MATH-LITERACY MANUAL

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1 Real Numbers

1.1 Sets

Sets are collections of objects. These "objects" are not necessarily physical objects; they may be imaginary, or, for instance, in mathematics we often deal with sets of numbers. Thus, we can talk about the set of all students in the classroom (at some particular time), of all names on the roster, of all objects on your desk, or of all positive integers less than 6. Sets are denoted by capital letters. Suppose A is the set of all students in the classroom. This sentence describes set A perfectly well, but there is a more formal notation, called the *set-builder notation*,

$$A = \{x \mid x \text{ is a student in the classroom}\}. \tag{1}$$

Each student in the classroom is an *element* of set A. Elements are denoted by lower-case letters. In (1), x is a letter used to name the general element of set A. The character '|' separates the general element from its description and can be read as "such that". Therefore, (1) has the following meaning: "A is a set of elements x such that x is a student in the classroom". Suppose a is a symbol we use to denote one particular student in the classroom. To indicate that *a is an element of set A*, we write $a \in A$. We read this also as *a is in A* or *a belongs to A*.

Similarly, if B is the set all positive integers less than 6, then we can write

$$B = \{n \mid n \text{ is a positive integer less than } 6\}. \tag{2}$$

Note that it is irrelevant how the elements of the set are denoted (we use x in (1) and n in (2)), but the description of the element has to refer to the chosen name. We read (2) as "B is a set of elements n such that n is a positive integer less than 6". We see therefore that the set-builder notation has the following structure:

$$\text{name of the set} = \{\text{name of the general element} \mid \text{description of the element}\}.$$

Since 0 is not positive, even though it is an integer less than 6, it is not an element of set B. We can write this symbolically as $0 \notin B$. On the other hand, $1 \in B$ and we can continue identifying the elements of B. There are 5 elements in B and this set can be described also as

$$B = \{1, 2, 3, 4, 5\}. \tag{3}$$

This *list description* is an alternative to the set-builder description in (2). Set B is an example of a *finite set*, that is, it a set which has a finite number of elements.

Set-builder notation is more convenient for sets with a large number of elements. If the number of elements in a set is very large, then it may be impractical to list each element individually. In this case, set-builder notation may be more useful. This is particularly so when the set is *infinite*, i.e. when it has infinitely many elements, like in the following example:

$$Q = \left\{ \frac{n}{m} \,\middle|\, n \text{ is an integer and } m \text{ is a positive integer} \right\}.$$

Nevertheless, some infinite sets can be described using a partial list of their elements. For instance, the set N,

$$N = \{1, 2, 3, 4, \ldots\}, \tag{4}$$

is a set of all positive integers and is therefore infinite. It is assumed here that the reader will realize the *pattern* that the elements of set N follow, and that he/she will be able to continue the list (the three dots indicate that the numbers go on and on).

The elements shown in (4) are ordered with the purpose of suggesting the desired pattern, but, in general, the order of elements in a set is unimportant. This is because any set is just a collection of objects. Therefore,

$$B = \{1, 2, 3, 4, 5\} = \{5, 4, 3, 2, 1\} = \{2, 5, 4, 1, 3\} = \cdots .$$

Moreover, no element occurs in a set more than once.

Example 1. The set D contains all letters in the word *algebra*. Write D listing its elements.
Solution. $D = \{a, l, g, e, b, r\}$. Notice that a is not repeated twice in D. □

The above defined set A is also finite, but its list description is less convenient than that of set B. Each element of A would have to be listed using phrases like "the student whose name is..." or "the student whose identification number is...". This is because there is a difference between each individual person and that person's name or i.d. number, which are just strings of letters/digits or sounds. For instance, the set S,

$$S = \{\text{Peter, Paul, Mary}\},$$

is a set of three words unless we state explicitly that by these three names we mean the actual specific persons. In general, it should be made clear that particular objects are meant when only their names are listed as elements of a set. Otherwise, the elements of the set are regarded as words. As another example, suppose you want to define a set C of all objects on your desk. If you come up with

$$C = \{\text{College Algebra textbook, notebook, pencil, calculator}\},$$

you have to say that by this you mean the actual objects which are on your desk, not some words and phrases which mean something in English, but nothing in many other languages. If you say nothing, C is to be understood as a set containing some strings of characters.

Sometimes there is no element in a set. Such a set is called *empty* and is denoted by $\emptyset$. If there is at least one element in a set, the set is called *non-empty*.

Example 2. Determine for each set whether it is empty or non-empty.

$$\begin{aligned} E &= \{x \mid x \text{ is a positive integer less than } 1\} \\ F &= \{x \mid x \text{ is a negative integer greater than } -3\} \\ G &= \{y \mid y \text{ is an integer solving the equation } y+1=y\} \end{aligned}$$

Solution. $E = \emptyset$, $G = \emptyset$, and F is non-empty ($F \neq \emptyset$) since $F = \{-2, -1\}$.
□

A subset of a set A is a set containing some elements of set A. For instance, the set $D = \{1, 3, 4\}$ is a subset of the set $B = \{1, 2, 3, 4, 5\}$ given in (3). We write this relationship using the symbol $\subset$, which can open also to the left, $\supset$,

$$D \subset B \quad \text{or} \quad B \supset D.$$

On the other hand, the set $E = \{-1, 2, 3\}$ is not a subset of B because $-1 \notin B$. We write this as

$$E \not\subset B \quad \text{or} \quad B \not\supset E.$$

We use the symbol $\subset$ in this text to mean that *a set is its own subset*, $A \subset A$. Many authors opt for $X \subset Y$ to mean that X is a subset of Y but X is not equal to Y. They use $X \subseteq Y$ if they want to permit that $X = Y$. This is just a question of what notation is more convenient. Note also that the empty set is a subset of any set, $\emptyset \subset A$.

Example 3. Given set $X = \{-1, 3, 4, 7, 10\}$, determine for each set below whether it is a subset of X or not and express this using the appropriate notation.

(a) $P = \{-1\}$

(b) $Q = \{3, 8, 10\}$

(c) X

(d) $\emptyset$

(e) $R = \{x \mid x \text{ is an integer greater than 3 and less than 8}\}$

Solution. $P \subset X$, $Q \not\subset X$, $X \subset X$, $\emptyset \subset X$, $R = \{4, 5, 6, 7\} \not\subset X$. □

We can do operations with sets. The only operations needed in this text are *union* and *intersection*. There are also other operations, but they will not be discussed. The union of two sets A and B, denoted by $A \cup B$, is the set of all elements of A and B together,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Obviously, $A \cup B = B \cup A$.

Example 4. Consider the sets B, D, and E, introduced earlier and repeated below for convenience,

$$B = \{1, 2, 3, 4, 5\}, \quad D = \{1, 3, 4\}, \quad E = \{-1, 2, 3\}.$$

Find $D \cup E$ and $B \cup D$.

Solution. $D \cup E = \{-1, 2, 3, 4\}$ (note that $D \subset D \cup E$ and $E \subset D \cup E$). $B \cup D = \{1, 2, 3, 4, 5\} = B$ since $D \subset B$. □

This example illustrates some general facts: $A \subset A \cup B$, $B \subset A \cup B$, and $A \cup B = B$ if $A \subset B$. Note also that $\emptyset \cup A = A$.

The intersection of two sets A and B, denoted by $A \cap B$, is the set containing all elements that A and B share,

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Example 5. Find $D \cap E$ and $B \cap D$ for the sets from the previous example.

Solution. $D \cap E = \{3\}$ and $B \cap D = \{1, 3, 4\} = D$. □

Intersection has the following general properties: $A \cap B = B \cap A$, $A \cap B \subset A$ and $A \cap B \subset B$, $\emptyset \cap A = \emptyset$, and $A \cap B = A$ if $A \subset B$.

Example 6. Let $A = \{0, 2, 4, 6\}$, $B = \{1, 3\}$, and $C = \{1, 2, 3, 6, 8, 9\}$. Find each of the following: $A \cup B$, $A \cap B$, $A \cup C$, $B \cup C$, $B \cap C$, and $A \cap C$.

Solution.

$$A \cup B = \{0, 1, 2, 3, 4, 6\}$$

$$A \cap B = \emptyset$$

$A \cup C = \{0, 1, 2, 3, 4, 6, 8, 9\}$

$B \cup C = C$

$B \cap C = B$

$A \cap C = \{2, 6\}$ □

EXERCISES 1.1

1. Write in set-builder notation.
   (a) S is the set of all citizens of the U.S.
   (b) A is the set of all integers greater than 3 and less then 10.
   (c) B is the set of all chairs at your home.
   (d) X is the set of all letters in the word *mathematics*.
2. Write the sets in Exercises 1(b) and 1(d) by listing their elements.
3. Write each set listing its elements.
   (a) A is the set of all even positive integers.
   (b) B is the set of all integers less than -5.
   (c) C is the set of all integers.
4. Use symbols $\in$ and $\notin$ to write whether each of the numbers -4, -1, 0, 3, and π belong to the set $S = \{-2, -1, 0, \sqrt{2}, \pi, 4, 5\}$ or not.
5. Determine for each set whether it is empty or non-empty.

$$\begin{aligned} H &= \{a \mid a \text{ is an odd integer greater than } 2\} \\ K &= \{b \mid b \text{ is a negative integer greater than } 5\} \\ M &= \{c \mid c \text{ is an integer between 8 and 9}\} \end{aligned}$$

6. Given set $Y = \{0, 2, 5, 6, 8\}$, determine for each set below whether it is a subset of Y or not. Express this using the appropriate notation.
   (a) $A = \{a \mid a \text{ is a positive even integer less than } 9\}$
   (b) $B = \{5, 8\}$

(c) $\emptyset$
(d) $C = \{5, 6, 7, 8\}$
(e) $D = \{0, 2, 5, 6, 8, 9\}$

7. Let $A = \{2, 5, 7\}$, $B = \{6, 7, 8, 9\}$, and $C = \{5, 9\}$. Find each of the following: $A \cup B$, $A \cup C$, $A \cap C$, $B \cap A$, $B \cup C$, and $C \cap B$.

1.2 Constants and Variables; Real Numbers

Constants are fixed numbers. Any specified number, like 3, -7.12, $\frac{1}{2}$, $\sqrt{2}$, etc., represents a fixed quantity, a numerical value that does not change, and is therefore called a *constant*. *Arithmetic* is a branch of mathematics that deals with constants. An *arithmetical expression* is formed by indicating various operations that should be performed on constants.

Example 1. An arithmetical expression:

$$1 + \frac{5 + \sqrt{3}}{2} \cdot (4^2 - 7) \tag{1}$$ □

A variable changes its numerical value. Most high school and college-level math classes deal with *algebra*, which makes use of *variables*, denoted by letters. The most frequently used variable is probably x. Any variable is a number, only this number is not specified. Therefore, generally speaking, a variable stands for *any* number. Sometimes, however, we may want to assign a specific value to the variable. We can do this by using the equal sign, like below:

$$x = 6.$$

In this case we say that the variable x has the (numerical) value of 6. Other times, there may be several constant values that the variable can have. This can be described using the set notation from the previous section:

$$x \in \left\{-4, \frac{3}{2}, 0, 23\right\}.$$

This means that x can have four possible values; it can equal -4, $\frac{3}{2}$, 0, or 23.

Expressions that involve operations with one or more variables are not arithmetical. Such expressions are called *algebraic*. Therefore, variables are used to form *algebraic expressions.*

Example 2. Algebraic expressions:

$$2x + 3y \tag{2}$$

$$1 + \frac{5 + \sqrt{x}}{x} \cdot (x^2 - 7) \tag{3}$$

$$z^3 \tag{4}$$

□

As we can see, an algebraic expression may contain constants as well. In general, even an expression like (1) can be referred to as algebraic – it is a special case of an algebraic expression without any variable. Any algebraic expression can be converted into an arithmetical expression by specifying its variable(s).

Example 3. Let $x = 3$, $y = -1$, and $z = 0$ in expressions (2), (3), and (4). Write the resulting arithmetical expressions.

Solution. In expression (2), replace x with 3 and y with -1 to get

$$2(3) + 3(-1).$$

We have to use parentheses around 3 and -1; without them, we would get $23+3-1$, which changes the numbers and operations indicated in the original expression. Instead of 2(3) we can write also $2 \cdot 3$, whereas the notation $3 \cdot -1$ is not customary and should be avoided since we do not write two basic operation symbols one after another. We can indicate that 3 and -1 are to be multiplied, but parentheses are still needed around -1, thus $3 \cdot (-1)$.

Expression (3) reduces to

$$1 + \frac{5 + \sqrt{3}}{3} \cdot (3^2 - 7).$$

Notice that in (1) we cannot substitute at the same time 3 for the x under the radical, 2 for the x under the fraction bar, and 4 for the x squared. In other words, we cannot get expression (1) from expression (3) in this way. x cannot be 3, 2, and 4 at the same time; it is a variable, but it can take on only one specific value at a time.

Finally, from expression (4), we get 0^3. □

What is a number? When we say that a variable stands for any number, we mean any *real number*. Very often people think in terms of integers, but the set of all real numbers, denoted by $\mathbb{R}$, contains many more numbers, not just integers. The set of *integers* is

$$\{\ldots,-3,-2,-1,0,1,2,3,4,\ldots\}.$$

Thus, an integer can be positive, negative, or zero (which is neither negative nor positive). Real numbers, on the other hand, consist of rational and irrational numbers. *Rational numbers* are the numbers that can be written as fractions, for instance, $\frac{1}{2}$, $\frac{-11}{5}$, $\frac{4}{12}$, etc. The fraction bar means nothing else but division, $\frac{1}{2} = 1 \div 2$. In a rational number, both the numerator (the number above the fraction bar) and the denominator (the number below the bar) have to be integers. The name *denominator* comes from the fact that the bottom number gives the name to the fraction ("denominates" it) because it indicates what part of the whole is considered. Thus, the following fractions are all thirds:

$$\frac{1}{3}, \quad \frac{7}{3}, \quad \frac{36}{3}, \quad \frac{-5}{3}.$$

How many thirds there are is indicated by the top number which gives the count of (enumerates) the parts specified by the bottom number. Hence, the top number is called the *numerator*.

The integer in the denominator cannot be 0, since division by 0 is undefined. Therefore, to write something like $\frac{7}{0}$ is meaningless. We say that such a fraction *does not exist* or is *undefined*, which is to say that $\frac{7}{0}$ is not a number. The denominator may be a negative integer, but when this is the case, we usually transform the fraction to make the denominator positive:

$$\frac{4}{-9} = \frac{-4}{9} = -\frac{4}{9}, \qquad \frac{-4}{-9} = \frac{4}{9}.$$

Because of the above, we can always write a rational number so that both its numerator and denominator be positive numbers, with the negative sign possibly in front of the whole fraction, i.e. in front of the fraction bar. Therefore, any rational number other than zero can be written as either $\frac{a}{b}$ or $-\frac{a}{b}$, where a and b are positive integers.

There are several different ways so write the same number, e.g.

$$\frac{10}{2} = \frac{5}{1} = 5. \tag{5}$$

This shows that *integers are also rational numbers.* The above fractions $\frac{10}{2}$ and $\frac{5}{1}$ are examples of improper fractions. Consider positive numerators and denominators. Any time the numerator is greater or equal to the denominator, the fraction is called *improper.* An improper fraction can be either reduced to an integer like in (5), or to a *mixed number* like below:

$$\frac{12}{7} = 1\frac{5}{7}, \qquad -\frac{16}{3} = -5\frac{1}{3}.$$

The use of mixed numbers, however, should be avoided in general. This is because of the possible confusion between

$$-5\frac{1}{3} = -\left(5 + \frac{1}{3}\right) = -\frac{16}{3} \quad \text{and} \quad -5 \cdot \frac{1}{3} = -\frac{5}{3}.$$

Therefore, if a problem involves mixed numbers, it is recommended to convert them immediately to improper fractions. Only exceptionally can mixed numbers be more suitable than improper fractions. For instance, when plotting a point with a coordinate $\frac{9}{4}$, it is easier to switch to $2\frac{1}{4}$.

Fractions that are not improper are called *proper*, i.e. their numerators are less than their denominators (both are again considered positive in this discussion). Fractions $\frac{1}{2}$ and $-\frac{2}{19}$ are examples of proper fractions.

The slash fraction bar is sometimes used instead of the horizontal one, $1/2 = \frac{1}{2}$. This too should be avoided if possible. For instance, in an expression like $1/2x$, it may be unclear whether x is below the fraction bar or not. There is no such ambiguity when the horizontal bar is used:

$$\text{either} \quad \frac{1}{2x} \quad \text{or} \quad \frac{1}{2}x.$$

Parentheses can be used to clarify the situation with slash notation, but this is clumsy,

$$1/(2x) = \frac{1}{2x}, \qquad (1/2)x = \frac{1}{2}x.$$

The only reason for using the slash is when there is not enough space to write the fraction with a horizontal bar, like when the fraction is in the exponent, e.g. $x^{2/3}$.

Fractions can be written also as decimals,

$$\frac{3}{2} = 1.5, \qquad \frac{1}{8} = 0.125, \tag{6}$$

(the zero before the decimal point may be omitted, thus $0.125 = .125$). A fraction has either finitely many decimal places like in (6), or infinitely many decimal places like $\frac{1}{3} = 0.3333\ldots$, where the dots indicate that the digit 3 is repeated forever. In the case when the decimal representation of a fraction has infinitely many decimals, there is always a string consisting of one or more digits, which keeps on repeating. Such a decimal is called the *repeating decimal.* We usually overline the repeating string of digits:

$$\frac{1}{3} = 0.\bar{3}, \qquad \frac{2}{7} = 0.\overline{285714}.$$

If a decimal has infinitely many digits and no repeating part, then it is not a rational number, i.e. it is not a fraction. Such a number is called an *irrational number*. Many radicals are irrational numbers:

$$\sqrt{2} = 1.41421\ldots, \quad \sqrt{3} = 1.73205\ldots, \quad \sqrt[3]{15} = 2.46621\ldots.$$

Here, the dots indicate that the decimals keep on occurring. There is, however, no repeating sequence of decimals. An irrational number is also real although we can never know its exact value since we cannot know and write infinitely many digits. We can only know approximate values of irrational numbers: $\sqrt{2} \approx 1.41421$ ($\approx$ means *approximately equal*); note that $\sqrt{2} \neq 1.41421$ ($\neq$ means *not equal* or *different*). The more decimals we use, the better approximation of an irrational number we have: 1.41421356 is a better approximation of $\sqrt{2}$ than 1.41421, but is still not the exact value.

Irrational numbers have to be defined in some way other than referring to their value. For instance, $\sqrt{2}$ is such a number which gives 2 when squared. Some irrational numbers are very frequently used constants and are therefore denoted by special letters. The constant π is such an irrational number. It is said very often that π is 3.14, which is not true since 3.14, as any finite decimal, is a rational number, $3.14 = \frac{314}{100} = \frac{157}{50}$. π is only approximately equal to 3.14. π is defined as the ratio of the circumference and the diameter of any circle regardless of its size. This ratio is constant for all circles and the constant is denoted by π. It turns out that π is an irrational number, $\pi = 3.14159\ldots$, and its exact value cannot be known although millions of digits of π have been calculated. Another such constant is $e = 2.71828\ldots$, which is a very important number in mathematics, defined as the number the following expression approaches when x takes on greater and greater values (when x tends to infinity):

$$\left(1+\frac{1}{x}\right)^x.$$

Example 4. Given the set

$$S = \left\{-7, \frac{4}{5}, 0, \pi+1, 103, \sqrt{7}, \frac{e}{2}, -\frac{17}{9}, 2.\overline{123}\right\},$$

determine its subset containing all

(a) integers,

(b) rational numbers,

(c) irrational numbers,

(d) real numbers.

Solution.

(a) $\{-7, 0, 103\}$

(b) $\{-7, \frac{4}{5}, 0, 103, -\frac{17}{9}, 2.\overline{123}\}$

(c) $\{\pi+1, \sqrt{7}, \frac{e}{2}\}$

(d) S □

Constants too may be denoted by letters. As π and e illustrate, letters may be used not only to indicate variables but constants as well. Letters used for constants are usually reserved for those specific numbers; π normally denotes just the irrational number defined above and nothing else. There is however a need to refer to constants as opposed to variables in a more general way. For instance, if we want to talk about expressions like

$$x^2 + 2x - 3, \quad 5x^2 - x + 7, \quad -2x^2 + 4x - 9, \quad \text{or} \quad 3x^2 + x,$$

we can represent them all together as

$$ax^2 + bx + c, \tag{7}$$

where a, b, and c are constants ($a \neq 0$) and x is a variable. Thus, in (7) we use letters to represent constants. The meaning is that we always think of a, b, and c in (7) as some *fixed*, albeit unspecified, numbers, whereas x *changes* its value taking on any number from the set $\mathbb{R}$. Whether a letter denotes a constant or a variable should be clear from the context. Letters a, b, c, d, e, k, ℓ, m, or n are usually used to denote constants, whereas s, t, u, v, w, x, y, z normally indicate variables. The corresponding capital letters may be used as well to denote constants or variables accordingly.

Example 5. Assuming the letters are used according to the above standard notation, determine what letters represent constants and what letters represent variables.

(a) $mx + b$

(b) $\sqrt{\dfrac{ax + b}{cy + d}}$

(c) $At^2 + B$

Solution.

(a) constants: m, b; variables: x

(b) constants: a, b, c, d; variables: x, y

(c) constants: A, B; variables: t □

Real numbers can be graphed. The set of real numbers is represented graphically by the *real-number line*. There are as many real numbers as there are points on this straight line. Every point corresponds to some real number and the other way around.

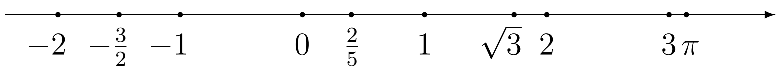

Figure 1. The real-number line with some real numbers indicated on it.

Figure 1 shows some real numbers on the real-number line. The real-number line is *oriented* in the sense that the numbers on it increase to the right. When plotting any two different numbers on the real-number line, we can position them anywhere we want to (provided, of course, that the greater number is to the right). This is simply a matter of choice. The location of any two different numbers determines the location of all other numbers. This is not to say that they can be plotted absolutely accurately in practice. In Figure 1, for instance, there is no guarantee that every indicated point is found with absolute precision. Our tools for measuring distance are not perfect, so we have to be satisfied with approximate accuracy when locating numbers on the real-number line. However, we know for certain that for each real number there is a unique point representing it. The only question is how accurately we can plot this point.

Suppose the positions of 0 (called the *origin*) and 1 are chosen. This is what is usually indicated on the real-number line. If we assume for a moment that we can measure distances perfectly, even some irrational numbers can be found accurately on the real-number line. For instance, $\sqrt{2}$ can be constructed geometrically as the length of the hypotenuse of the right triangle with both legs of length 1. However, there are also irrational numbers, like π and e, which cannot be constructed geometrically. Their position relative to 0 and 1 is impossible to determine exactly, even under ideal conditions.

Example 6. What right triangle should be constructed to find each length?

(a) $\sqrt{5}$

(b) $\sqrt{3}$

(c) $\sqrt{6}$

Solution. Let a and b denote the length of each leg and let c be the length of the hypotenuse of a right triangle. Then, by Pythagorean theorem, $c^2 = a^2 + b^2$. We should construct a right triangle with the side-lengths given below. The third side has the desired length.

(a) $a = 1$ and $b = 2$, giving $c = \sqrt{1^2 + 2^2} = \sqrt{5}$.

(b) $a = 1$ and $c = 2$, giving $b = \sqrt{2^2 - 1^2} = \sqrt{3}$.

(c) $a = 2$ and $b = \sqrt{2}$ (this has to be constructed separately as described in the text above), giving $c = \sqrt{2^2 + (\sqrt{2})^2} = \sqrt{6}$. Or, $a = \sqrt{3}$ (constructed like in (b)) and $c = 3$, giving $b = \sqrt{3^2 - (\sqrt{3})^2} = \sqrt{6}$. □

Distances on the real-number line can be expressed using the absolute value. Both 3 and -3 are at the distance of 3 units from the origin. We can state this using absolute value: $|3| = |-3| = 3$. Therefore, $|x|$ means the distance of the real number x from the origin. Since the physical distance cannot be negative, $|x|$ is non-negative for all $x \in \mathbb{R}$. The only time $|x| = 0$ is when $x = 0$ (indeed, the distance between 0 and the origin is 0). More generally, the distance between any two real numbers can be expressed using the absolute value: $|a - b|$ *is the distance between* a *and* b *on the real-number line.*

Example 7. Express the distance between the given numbers using the absolute value.

(a) 0 and -10

(b) t and 0

(c) 3 and 7

(d) -11 and -1

(e) x and 5

(f) x and -6

Solution.

(a) $|-10|$

(b) $|t|$

(c) $|3 - 7|$

(d) $|-11-(-1)|$

(e) $|x-5|$

(f) $|x-(-6)|$ □

EXERCISES 1.2

1. Determine for each expression whether it is arithmetical or not.

   (a) $\sqrt{5+\frac{x+3}{7}}$

   (b) $\pi - 3(29-45)^2$

   (c) $7 - e^5\sqrt{11}$

   (d) $x(3y-4)+27-5$

2. Use $x = 0$, $y = \frac{1}{3}$, and $z = -2$ to write each algebraic expression as an arithmetical one.

   (a) $\frac{z^2+8}{z+13}$

   (b) $6^z - 3x$

   (c) $y(2z+7)^x$

   (d) $4x^2 - z^y$

3. Identify the numerator and the denominator of each fraction given below:
$$\frac{14}{3},\quad \frac{-7}{5},\quad \frac{-9}{11}.$$

4. Write each fraction in the form $\frac{a}{b}$ or $-\frac{a}{b}$, where both a and b are positive integers,
$$\frac{-12}{5},\quad \frac{-9}{-8},\quad \frac{14}{-3},\quad \frac{1}{-20}.$$

5. Determine for each fraction whether it is proper or improper,
$$\frac{5}{16},\quad -\frac{1}{6},\quad \frac{7}{2},\quad \frac{3}{11}.$$

6. Write each fraction given in problem 5 as a decimal. Indicate any repeating part.

7. Given the sets
$$A = \left\{ e-2, \frac{-1}{6}, 18, 2\pi, \sqrt[3]{2}, \frac{15}{2} \right\}$$
and
$$B = \left\{ 12.12\bar{5}, 3.14, \frac{1}{3}, -0.\overline{17}, \sqrt{4} \right\},$$
determine their subsets containing all
   (a) integers,
   (b) rational numbers,
   (c) irrational numbers,
   (d) real numbers.

8. Assuming the letters are used according to the standard notation, determine what letters represent constants and what letters represent variables.
   (a) $u^2 + bu + c$
   (b) $\sqrt{(x-a)^2 + (y-b)^2}$
   (c) $b^2 - 4ac$
   (d) $\dfrac{x+y}{v+w}$

9. What right triangle should be constructed to find each length?
   (a) $\sqrt{7}$
   (b) $\sqrt{8}$
   (c) $\sqrt{11}$

10. Express the distance between the given numbers using the absolute value.
   (a) x and 0
   (b) x and 9
   (c) 4 and 11
   (d) -6 and 10
   (e) 2 and $-a$
   (f) 0 and -3

1.3 Operations with Numbers

The four basic operations are addition, subtraction, multiplication, and division. They all operate between two numbers, as illustrated below,

$$11 + \pi, \quad x - y, \quad 3 \times x = 3 \cdot x = 3x, \quad a \div b = a/b = \frac{a}{b}$$

(there are different ways of expressing multiplication and division). The result of each basic operation is named in Table 1.

Operation	Result
addition	sum
subtraction	difference
multiplication	product
division	quotient

Table 1. Results of basic operations.

As we have already seen when discussing fractions in the previous section, *division by* 0 *is undefined*. This means that we cannot do this operation. When we look at addition, subtraction, multiplication, and even division by a non-zero number, we can always perform these operations, i.e. we always get some real number as their result. However, $3 \div 0$ (this can be written also as $3/0$ or $\frac{3}{0}$), for instance, is not a number. Division by 0 cannot be defined in a meaningful way, but we know how to divide 0 by a non-zero number and the result is 0, e.g. $0 \div 3 = 0$ ($0 \div 0$ is still undefined).

The minus, '$-$', may seem an ambiguous symbol because it is used not only for subtraction but for negative numbers as well. In the arithmetical expression $-5 - 3$, the first minus indicates the negative sign, whereas the second minus means subtraction. However, even the negative sign may be viewed as subtraction since $-5 = 0 - 5$. Or, another way to look at subtraction is through addition because of $-5 - 3 = -5 + (-3)$, where subtraction is expressed as addition of the negative number. In general, $x - y = x + (-y)$. Therefore, the seemingly different meanings of the minus can be reduced to the same. When the minus is used to indicate a negative number, like in -3, we read this as "negative three". This is usually transferred to variables and we say "negative y" for $-y$. Unfortunately, this often leads to the misinterpretation that $-y$ is negative. y, however, is a variable, and if nothing more specific is said of y, then it stands for *any* real number. If, for instance, $y = -5$, then $-y = 5$, so "negative y" may be a *positive* number. It is probably better to read $-y$ as "minus y" and, if you associate the word "minus" with the operation of "minusing" (that is, subtracting), you still *are* "minusing" because $-y = 0 - y$.

When 0 is added or subtracted, the result of the remaining operations is not changed, and 0-addition or 0-subtraction are normally not written,

$$0 + 3 + x = 3 + x, \qquad v - 0 = v, \qquad 0 - w + 7 + 0 - z - 0 = -w + 7 - z.$$

However, if the final result is 0, it should be written. When nothing is written, this does not mean 0.

The same way subtraction can be expressed in terms of addition, division can be expressed in terms of multiplication. *To divide by a number (not equal to zero, of course) is the same as to multiply by its reciprocal*, e.g.

$$\frac{a}{b} = a \cdot \frac{1}{b}, \qquad \frac{x}{6} = \frac{1}{6}x.$$

Multiplication by 1 and division by 1 should not be written in normal situations, since these operations do not change the result,

$$1 \cdot x = x, \qquad \frac{1x}{1 \cdot 5} = \frac{x}{5}, \qquad \frac{x+3}{1} = x + 3.$$

There are also operations that require only one number. Squaring, cubing, taking the square root or the cube root are some examples of such operations. The expression x^n is a *power*, which indicates that the number x, called the *base*, is raised to the number n, called the *exponent*. The word "power" is used also to mean "exponent" – we read x^5 as "x raised to the fifth power", or simply "x to the fifth". It is not necessary to write the exponent when it is equal to 1, $x^1 = x$. Raising to the second power is called *squaring* and raising to the third power is *cubing*. What have x^2 and x^3 to do with the square and the cube as geometrical objects? One of the things that comes to mind first when the square is mentioned is that it has four sides of equal length, so why does not "x squared" mean x^4? The answer is simply that the square is a two-dimensional object, and if its sides measure x length units, we say that it is an "x-by-x square", which we write also as "$x \times x$ square". However, $x \times x = x \cdot x = x^2$, which is at the same time the area of the square. Similarly, x^3 is related to the cube through the three dimensions of this object that can be described as $x \times x \times x$. At the same time, x^3 indicates the volume of such a cube. By the way, the same kind of analogy explains why the word 'linear' is used in mathematics to describe the first power. A line segment is a one-dimensional geometrical object. If its length is x units, x being a positive real number, then $x = x^1$ represents the only measurable property of this *linear* figure.

Although in the above examples the exponent is always a positive integer, it can be any real number in general (provided the base is positive). When the exponent is a fraction, the power is identified with a *radical* (this word comes from the Latin "radix", which means nothing else but "root"). If n is an integer greater than 1, then $x^{1/n}$ means *the nth root of x*,

$$x^{1/n} = \sqrt[n]{x}.$$

In the expression $\sqrt[n]{x}$, n is called the *index* and x is the *radicand*. The whole expression is the nth root or radical. When the index is 2, it is not written: $\sqrt[2]{x} = \sqrt{x}$. In view of the above discussion of the second power, it is not surprising that $\sqrt{x}$ is called *the square root*. All other indices (plural of "index") greater than 2 should be written; if we omit them, the radical means a square root. The index does make a difference,

$$\sqrt{4} = 2 \neq \sqrt[3]{4}.$$

Of course, when the index is 3, the root is a *cube root*. Pay attention where the index is supposed to be written:

$$\sqrt[3]{x}, \qquad \sqrt[4]{2}, \qquad \sqrt[5]{y}.$$

If the index is placed in front of the symbol '$\sqrt{\ }$' it may be confused for a number multiplying the radical. For instance, $3\sqrt{x}$ means $3 \cdot \sqrt{x}$, not $\sqrt[3]{x}$. Also, note the difference between $x^3\sqrt{x}$ and $x\sqrt[3]{x}$. The former means $x^3 \cdot \sqrt{x}$ and the latter $x \cdot \sqrt[3]{x}$.

Another important difference between various indices is that *even roots (i.e., those with an even index) are undefined for negative numbers*, whereas odd roots (those with odd indices) are always defined,

$$\sqrt{-64} \quad \text{is undefined (not a real number), but} \quad \sqrt[3]{-64} = -\sqrt[3]{64} = -4.$$

More generally, if m and n are integers, $m \geq 1$, $n \geq 2$, and the fraction $\frac{m}{n}$ is reduced, then $x^{m/n}$ means the following:

(1) $$x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m,$$

provided $\sqrt[n]{x}$ is defined (i.e. $x \geq 0$ if n is even).

If there are several operations in an expression, they have to be performed in a certain order. Among the operations discussed so far, the highest priority is given to exponents. Exponents include radicals because of (1). This means that the first operation that should be done in an expression is *exponentiation* (raising to the exponent). After this, we should multiply or divide, these two operations having equal priority. Finally, the lowest priority is shared by addition and subtraction. There is one more rule that applies to equally ranked operations – they should be done from left to right, the way we (but not the whole world) read and write.

Example 1. Using words, describe the order of operations in each expression.

(a) $1 - 3^4$

(b) $5\sqrt{x} + 4y - 7$

(c) $3y \div 4$

(d) $1 - 10 - 3$

(e) $36 \div 4 \div 2$

Solution.

(a) First, raise 3 to the 4th and then subtract from 1. A typical mistake here is to raise -3 to the fourth, but this is not what the operations indicate.

(b) Take the square root of x and multiply it by 5; multiply 4 and y separately and add this to the previous; finally subtract 7. This shows that operations are not done consecutively from left to right if they don't have the same priority – the addition has to wait until we do the operations in $5\sqrt{x}$ and $4y$. Note also that since x is not a specific number, we cannot actually find the result for $\sqrt{x}$ or for the whole expression.

We would be able to do this if we were told to substitute some constant values for x and y. For instance, if $x = 9$ and $y = -2$,

$$5\sqrt{9} + 4(-2) - 7 = 5 \cdot 3 - 8 - 7 = 15 - 8 - 7 = 7 - 7 = 0.$$

(c) Multiply 3 and y and then divide by 4.

(d) Subtract 10 from 1 to get $1 - 10 = -9$, then subtract 3: $-9 - 3 = -12$. Note that if we do $10 - 3$ first, we violate the rule that operations of equal rank should be done from left to right and we get a wrong result, $1 - 7 = -6$.

(e) Again, do the two divisions from left to right: $36 \div 4 = 9$, $9 \div 2 = \frac{9}{2}$ or 4.5. The order $4 \div 2 = 2$ and $36 \div 2 = 18$ gives a nice-looking but wrong answer. □

Parentheses are used to change the standard order of operations. Consider Example 1(d) again. If our intention is to subtract 3 from 10 first, we have to indicate this using *parentheses* (this is the plural of "parenthesis", a word that comes from Greek and means "insertion"). Thus,

$$1 - (10 - 3) = 1 - 7 = -6 \quad \text{as opposed to} \quad 1 - 10 - 3 = -9 - 3 = -12.$$

Compare also

$$36 \div (4 \div 2) = 36 \div 2 = 18 \quad \text{to} \quad 36 \div 4 \div 2 = 9 \div 2 = \frac{9}{2}.$$

Parentheses are always paired. For each parenthesis opening to the left, i.e. '(', there should be one opening to the right, i.e. ')'. Parentheses are given the highest priority when doing operations – operations in parentheses should be done first. Table 2 summarizes the order of operations.

Priority	Operation
highest	in parentheses
	exponentiation
	multiplication or division
lowest	addition or subtraction
Equally ranked operations are done from left to right.	

Table 2. Order of operations.

When there are many parentheses in an expression, its legibility is improved if we use parentheses of different sizes, e.g.

$$2\big(3(x+1)^2 - 5(2x-7)\big) \quad \text{instead of} \quad 2(3(x+1)^2 - 5(2x-7)),$$

or, even better, brackets [ ] or braces (curly brackets) { } (both come in different sizes too),

$$2\left[3(x+1)^2 - 5(2x-7)\right].$$

The common name for parentheses, brackets, braces, and other similar mathematical symbols that are used in pairs around an expression is *delimiters*. The notation for the absolute value uses another pair of delimiters. This means that the absolute value should be treated like parentheses regarding the order of operations. *When there are several delimiters in the expression, we should start doing operations from the innermost pair of delimiters.*

Example 2. Using words, describe the order of operations in each expression.

(a) $2\{4 - [5 + 2(21 + 17)]\}$

(b) $7(2 \cdot 5 - x) - y^2$

(c) $[3(x-1)^2 + 8|1 - 2 \cdot 5|] \div 6$

(d) $(4a - 2)^3$

(e) $4a - 2^3$

Solution.

(a) Add 21 and 17 since this is the operation in the innermost parentheses. Then multiply the previous result by 2, add 5 to this and subtract everything from 4; finally, multiply by 2.

(b) Multiply 2 and 5 and then subtract x, multiply the result by 7, square y and subtract from the previous. As before, we discuss the order of operations regardless of the fact that for some of them we cannot provide the actual answer since the variables used are not specified.

(c) Subtract 1 from x and square, multiply by 3, and add to this the result of the following operations: $8|1-2\cdot 5| = 8|1-10| = 8\cdot 9 = 72$. Finally, divide everything by 6.

(d) Multiply 4 and a, subtract 2, then cube the previous.

(e) Multiply 4 and a, cube 2 separately, and subtract these two results (compare to (d)). Note that we can multiply here before cubing only because these operations are to be done independently before the subtraction. □

Let us take another look at the absolute value in problem (c) of Example 2, $|1 - 2 \cdot 5|$. The following is a typical misinterpretation:

$$|1 - 2 \cdot 5| = 1 + 2 \cdot 5 = 1 + 10 = 11. \quad \text{WRONG!}$$

The sign in front of 2 is changed "because the absolute value changes the negative number into the positive one". However, what we have above is the absolute value of an arithmetical expression which should be evaluated *before* absolute value is taken. Thus, $1 - 2 \cdot 5 = 1 - 10 = -9$ and then we have $|-9| = 9$. Note that the absolute value is in fact an operation (since it may change the number inside) which is denoted using delimiters.

Parentheses are omitted in certain cases without changing the meaning of the expression. Consider the expression $\sqrt{3+1}$ for instance. According to the order of operations, radicals (being fractional exponents) should be done first. But, what should we take the square root of? The intention here is to add 3 and 1 first and then to take the square root, $\sqrt{3+1} = \sqrt{4} = 2$. Therefore, the meaning of $\sqrt{3+1}$ is $\sqrt{(3+1)}$, but it is customary not to use parentheses in such cases. By the way, the presence of parentheses and the right order of operations is revealed in the alternative notation,

$$(3 + 1)^{1/2} = \sqrt{3+1}.$$

The situation with horizontal fraction bars is similar,

$$\frac{9-3}{4+6} = \frac{(9-3)}{(4+6)} = (9-3)/(4+6) = (9-3) \div (4+6).$$

Whereas the parentheses in the second expression above are normally omitted and the first expression is used instead, it is impossible to omit parentheses in the third and fourth expressions without changing their meaning. Therefore, parentheses may be needed around the numerator and the denominator of a fraction written with a slash fraction bar, but they are not used when the fraction bar is horizontal.

The correct use of parentheses with the slash fraction bar should be kept in mind when typing expressions in older versions of graphing calculator. For instance, to type $\frac{x+4}{2x-7}$, we use the following keystrokes: $(x+4)/(2x-7)$. If the numerator or the denominator are single-term expressions, parentheses are not necessary around them, but there is no harm if we use them. Thus, $3x/(2x-7)$ and $(x+4)/x$ work equally well as $(3x)/(2x-7)$ and $(x+4)/(x)$ respectively. Some calculators may also require parentheses around radicands. When the square–root key is pressed and the calculator displays $\surd($, we need to type $\surd(4x+5)$ for $\sqrt{4x+5}$. If the calculator does not open the left parenthesis automatically after the square-root symbol, we have to type it ourselves. Like in the case of single-term numerators or denominators, parentheses may be omitted around single-term radicands. However, it is a bad habit not to close a parenthesis once the left one is opened, so it is recommended to type $\surd(5)$ and not just $\surd(5$.

Example 3. Rewrite each expression the way it should be typed in the calculator that uses a single line for its input.

(a) $\sqrt{7x+11}$

(b) $\dfrac{4+5x}{9}$

(c) $\dfrac{1+\sqrt{x-6}}{2+x}$

(d) $\sqrt{\dfrac{3}{2x-1}}$

Solution.

(a) $\surd(7x+11)$

(b) $(4+5x)/9$

(c) $(1+\surd(x-6))/(2+x)$

(d) $\surd(3/(2x-1))$ □

EXERCISES 1.3

1. Express each subtraction as addition.
   (a) $x - 9$
   (b) $-a - b$
   (c) $4 - z$
2. Is $-x$ negative or positive when $x = 3$, $x = -7$, $x = -\pi$, or $x = \sqrt{7}$?
3. Simplify.
   (a) $s - 0$
   (b) $0 + a - b$
   (c) $p + 3 - 0 + q + 0$
4. Express each division as multiplication.
   (a) $\frac{5}{a}$
   (b) $\frac{-3}{8}$
   (c) $\frac{u}{10}$
   (d) $x \div 7$
5. Simplify.
   (a) $\frac{1 \cdot x - 1}{1}$
   (b) $1a + 0$
   (c) $b^1 - 0 + 3$
   (d) $\frac{0 - y^1}{1}$
6. In each expression, identify the appropriate quantities: the base and the exponent, or the radicand and the index.
   (a) 3^t
   (b) $\sqrt{x + 1}$
   (c) $\sqrt[4]{7}$
   (d) $(3s - 5)^4$

(e) $\left(\frac{1}{2}\right)^{2-x}$

(f) $\sqrt[m]{z}$

7. Identify the index of the radical in each expression.

   (a) $4\sqrt{x}$

   (b) $2\sqrt[3]{x}$

   (c) $x\sqrt[4]{x}$

   (d) $2^3\sqrt{x}$

8. Using words, describe the order of operations in each expression.

   (a) $9x^2 - 7^2 - 2$

   (b) $8 - 5\sqrt[3]{27}$

   (c) $\frac{3}{z} + 4 \cdot (-7)$

   (d) $1 + 2 \cdot 5^2$

   (e) $4 - 7 + 6 - 2$

9. Using words, describe the order of operations in each expression.

   (a) $[6(3 - 2 \cdot 8)]^4$

   (b) $6(3 - 2) + 8^4$

   (c) $2[(x - 7) \div 3 + 5]$

   (d) $|8 \div 2 - 13|(3y - 4)^2$

   (e) $2x + 5(3y - 4)^2$

10. Rewrite each expression the way it should be typed in the calculator that uses a single line for its input.

    (a) $\dfrac{7x + 11}{\sqrt{3x - 5}}$

    (b) $\sqrt{4 + \dfrac{x - 1}{x + 1}}$

    (c) $\dfrac{x^2 - 3x + 1}{2x^2 + 8x - 3}$

    (d) $\dfrac{7 - 2\sqrt{x}}{x^2 + 1}$

(e) $\sqrt{x+3}+\dfrac{x-3}{x+3}$