MATH-LITERACY MANUAL

Dr. Relja Vulanovic
Professor of Mathematics
Kent State University at Stark

2 Algebraic Expressions

2.1 Terms and Factors

Term or factor? – That is the question. We have introduced algebraic expressions already in Section 1.2. As we shall see soon, the ability to distinguish between terms and factors within an expression is one the most crucial ones in algebra. *Terms are separated by + or − signs, the sign belonging to the term together with the quantity after the sign.*

Example 1. List all terms in $2x^4 - 8x^3 + x^2 + 2x - 7$.

Solution. The terms are $2x^4$, $-8x^3$, x^2 (the + sign does not have to be written and usually is not), $2x$, and -7. □

When describing what $8x^3$ is in the above example, we can say that it is a term since there is a minus and a plus separating it from the preceding and following terms respectively. However, the full description of this term is $-8x^3$.

Factors are separated by multiplication or division. In many sources, only multiplication is mentioned when factors are defined, but there is no reason for excluding division since we have seen in Section 1.3 that division can be viewed as multiplication by the reciprocal. Constant factors are called *coefficients.*

Example 2. List all factors in each expression. What factors are coefficients?

(a) $3x^2y$

(b) $\frac{7x}{5y^4}$

Solution.

(a) Factors: 3, x^2, y; coefficients: 3.

(b) Factors: 7, x, 5, y^4; coefficients: 7, 5, and even $\frac{7}{5}$. In this example, 7 is the coefficient of x, 5 is the coefficient of y^4, and $\frac{7}{5}$ is the coefficient of $\frac{x}{y^4}$. □

Example 3. What are x and -3 in each expression, terms or factors?

(a) $x - 3$

(b) $x(-3)$

(c) $\frac{-3}{x}$

(d) $\frac{-3+x}{2}$

Solution. Terms in (a) and (d) and factors in (b) and (c). □

Terms and factors may be more complicated expressions. In the expressions $A + B$ or $A - B$, A and B are terms, whereas in AB or A/B, they are factors. However, A and B may stand here for longer expressions that themselves have terms and factors.

Example 4. Is the specified expression a term or a factor in the bigger expression it is part of?

(a) $x^2 + 1$ in $7 - 3(x^2 + 1)$

(b) $5(x - 2)$ in $\frac{5(x-2)+4(x+1)}{(x+1)(x-2)}$

(c) $2x - 3$ in $\sqrt{(2x-3)(1-x)}$

(d) $4(x+7)$ in $\sqrt[3]{9 - 4(x+7)}$

Solution.

(a) It is a factor since it is multiplied by -3. Note that $x^2 + 1$ is not a factor of the whole given expression, but of $-3(x^2+1)$ only.

(b) It is a term added to $4(x+1)$.

(c) It is a factor multiplied by $1 - x$.

(d) It is a term, being subtracted from 9 (which is another term). More precisely, this term is $-4(x+7)$. □

EXERCISES 2.1

1. List all terms in each expression.

   (a) $-7x^5 + 2x^3 - \frac{2}{3}x^2 - x + 1$

   (b) $4x + 2\sqrt{3y} - \frac{1}{5}$

   (c) $\dfrac{8a}{3} - \dfrac{b}{6} + 2c$

2. List all factors in each expression. What factors are coefficients?

   (a) $-4x^3 + \frac{5}{2}x^2 - 7x$

   (b) $(4x^2)(-5y)$

   (c) $\dfrac{8a}{3b^3}$

3. What are x^3 and -8 in each expression, terms or factors?

   (a) $-8x^3$

   (b) $-8 + x^3$

   (c) $x^3(-8)$

   (d) $\dfrac{x}{x^3 - 8}$

4. Is $3x - 5$ a term or a factor in the bigger expression it is part of?

(a) $\dfrac{x+2-4(3x-5)}{(3x-5)(x+2)}$

(b) $(4x^2-6x+1)-(3x-5)$

(c) $\sqrt{\dfrac{3x-5}{x^2+1}}$

5. Is $2(x^2+1)$ a term or a factor in the bigger expression it is part of?

(a) $2(x^2+1)[3x+(x-1)(5x^2+1)]$

(b) $\sqrt[4]{2(x^2+1)+9}$

(c) $\dfrac{7(4x+3)+2(x^2+1)}{x}$

2.2 Types of Algebraic Expressions

Algebraic expressions are classified by operations performed on variables. Main types of algebraic expressions are:

- polynomials,
- rational expressions, and
- irrational expressions.

Polynomials are the simplest algebraic expressions. The Greek word “polynomial” can be loosely translated by the phrase “many-term expression”. However, terms forming a polynomial are not arbitrary – they are either constants or constants multiplied by variables raised to positive integers.

Example 1. Determine for each expression whether it is a polynomial or not.

(a) x^3-2x^2+x-9

(b) $\frac{3}{5}x^2-7xy$

(c) $x^2-\sqrt{3x}+1$

(d) $x^2-\sqrt{3}x+1$

(e) $x^4 - \dfrac{6}{x} + 4$

(f) $\sqrt{x^6 - x^2 + 2}$

(g) $8x^{-2} - 3y + 9$

Solution. Expressions (a), (b), and (d) are polynomials. The others are not. Note that fractions and radicals are permitted in a polynomial only as coefficients or constant terms; in other words, polynomials do not have variables under fraction bars or radicals. In order to avoid the possible confusion between expressions like (c) and (d), we often write (d) as $x^2 - x\sqrt{3} + 1$. □

There are two main further classifications of polynomials. One is based on the number of terms. *Monomial* is a polynomial with one term ('mono' means 'alone' in Greek). *Binomials* and *trinomials* are polynomials with two and three terms respectively. The other classification criterion is the *degree* of the polynomial. For polynomials of one variable, the degree is the highest power of the variable. For polynomials of several variables, if there is a term with two or more variables, their powers are added to determine the degree of this term. Then the highest degree of all term degrees is the degree of the polynomial.

Example 2. Determine the degree of each polynomial.

(a) $-2x^4 - 5x^2 + \frac{1}{2}x - 12$

(b) $4x^2y + x^2 - 6xy + 2y^2 - 7xy^2$

(c) $1 - 3x + 5x^2 - 4x^3 + 8x^4 - 2x^5$

Solution. Degrees from (a) to (c) are: 4, 3, 5. □

A single constant term, e.g. 3, can be viewed as a polynomial as well since $3 = 3 \cdot x^0$. This is a constant monomial and its degree is 0. Polynomials of first degree are called *linear* (recall from Section 1.3 that the word 'linear' signifies first power). Second-degree polynomials are called *quadratic* (the Latin word 'quadratum' means 'square'), and polynomials of third degree are *cubic*.

Example 3. Classify each polynomial using any of the following words if applicable: monomial, binomial, trinomial, constant, linear, quadratic, cubic.

(a) $2x - 7x^6$

(b) $4x^3 - x^2 + 6x + 2$

(c) $1 + x - 5x^2 - 3x^4$

(d) $-x^2 + 2x + \frac{7}{3}$

(e) $3x - 27$

(f) π

(g) $8x^5$

Solution. (a) binomial, (b) cubic, (c) nothing applies, (d) quadratic trinomial, (e) linear binomial, (f) constant monomial, (g) monomial. □

A rational expression is any algebraic expression that can be written as a quotient of two polynomials. This is why rational expressions are called also *algebraic fractions.* A rational expression does not have to be given as a quotient of two polynomials but it should be possible to rewrite it as such. For instance,

$$x + \frac{1}{x} \quad \text{is rational since} \quad x + \frac{1}{x} = \frac{x^2 + 1}{x}.$$

An irrational expression is any algebraic expression that has a variable under a radical.

Example 4. Classify each expression as a polynomial, rational expression, or irrational expression.

(a) $x + 1 - \sqrt{x^2 + 5x - 1}$

(b) $3 - x\sqrt{5}$

(c) $\frac{2x^2 - 1}{3x + 8}$

(d) $\frac{6}{5}x^3 - 2x^2 + 1$

(e) $\frac{6}{5x^3} - 2x^2 + 1$

(f) $\frac{x + 1}{\sqrt{x}}$

Solution. Polynomials: (b), (d). Rational expressions: (c), (e). Irrational expressions: (a), (f). □

EXERCISES 2.2

1. Determine for each expression whether it is a polynomial or not.

   (a) $\sqrt[3]{x} - 2x^2 - 9$

   (b) $4x^2yz$

   (c) $\dfrac{5x+9}{3x+4}$

   (d) $x^2 - x + \frac{1}{7}$

   (e) $3xy + 5x^2z^{-1}$

   (f) $1 - x^2\sqrt{7}$

   (g) $1 - x^2\sqrt{x}$

2. Determine the degree of each polynomial.

   (a) $2xyz - 5x^2y + x^2z^2$

   (b) $-4x^2 + 7x + 10^6$

   (c) $x - 6x^3 + 3x^5 - 9x^7$

3. Classify each polynomial using any of the following words if applicable: monomial, binomial, trinomial, constant, linear, quadratic, cubic.

   (a) $-2x^4 - 5x^2 + \frac{1}{2}x - 12$

   (b) $4x^2 - 6x + 2$

   (c) $1 - 3x$

   (d) $11x^7 - 3x^2$

   (e) $-\frac{4}{9}x$

4. Classify each expression as a polynomial, rational expression, or irrational expression.

   (a) $\dfrac{x+2}{x^2} - 3x + 5$

   (b) $\dfrac{1-\sqrt{x}}{x^2+3}$

(c) $\dfrac{7x+\sqrt{3}}{1-4x}$

(d) $6x^3 - 2x^2 + \frac{11}{3}$

(e) $\sqrt[3]{x^2 - 3x + 9}$

2.3 Transforming Algebraic Expressions

Algebraic expressions are all about numbers. Very often, expressions have to be transformed, like in (1),

$$x + \frac{1}{x} = \frac{x^2+1}{x}, \tag{1}$$

because different forms are needed for different purposes. *Whenever an algebraic expression is transformed, its numerical value has to be preserved.* Algebraic expressions are, after all, numbers, only those numbers are usually left unspecified. If we transform an algebraic expression, we should still get the same number. The two expressions stated equal in (1) are indeed equal as numbers for any real number x different from 0 (when $x = 0$, both expressions are undefined). For instance, if $x = 7$,

$$7 + \frac{1}{7} = \frac{50}{7} \quad \text{and} \quad \frac{7^2+1}{7} = \frac{50}{7}.$$

This is why we write the equality symbol '=' between different forms of an algebraic expression that we are transforming. The above example can be done in more than one step if we want to provide all the details of the transformation,

$$x + \frac{1}{x} = \frac{x}{1} + \frac{1}{x} = \frac{x^2}{x} + \frac{1}{x} = \frac{x^2+1}{x}. \tag{2}$$

In this procedure, all expressions are equal from the beginning to the end and we put equal signs between them. We can write (2) aligning the transformation steps vertically, but this does not mean that we should omit equal signs,

$$\begin{aligned} x + \frac{1}{x} &= \frac{x}{1} + \frac{1}{x} \\ &= \frac{x^2}{x} + \frac{1}{x} \\ &= \frac{x^2+1}{x} \end{aligned} \tag{3}$$

It is understood here that each new line continues the previous one, and therefore the meaning of (3) is the same as in (2). An alternative notation is to write equal signs *after* each line but the last. However, omitting equal signs, like below, is not acceptable,

$$\begin{aligned}
&x+\frac{1}{x}\\
&\frac{x}{1}+\frac{1}{x}\\
&\frac{x^2}{x}+\frac{1}{x}\\
&\frac{x^2+1}{x} \quad \text{WRONG!}
\end{aligned}$$

What we see here is four expression which seem to be disconnected and, since we are not told anything else, may be unrelated. *If the expressions we work with are equal, we should say so.* Different kinds of arrows ($\to$, $\Rightarrow$) are not substitutes for '='. Therefore, the notation given below is strongly discouraged,

$$x+\frac{1}{x} \to \frac{x}{1}+\frac{1}{x} \to \frac{x^2}{x}+\frac{1}{x} \to \frac{x^2+1}{x} \quad \text{WRONG!}$$

or

$$\begin{aligned}
&x+\frac{1}{x}\\
\Rightarrow\ &\frac{x}{1}+\frac{1}{x}\\
\Rightarrow\ &\frac{x^2}{x}+\frac{1}{x}\\
\Rightarrow\ &\frac{x^2+1}{x} \quad \text{WRONG!}
\end{aligned}$$

Arrows make no sense in the above contexts since they have special use in mathematics. The symbol '$\Rightarrow$' is meant to indicate an *implication*, that is, when something follows from the previous *statement*. For instance, when we say: "if $x+2=3$, then $x=1$," this can be written more symbolically as $x+2=3 \Rightarrow x=1$. The other type of arrow, $\to$, can mean implications as well, but it is mainly used in calculus to denote the process in which the variable approaches some number. The meaning of arrows is therefore completely different from the equal sign. Even worse, *expressions are not statements* in the first place, so we cannot use arrows between them. When we write $x + 2 = 3$, this is a statement because we are saying that some quantities are equal, whereas in $x + \frac{1}{x}$ we are just indicating some operations to be performed on x. This is also why an expression itself is not enough to describe a mathematical problem. Suppose you are given this "problem": $x + \frac{1}{x}$. What are you supposed to do with it? This you have to be told, since there are different things (including nothing) you can do with an algebraic expression. Thus, "perform the indicated addition in $x + \frac{1}{x}$", or "write $x + \frac{1}{x}$ as an algebraic fraction", constitute valid problems.

There is a concern that the transformation as written in (3) can be mistaken for the procedure used for solving equations. It should be pointed out that there is nothing written on the left side of equal signs and that there has to be something on both sides if an equation is meant. If the left sides of equal signs in (3) are misunderstood as zeros, this too is incorrect, since zero is a number which does not mean a blank space. In other words, if we wanted a zero there, we would have written it.

Equal expressions are equal only when all of them are defined. To be honest, and pedantic, sometimes algebraic expressions are not equal even though we say they are. Already in (1), the two expressions are undefined when $x = 0$. This means that they are not numbers for $x = 0$, but *the equal sign is used only between numbers.* Another example is the following simplification:

$$\frac{x^2 - 1}{x^2 - 3x + 2} = \frac{(x-1)(x+1)}{(x-1)(x-2)} = \frac{x+1}{x-2}. \tag{4}$$

Suppose $x = 1$. The first and the second expressions are undefined since their denominators equal zero, whereas the last expression equals -1. How can -1 equal something that is undefined, that is, how can a number equal something that is not a number? It can't. It is understood here that the three expressions in (4) are equal when *all of them are defined.* This means that they are equal for all real numbers x other than, not only $x = 1$, but $x = 2$ as well. When $x = 2$, all three expressions are undefined.

Example 1. In each problem below, the two expressions are equal when both are defined. For what values of the variable are the equalities valid?

(a) $\dfrac{x(2x-7)}{x} = 2x - 7$

(b) $\sqrt{4x^3} = 2x\sqrt{x}$

(c) $\dfrac{(2x-3)(x-5)}{(2x-3)(x+5)} = \dfrac{x-5}{x+5}$

(d) $\sqrt[3]{z^4} = z\sqrt[3]{z}$

Solution.

(a) For all real numbers x, $x \neq 0$.

(b) For all real numbers x, $x \geq 0$ (recall that even radicals are undefined for negative numbers).

(c) For all real numbers x, $x \neq \frac{3}{2}, -5$.

(d) For all real numbers z (recall that odd radicals are always defined). □

Again, algebraic expression are all about numbers. This means that we cannot transform an algebraic expression by applying to it operations that will change the numerical value of the expression. For instance, the following mistake often happens in *rationalization* problems (i.e. when attempting to remove a radical):

$$\frac{3}{\sqrt{5}} = \left(\frac{3}{\sqrt{5}}\right)^2 = \frac{3^2}{\left(\sqrt{5}\right)^2} = \frac{9}{5}. \quad \text{WRONG!}$$

This certainly eliminates the radical but it does not preserve the value of the fraction – if we square the numerator and the denominator of a fraction, we get a different number, e.g.,

$$\frac{1}{2} \neq \frac{1^2}{2^2} = \frac{1}{4}.$$

It is equally impossible to multiply an expression by a number in order to clear fractions or decimals. By the way, this operation, as well as squaring both sides, can be performed on *equations*, but expressions and equations are completely different, as we shall see in Chapter 3. Thus, if we have an expression like $0.1x^2 - 2x + 2.5$, there is no way we can transform it to

$x^2 - 20x + 25$ after multiplication by 10. If we multiply a non-zero number by 10, we get a different number… Likewise,

$$\frac{x}{4} + \frac{1}{3} \neq 12\left(\frac{x}{4} + \frac{1}{3}\right) = 3x + 4,$$

therefore, fractions cannot be cleared from an expression unless they are completely reducible themselves, like in

$$\frac{8x}{4} + \frac{12}{3} = 2x + 4.$$

Expressions can be transformed in different ways. One of the transformations is *simplification.* What is simpler, is a relative thing, but simplification normally means *expanding* (i.e. removing parentheses) and *combining like terms*, and it does not mean *factoring* (i.e. writing the expression as a product). Since it may be debatable whether $x^2 + x$ is simpler than $x(x+1)$ or not, it is better to specify what kind of simplification is needed. Thus, we can say 'expand', or 'perform the indicated operations', and it goes without saying that like terms should always be combined in the final answer. *Like terms have the same variables raised to the same exponents.* In other words, the only difference between like terms may be in their coefficients. Constant terms are like terms.

Example 2. Simplify each expression by combining like terms.

(a) $3x^2y + 3xy$

(b) $-2x^4 - 5x^4$

(c) $3x^2 - 2x + 1 - x^2 + 2x$

(d) $7xy - y^2 + 2x + 6y^2 - 3xy$

(e) $\frac{1 - 8x + 4x + 3}{x - 7}$

Solution.

(a) The two terms in this expression are unlike and there is nothing to simplify.

(b) $-2x^4 - 5x^4 = -7x^4$. We can see that when we combine like terms, we get another like term, i.e. the variables and their exponents do not change, only the coefficient does.

(c) $3x^2 - 2x + 1 - x^2 + 2x = 2x^2 + 1$. In this example, the like terms $-2x$ and $2x$ have opposite coefficients, so that $-2x + 2x = 0x = 0$, which, of course, is not written in the answer. In this case, we say that the $2x$ and $-2x$ are *opposite terms* which *cancel*.

(d) $7xy - y^2 + 2x + 6y^2 - 3xy = 4xy + 5y^2 + 2x$

(e) $\dfrac{1 - 8x + 4x + 3}{x - 7} = \dfrac{4 - 4x}{x - 7}$ □

When expanding, we do the operations following their prescribed order, see Section 1.3. Again, if an operation cannot be performed because it involves variables, we move on to the next operation. If an expression containing several terms is to be multiplied by a monomial, like in

$$3x(2x^2 - x + 4),$$

we remove parentheses by *distributing* the monomial *over the terms* inside parentheses:

$$3x(2x^2 - x + 4) = 3x \cdot 2x^2 + 3x \cdot (-x) + 3x \cdot 4 = 6x^3 - 3x^2 + 12x.$$

When two multiple-term expressions are factors of a product, we remove parentheses by distributing each term of the first expression over all terms of the other expression.

Example 3. Expand $-\left[3(x + 1)^2 - 5(7 - 2x)\right]$ indicating each step.

Solution. There is nothing we can do with $x + 1$ and $7 - 2x$ in the innermost parentheses, so we square $x + 1$ and distribute -5 over $7 - 2x$. For $(x + 1)^2$, we either use the formula for the binomial squared, $(a + b)^2 = a^2 + 2ab + b^2$, or multiply $(x + 1) \cdot (x + 1)$ to get the same result (the latter is known as *foiling*). Thus,

$$-\left[3(x + 1)^2 - 5(7 - 2x)\right] = -\left[3(x^2 + 2x + 1) - 35 + 10x\right].$$

Since $-5(7) = -35$, it is clear that the resulting term -35 should be written as such. However, there is sometimes confusion how to write the result of

$-5(-2x)$. This is simply $10x$, but we should write it as $+10x$ in order to separate it from the preceding terms. In other words, $-2x$ is a term in $(7-2x)$ and it remains a term after it is multiplied by -5. The + sign before $10x$ is needed to show this.

We continue by multiplying 3 and $x^2 + 2x + 1$ and then simplifying the expression in brackets. Finally, the minus in front of the brackets can be viewed as a coefficient equal to -1, thus we distribute -1 over the expression in brackets. We can refer to this as *distributing the minus.*

$$\begin{aligned}
-\left[3(x+1)^2 - 5(7-2x)\right] &= -\left[3(x^2+2x+1) - 35 + 10x\right] \\
&= -\left[3x^2 + 6x + 3 - 35 + 10x\right] \\
&= -\left[3x^2 + 16x - 32\right] \\
&= -x^2 - 16x + 32
\end{aligned}$$ □

In factoring problems, it is usually required to factor completely. This means that each factor is a *prime*, i.e. it cannot be factored any further. If there is a coefficient which is not a prime, it is usually left unfactored. Thus, in

$$24(2x+5)(x^2+x+1),$$

24 is not a prime number, but $2x + 5$ and $x^2 + x + 1$ are prime expressions. Therefore, the above expression is considered completely factored. We normally factor only relative to integers, which means that we do not use any other numbers when factoring.

Example 4. Determine for each expression whether it is factored or not. If it is, say whether the factorization is complete or not.

(a) $3(x-1) + 5(x^2-1)$

(b) $(8x+4)(x-7)$

(c) $x(2x-1) + 3$

(d) $14(3x+5)(3x-5)$

(e) $2x^2 - x + 4$

Solution.

(a) This expression is not factored since $3(x-1)$ and $5(x^2-1)$ are separated by a +.

(b) Factored, but not completely: $(8x+4)$ is still factorable as $4(2x+1)$.

(c) Not factored.

(d) Factored completely.

(e) Not factored (but not factorable either – this quadratic trinomial is a prime). □

Any quantity has a factor equal to 1. This is important when factoring expressions like x^3+2x^2+x. The GCF (*greatest common factor*) of the terms x^3, $2x^2$, and x is GCF=x. When we *factor the GCF out*, we get $x\,(x^2+2x+1)$. A typical mistake is to give here $x\,(x^2+2x)$ for the answer. This should be checked by distributing x over x^2 and $2x$ to get x^3+2x^2, which is not the original expression. As we can see, distributing and factoring the GCF out are two opposite procedures. *When factoring the GCF out, we get inside parentheses the same number of terms as in the original expression.* Therefore, $x(x^2+2x)$ cannot be right since there are two terms in parentheses, whereas the original expression has three terms. Every term of the expression has to leave a "trace" when the GCF is "pulled" out. The trace of the term x in this example is 1, because $x = 1\cdot x$.

Example 5. Determine for each factorization whether the GCF is factored out correctly or not.

(a) $-5x^2+3x-1 = -(5x^2-3x+1)$

(b) $x^6-2x^4+x^2 = x^2(x^4-2x^2)$

(c) $2x^2-6x+2 = 2(x^2-3x+1)$

(d) $5xy+5x^2y-5xy^2 = 5xy(x-y)$

Solution. (a) and (c) are correct factorizations, (b) and (d) are not. □

When an algebraic expression is transformed, some rules apply to terms and other rules apply to factors. This is why it is important to be able to tell them apart. Four typical situations are discussed below.

We distribute over terms, not over factors. When $3(x+2)(x-1)$ is to be expanded, 3 is sometimes distributed not only over x and 2, but also over x and -1. This results in $(3x+6)(3x-3)$, which is wrong. The binomials $x+2$ and $x-1$ are *factors* of this expression. We can only distribute 3 over the terms of one of the two binomials. Therefore, either $(3x+6)(x-1)$ or $(x+2)(3x-3)$ follow correctly from the original expression. (This is not the end of the expansion, which, by the way, is better to do by foiling $(x+2)(x-1)$ first and then finally distributing 3.) This kind of mistake is of the same type as in

$$2(xy) = 2x \cdot 2y. \quad \text{WRONG!}$$

Here, the factors x and y are mistaken for terms, since the following would be correct:

$$2(x+y) = 2x + 2y.$$

When reducing fractions, only factors, not terms, can be crossed out. The following transformation is wrong:

$$\frac{3+x}{2+x} = \frac{3+\not{x}}{2+\not{x}} = \frac{3}{2}. \quad \text{WRONG!}$$

Incidentally, if $x=0$ in the above expression, then

$$\frac{3+0}{2+0} = \frac{3}{2}.$$

However, if $x=1$,

$$\frac{3+1}{2+1} = \frac{4}{3} \neq \frac{3}{2}.$$

Remember that if we state that two expressions are equal, then this means that they are equal for *all* values of the variables for which both expressions are defined. This is why the above way of reducing the fraction is wrong – terms are mistaken for factors, i.e. the original expression $\frac{3+x}{2+x}$ is mistaken for $\frac{3x}{2x}$, which is safe to reduce,

$$\frac{3x}{2x} = \frac{3\not{x}}{2\not{x}} = \frac{3}{2}.$$

More on this in Section 2.4.

Exponents and radicals can be applied only factor by factor, not term by term. The following is correct:

$$(3xy)^2 = 9x^2y^2, \qquad \sqrt{25x} = 5\sqrt{x}.$$

However, if terms are mistaken for factors, we get the wrong transformations

$$(3x+y)^2 = 9x^2 + y^2, \qquad \sqrt{25+x} = 5 + \sqrt{x}. \quad \text{WRONG!}$$

When simplifying fractional expressions with factors raised to negative exponents, the following shortcut may be taken:

$$\frac{2x^{-2}}{3y^{-1}} = \frac{2y}{3x^2}.$$

A factor with a negative exponent can be moved to the opposite side of the fraction bar by changing the sign of the exponent. This cannot be extended to terms with negative exponents. The following is therefore wrong:

$$\frac{2+x^{-2}}{3+y^{-1}} = \frac{2+y}{3+x^2}. \quad \text{WRONG!} \tag{5}$$

We shall see in Section 2.4 that expressions like the one on the left side in (5) give rise to compound fractions, which are simplified completely differently.

EXRECISES 2.3

1. Rewrite each transformation using vertical alignment.

   (a) $x(x+5)^2 = x(x^2+10x+25) = x^3 + 10x^2 + 25x$

   (b) $\dfrac{x^3+2x^2-6x+9}{x} = \dfrac{x^3}{x} + \dfrac{2x^2}{x} - \dfrac{6x}{x} + \dfrac{9}{x} = x^2 + 2x + 6 + \dfrac{9}{x}$

   (c) $\sqrt{32} - 2\sqrt{18} + 2\sqrt{2} = \sqrt{16\cdot 2} - 2\sqrt{9\cdot 2} + 2\sqrt{2} = \sqrt{16}\sqrt{2} - 2\sqrt{9}\sqrt{2} + 2\sqrt{2} = 4\sqrt{2} - 2\cdot 3\sqrt{2} + 2\sqrt{2} = 4\sqrt{2} - 6\sqrt{2} + 2\sqrt{2} = 0$

2. In each problem below, the two expressions are equal when both are defined. For what values of the variable are the equalities valid?

   (a) $\sqrt[5]{2x^5} = x\sqrt[5]{2}$

(b) $\sqrt[4]{16x} = 2\sqrt[4]{x}$

(c) $\dfrac{(x-8)(x+1)}{(x^2+1)(x-8)} = \dfrac{x+1}{x^2+1}$

(d) $\dfrac{t^3+t^2}{t(t+1)^2} = \dfrac{t}{t+1}$

(e) $\dfrac{x^2}{x^2(7-x)} = \dfrac{1}{7-x}$

3. Simplify each expression by combining like terms.

(a) $3x^2y^3 - 5x^2y^3$

(b) $-2x^4y - 5x^4z$

(c) $\dfrac{x^3-1+x^2+7}{x^2+3}$

(d) $5x + \sqrt{x} - 4 - 6x + 4$

(e) $2y^2 + 7y - 1 + y - 2y^2 - 8y + 1$

4. Expand $2(x^2-3) - [(x+1)(2x-1) - 5]$ indicating each step.

5. Determine for each expression whether it is factored or not. If it is, say whether the factorization is complete or not.

(a) $15(x-1)(x^2-1)$

(b) $(3x+4)(x-7)$

(c) $x(x^2-5x+6)$

(d) $x(3x+5) - 2(3x+5)$

(e) $7x+1$

6. Determine for each factorization whether the GCF is factored out correctly or not.

(a) $-3x^2+6x-3 = -3(x^2-2)$

(b) $y^3-5y^2+y = y(y^2-5y+1)$

(c) $-2x^2y+8xy-2y = -2y(x^2-4x+1)$

(d) $7+14x-7x^2 = 7(2x-x^2)$

7. Determine whether the given transformation is correct or not.

   (a) $\dfrac{x+y^{-2}}{x+2} = \dfrac{x}{(x+2)y^2}$

   (b) $\dfrac{x+y^2}{x+2} = \dfrac{y^2}{2}$

   (c) $x(2x-y) = 2x^2 - xy$

   (d) $\dfrac{4^{-2}a}{b^{-3}} = \dfrac{ab^3}{16}$

   (e) $5(1-x)(3x+2) = (5-5x)(15x+10)$

   (f) $\dfrac{7(x+1)}{14(x-1)(x+1)} = \dfrac{1}{2(x-1)}$

   (g) $(1-x)^3 = 1 - x^3$

   (h) $\sqrt[3]{8z^6} = 2z^2$

2.4 Rational Expressions

The fraction bar is the "axis" of a rational expression. As we have seen in Section 2.2, rational expressions are those that can be written as fractions with polynomials in both the numerator and the denominator. It is important to realize where the fraction bar should be placed relative to other symbols, particularly equal sign and operation symbols, or relative to the surrounding expressions and text.

Example 1. Rational expressions with correctly written fraction bars:

(a) $2x + 4 + \dfrac{3x-5}{x^2+1}$

(b) Simplify $\dfrac{2x-1}{4x^2-1}$.

(c) $\dfrac{5x+4}{3x^2}$

(d) $\dfrac{2x}{4x+6} = \dfrac{2x}{2(2x+3)} = \dfrac{x}{2x+3}$ □

Example 2. Rational expressions from Example 1 with incorrectly written fraction bars:

(a) $2x + 4 + \frac{3x-5}{x^2+1}$

(b) Simplify $\frac{2x-1}{4x^2-1}$.

(c) $\frac{5x+4}{3x^2}$

(d) $\frac{2x}{4x+6} = \frac{2x}{2(2x+3)} = \frac{x}{2x+3}$ □

In Examples 2(a) and 2(b), the fraction bar is placed too low – it should be a continuation of the imaginary horizontal line that goes through the middle of $2x + 4+$ or the word "Simplify". This is an illustration of what we mean by saying that the fraction bar should be the axis of the expression. The same criticism applies to Examples 2(c) and 2(d), but in 2(c) the fraction bar is also too short. In 2(d), we can say that the equality signs are too high since they should be placed along the same imaginary horizontal line (horizontal axis) with fraction bars, like in Example 1(d).

The relative position of the fraction bar is particularly important for compound fractions. A compound fraction has fractions above and/or below the *main fraction bar*, like in the following examples:

$$\frac{1+\frac{x+2}{x-3}}{x^2-9}, \quad \frac{\frac{1}{x}-\frac{1}{y}}{\frac{1}{x}+1}, \quad \frac{\frac{2x+5}{x}}{\frac{3x-1}{x+2}}. \tag{1}$$

We could say that the main fraction bar is the longest one, but the way compound fractions are typeset in many books, it is either impossible or very hard to tell that there is any difference between fraction bar lengths in compound fractions. We mean here compound fractions like the third one in (1) because it is clear in the first and the second ones what fraction bar is the longest and therefore main. We have to take a very close look at the third expression in (1) to realize that the second fraction bar from above is ever so slightly longer than the other two. This is unfortunate; a pronouncedly longer main fraction bar would help us distinguish it more easily from the other fraction bars, compare

$$\frac{\;\frac{2x+5}{x}\;}{\;\frac{3x-1}{x+2}\;} \quad \text{to} \quad \frac{\frac{2x+5}{x}}{\frac{3x-1}{x+2}}.$$

However, there are other ways to tell what fraction bar is the main one. We have to pay attention to the horizontal axis of the expression and the surrounding symbols and text. For instance, the main fraction bar below lies on the same horizontal axis as the period (the punctuation mark),

$$\frac{\frac{2x+5}{x}}{\frac{3x-1}{x+2}}. \tag{2}$$

Why is the main fraction bar important at all? The following example illustrates this.

Example 3. Simplify $\dfrac{\frac{2x-1}{3x+1}}{x}$ and $\dfrac{2x-1}{\frac{3x+1}{x}}$.

Solution. It looks like we are asked here to simplify the same expression twice. However, the two expressions are not identical since they have different main fraction bars. Even though there is a minimal difference in the fraction bar lengths, we can identify the main fraction bars by the horizontal axis of the whole sentence. Therefore,

$$\frac{\frac{2x-1}{3x+1}}{x} = \frac{2x-1}{3x+1} \div x = \frac{2x-1}{3x+1} \cdot \frac{1}{x} = \frac{2x-1}{x(3x+1)} \tag{3}$$

and

$$\frac{2x-1}{\frac{3x+1}{x}} = (2x-1) \div \frac{3x+1}{x} = (2x-1) \cdot \frac{x}{3x+1} = \frac{x(2x-1)}{3x+1}. \tag{4}$$

We see in (3) and (4) that equal signs stand by the main fraction bars. □

In other words, Example 3 shows that the following problem is ambiguous: *Simplify*

$$\begin{array}{c} 2x-1 \\ \hline 3x+1 \\ \hline x \end{array}$$

The two fraction bars are of equal length and there is no other symbol or text (not even a punctuation mark) to indicate the main fraction bar. The best remedy is to use an easily recognizable length for the main fraction bar. Thus the problem *Simplify*

$$\frac{\;\frac{2x-1}{3x+1}\;}{x}$$

is not ambiguous and it should be solved like in (3), as opposed to *Simplify*

$$\frac{2x-1}{\frac{3x+1}{x}}$$

which should be solved like in (4).

When compound fractions have to be written out in one line (like when typing them in the graphing calculator, see Section 1.3), the order of operations and the proper use of parentheses remove any ambiguity. Thus, the following two expressions correspond to the original compound fractions in (3) and (4) respectively:

$$(2x-1)/(3x+1)/x \quad \text{and} \quad (2x-1)/[(3x+1)/x].$$

In the above discussion of possible ambiguity involving compound fractions with fraction bars of equal length, examples with two fraction bars are used for simplicity. The same problems remain for compound fractions with three fraction bars, like the one in (2). Moreover, when simplifying compound fractions like the two first ones in (1), one of the standard methods is to transform them initially to the structure

$$\frac{\frac{A}{B}}{\frac{C}{D}},$$

where A, B, C, and D are some polynomials, and where it is again important to recognize the main fraction bar. If there is no indication what fraction bar is the main one, there are five possible ways to interpret this kind of structure:

$$\begin{array}{l} (A/B)/(C/D), \quad A/[(B/C)/D] = A/(B/C/D), \quad A/[B/(C/D)], \\ \left[A/(B/C)\right]/D, \quad [(A/B)/C]/D = A/B/C/D. \end{array} \tag{5}$$

Two of the above expressions are simplified by reducing the number of parentheses because the indicated divisions are done from left to right.

To divide by a number is the same as to multiply by its reciprocal. This has already been mentioned in Section 1.3. We see new examples of this fact in the transformations (3) and (4). This is used below to show that the five expression in (5) reduce to four different forms:

$$(A/B)/(C/D) = \frac{A}{B} \cdot \frac{D}{C} = \frac{AD}{BC},$$

$$A/[(B/C)/D] = A \cdot \frac{D}{\frac{B}{C}} = A \cdot D \cdot \frac{C}{B} = \frac{ACD}{B},$$

$$A/[B/(C/D)] = A \cdot \frac{\frac{C}{D}}{B} = A \cdot \frac{C}{D} \cdot \frac{1}{B} = \frac{AC}{BD},$$

$$[A/(B/C)]/D = \frac{A}{\frac{B}{C}} \cdot \frac{1}{D} = A \cdot \frac{C}{B} \cdot \frac{1}{D} = \frac{AC}{BD},$$

$$[(A/B)/C]/D = \frac{\frac{A}{B}}{C} \cdot \frac{1}{D} = \frac{A}{B} \cdot \frac{1}{C} \cdot \frac{1}{D} = \frac{A}{BCD}.$$

Different operations with rational expressions require different approaches. This is why it is important to recognize what kind of operation should be performed between two or more fractions.

Example 4. Classify each expression according to the operation indicated between two algebraic fractions. Use the following phrases: multiplication problem, division problem, addition problem, or subtraction problem.

(a) $\frac{3}{2x+7} - \frac{x+4}{5-x}$

(b) $\frac{x^2+4}{x^2-2x+4} \div \frac{7x}{x^2-4}$

(c) $\frac{x^3}{x-9} + \frac{2x^2-3}{4x+5}$

(d) $\frac{\frac{5x-1}{x^2}}{\frac{1-2x}{x+3}}$

(e) $\frac{x^3-1}{x^2-x} \cdot \frac{2x-6}{x^2+x}$

(f) $\frac{1}{x} + \frac{8}{y}$

Solution. Multiplication problems: (e); division problems: (b), (d); addition problems: (c), (f); subtraction problems: (a). □

Finding the least common denominator (LCD) is needed only when adding and subtracting rational expressions with unlike denominators. The phrase "like denominators" indicates that all denominators are identical polynomials even though they may look differently. For instance, if $x(x+1)$ and x^2+x are the denominators of two rational expressions, they are like denominators since $x^2+x = x(x+1)$. Thus, if among all the denominators in an expression, there are two which are non-identical polynomials, then we have algebraic fractions with unlike denominators.

Example 5. Determine for each expression whether we need to find the LCD when performing the indicated operations.

(a) $\dfrac{x^2-1}{3x} - \dfrac{x+4}{3x}$

(b) $\dfrac{7x}{5x-1} + \dfrac{x+6}{2x+3}$

(c) $\dfrac{x}{2x^2-x+1} \cdot \dfrac{x+1}{x^3-8}$

(d) $\dfrac{4x-1}{x+3} + \dfrac{1}{3+x} - \dfrac{x^2-2x+6}{x+1}$

(e) $\dfrac{3-2x}{x+x^2} \div \dfrac{7x+1}{x-9}$

Solution. The LCD is needed in problems (b) and (d). Note that the first and the second denominators in (d) are like, but the third one is unlike. The LCD is not needed in problems (c) and (e) since it is never required for multiplication or division of algebraic fractions. Although (a) is a subtraction problem, the two rational expressions have like denominators and their LCD (LCD = $3x$) is already present; it does not have to be formed separately. □

Factor first, then cross out. We have seen already in the previous section that only factors, not terms, can be crossed out when reducing a fraction. Both the numerator and the denominator should be factored completely and only then can we cross out identical factors on opposite sides of the fraction bar. This is also how we approach multiplication of rational expressions – each numerator and each denominator should be completely factored and then identical factors on opposite sides of either fraction bar should be crossed out.

Example 6. Determine whether $2x-3$ can be crossed out of each expression.

(a) $\dfrac{2x-3}{x^2+1} + \dfrac{4x-7}{2x-3}$

(b) $\dfrac{x(2x-3)}{(2x-3)(x+8)}$

(c) $\dfrac{x(2x-3)+(x^2+2)(x+1)}{(x+1)(2x-3)}$

(d) $\dfrac{(x-2)(x+3)}{2x-3} \cdot \dfrac{(2x-3)(4x+1)}{x(x^2+1)}$

(e) $\dfrac{(2x-3)x}{(2x+1)(x-5)} \div \dfrac{5x-9}{(2x-3)(3x+1)}$

Solution.

(a) No, this is an addition, not a multiplication problem.

(b) Yes, $2x-3$ is a factor of both the numerator and the denominator.

(c) No, although $2x-3$ is a factor of the denominator and even of a part of the numerator (it is a factor of $x(2x-3)$), it is not a factor of the *whole* numerator. The numerator is not factored at this stage, so it is too early to start crossing out.

(d) Yes, this is a multiplication problem and $2x-3$ is a factor of the second numerator and it is the entire first denominator. We can say that it is a factor of the first denominator because we can think of $1 \cdot (2x-3)$ there.

(e) No, this is a division problem, and, as we have seen already in this section, to divide by an algebraic fraction, we multiply by its reciprocal. This example therefore becomes

$$\frac{(2x-3)x}{(2x+1)(x-5)} \cdot \frac{(2x-3)(3x+1)}{5x-9}$$

and both $(2x-3)$-factors are in the numerators. □

Example 6(d) reminds us also that 1 is a factor of any quantity, see Section 2.3. Therefore, when $2x-3$ is crossed out of this expression, 1 remains in the first denominator before we get the final answer:

$$\begin{aligned} \frac{(x-2)(x+3)}{2x-3} \cdot \frac{(2x-3)(4x+1)}{x(x^2+1)} &= \frac{(x-2)(x+3)}{1} \cdot \frac{(4x+1)}{x(x^2+1)} \\ &= \frac{(x-2)(x+3)(4x+1)}{x(x^2+1)}. \end{aligned}$$

Compare this example to

$$\frac{(x-2)(x+3)}{3-2x}\cdot\frac{(2x-3)(4x+1)}{x(x^2+1)} = \frac{(x-2)(x+3)}{-1}\cdot\frac{(4x+1)}{x(x^2+1)} = -\frac{(x-2)(x+3)(4x+1)}{x(x^2+1)}. \tag{6}$$

How can we cross out factors $3-2x$ and $2x-3$ which are not identical? When we interchange two numbers with the operation of subtraction between them, we get different results: $3-8=-5$ and $8-3=5$. However, we can see that the only difference between the results is the sign. Therefore $3-2x=-(2x-3)$ and this why we still can cross $2x-3$ out, but -1 is left in the first denominator in (6).

In order to add or subtract fractions with unlike denominators, we have to make them more complicated. Consider the following addition problem:

$$\frac{x}{x+1}+\frac{x^2+2}{2x-3}. \tag{7}$$

Since the LCD is needed here, we change each fraction,

$$\frac{x(2x-3)}{(x+1)(2x-3)}+\frac{(x^2+2)(x+1)}{(2x-3)(x+1)}. \tag{8}$$

Note that in order to preserve the value of a fraction, we have to multiply both its numerator and denominator by the same quantity. Each fraction in (8) is immediately possible to reduce, but if we do this, we will return to the beginning, that is, (7). Therefore, we make each fraction more complicated on purpose – this is the only way to get like denominators and add the fractions. We have to be patient and refrain from reducing fractions too early. After step (8), we switch to one fraction bar,

$$\frac{x(2x-3)+(x^2+2)(x+1)}{(2x-3)(x+1)}.$$

A typical mistake now is to cross out $2x-3$ and $x+1$ and get $x+x^2+2$, that is, x^2+x+2 for the final answer. This is wrong because, as pointed out in Example 6(c), the numerator is not factored yet and there is nothing to cross out at this stage.

The above discussion applies equally to subtraction problems. There is, however, an additional dangerous spot in these problems if the second fraction has more than one term in the numerator. The following two fractions already have like denominators,

$$\frac{x-1}{5x} - \frac{2x+3}{5x}, \tag{9}$$

so we are ready to write this expression with one fraction bar. The following is a frequent mistake:

$$\frac{x-1-2x+3}{5x} \quad \text{WRONG!} \tag{10}$$

We should recall here from Section 1.3 that the meaning of the operations in $\frac{2x+3}{5x}$ is better expressed as $\frac{(2x+3)}{(5x)}$, but that it is normal practice to omit parentheses. The parentheses around the numerator should now be reinstated in order to do the subtraction in (9) correctly. The answer is

$$\frac{x-1}{5x} - \frac{2x+3}{5x} = \frac{x-1-(2x+3)}{5x} = \frac{x-1-2x-3}{5x} = \frac{-x-4}{5x}. \tag{11}$$

The point here is that the *whole second numerator should be subtracted* from the first one, whereas in (10), only the first term, $2x$, is subtracted. Another way of avoiding the mistake in (10) is to rewrite the original problem (9) in the form

$$\frac{x-1}{5x} + \frac{-(2x+3)}{5x} \tag{12}$$

and then proceed like in (11). We have already stated in Section 1.3 that subtraction can be viewed as addition of the opposite number. This is why expressions (9) and (12) are equal.

EXERCISES 2.4

1. Some of the rational expressions given below are written with incorrect fraction bars. Identify them and suggest possible corrections.

   (a) Reduce $\frac{x^2-4x}{x^2-16}$.

(b) Simplify $\dfrac{(x-1)(x+3)}{x^2+x-6}$.

(c) $2x-7+\dfrac{3x-1}{x^2+x-9}$

(d) $\dfrac{x^2+x}{x^2-1} \quad = \quad \dfrac{x}{x-1}$

(e) $\dfrac{2x-8}{7x^3}$

2. Simplify $\dfrac{\dfrac{3x+5}{x-1}}{4-x}$ and $\dfrac{3x+5}{\dfrac{x-1}{4-x}}$.

3. Simplify the compound fraction using all five possible interpretations.

$$\frac{\frac{\frac{x}{2x+7}}{5-3x}}{9x+4}$$

4. Classify each expression according to the operation indicated between two algebraic fractions. Use the following phrases: multiplication problem, division problem, addition problem, or subtraction problem.

(a) $\dfrac{3}{2x+7}+\dfrac{x+4}{5-x}$

(b) $\dfrac{\frac{x}{3}}{\frac{x+y}{x-y}}$

(c) $\dfrac{x^2+4}{x^2-2x+4}\cdot\dfrac{7x}{x^2-4}$

(d) $\dfrac{x^3}{x-9}-\dfrac{2x^2-3}{4x+5}$

(e) $\dfrac{x^3-1}{x^2-x}-\dfrac{2x-6}{x^2+x}$

(f) $\frac{1}{x}\div\frac{8}{x^2}$

5. Determine for each expression whether the LCD is needed when performing the indicated operations.

(a) $\dfrac{x^2-1}{x+3}\div\dfrac{x+4}{3x}$

(b) $\dfrac{7x}{5x^2 - x} - \dfrac{x+6}{x(5x-1)}$

(c) $\dfrac{x}{2x^2 - x + 1} + \dfrac{x+1}{x^3 - 8}$

(d) $\dfrac{4x-1}{x-3} - \dfrac{1}{x^2+1} + \dfrac{x^2 - 2x + 6}{x^2+1}$

(e) $\dfrac{3-2x}{x+x^2} \cdot \dfrac{7x+1}{x-9}$

6. Determine whether $x^2 + 4$ can be crossed out of each expression.

(a) $\dfrac{x^2+4}{2x+1} \cdot \dfrac{5x^2 - x}{x^2+4}$

(b) $\dfrac{x(x^2+4)}{1 + (x^2+4)(x+8)}$

(c) $\dfrac{(3x-2)(2x-3)}{x^2+4} - \dfrac{x^2+4}{7x-3}$

(d) $\dfrac{(x-2)(x+3)}{(x-1)(x^2+4)} \div \dfrac{x^2+4}{(6x-5)(x^2+1)}$

(e) $\dfrac{(x^2+4)(2x-3)}{(2x+1)(x^2+4)}$

7. Simplify by crossing out appropriate factors if any.

(a) $\dfrac{(4x-7)(2x+1)}{(1+2x)(3x^2 - x + 8)}$

(b) $\dfrac{(x^2-3)(2x+1)}{5x-9} \cdot \dfrac{x^3-2}{x(x-8)}$

(c) $\dfrac{(4x-7)(2x+1)}{(7-4x)(3x^2 - x + 8)}$

(d) $\dfrac{x}{x^2(x-5)} \cdot \dfrac{x-5}{3x(2x+1)}$

(e) $\dfrac{x^2+1}{x(x-5)} \cdot \dfrac{2(5-x)}{3x(x^2+1)}$

8. Determine for each transformation given below whether it is correct or not. Correct those that are wrong.

(a) $\dfrac{x+1}{x} + \dfrac{2x-7}{3} = \dfrac{\cancel{3}(x+1) + \cancel{x}(2x-7)}{\cancel{3}\,\cancel{x}} = 3x - 6.$

(b) $\frac{3}{4} - \frac{1}{x} = \frac{3x-4}{4x}$

(c) $\frac{3}{4} - \frac{1-x}{4} = \frac{3-1-x}{4} = \frac{2-x}{4}$

(d) $\frac{x}{2x+1} + \frac{x+1}{2x+1} = \frac{2x+1}{2x+1} = 1$

(e) $\frac{x}{2x+1} - \frac{x+1}{2x+1} = \frac{-1}{2x+1}$