 0 \rangle}$$

Example: Let $u = \langle 0,1,0 \rangle$,

$v = \langle 1,0,0 \rangle$

Find the ~~vect~~ unit vector which is perpendicular to u and v.

Solution: It is clear that this vector is going to be

$$\frac{u \times v}{|u \times v|}.$$ Therefor $u \times v = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} =$

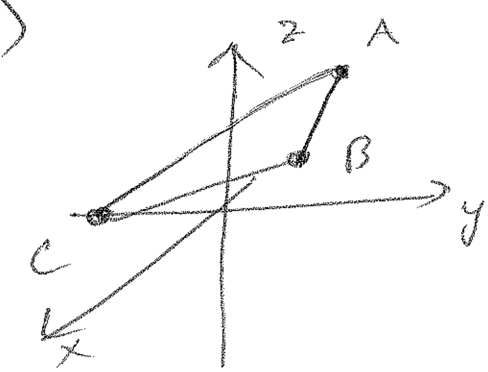
$$= i \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} - j \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} + k \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = k = \langle 0,0,1 \rangle$$

Therefore $$\frac{u \times v}{|u \times v|} = \frac{\langle 0,0,1 \rangle}{\sqrt{1^2+0^2+0^2}} = \langle 0,0,1 \rangle$$

Example:

Let $A = \langle 0,1,2 \rangle$, $B = \langle 1,1,1 \rangle$,

$C = \langle -1,0,0 \rangle$



Find the area of a triangle ABC!

Solution:

$$S_{ABC} = \frac{|\overrightarrow{CA} \times \overrightarrow{CB}|}{2}$$ ← is it clear? if not, see definition of cross product

Hence $\overrightarrow{CA} = \langle 1,1,2 \rangle$, $\overrightarrow{CB} = \langle 2,1,1 \rangle$

$$\overrightarrow{CA} \times \overrightarrow{CB} = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{vmatrix} = i\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} - j\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + k\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} =$$

$$= i(1-2) - j(1-4) + k(1-2) = -i + 3j - k = \langle -1, 3, -1 \rangle$$

Hence $S_{ABC} = \dfrac{|\langle -1,3,-1 \rangle|}{2} = \dfrac{\sqrt{1+9+1}}{2} = \dfrac{\sqrt{11}}{2}$.